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Edge addition and edge deletion of graphs*

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Abstract: Let $P(t,d)$ (resp. $C(t,d)$) be the minimum diameter of a connected graph obtained from a single path (resp. from a single circle) of length d by adding t extra edges. It is proved that $P(t,d)=\left\lceil\frac{d-2}{t+1}\right\rceil+1$ if t and d satisfy the following conditions: $t\geqslant 4$ and $t+4\leqslant d\leqslant t+7$, $t=4$ and $d=10k+1$ $(k\geqslant 1)$. For some t and d, the exact value and the best lower bound of $C(t,d)$ are determined, and so the conjecture of Schoone, et al [J. Graph Theory, 1987, 11: 409-427] is settled partially.

Key words: diameter; altered graph; edge addition; edge deletion; Schoone et al's conjecture

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图的边添加和减少

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摘要:用 $P(t,d)$ (或者 $C(t,d)$)表示从一条长为 d 的简单路(或者简单圈)通过添加 t 条边后得到图的最小直径.证明了:如果 t 和 d 满足条件 $t\geqslant 4$ 且 $t+4\leqslant d\leqslant t+7$,或者 $t=4$ 且 $d=10k+1$ ($k\geqslant 1$),那么 $P(t,d)=\left\lceil\frac{d-2}{t+1}\right\rceil+1$. 对某些 t 和 d,确定了 $C(t,d)$ 的值和最好下界,部分地解决了 Schoone 等的猜想[J. Graph Theory, 1987, 11: 409—427].

关键词:直径;变更图;边添加;边减少;Schoone 等的猜想

0 Introduction

We follow ref. [1] for graph-theoretical terminology and notation not defined here. Let G be a simple undirected graph with vertex-set $V=V(G)$ and edge-set $E=E(G)$.

Let $P(t,d)$ denote the minimum diameter of a graph obtained by adding t extra edges to a path $P=(x_1,x_2,\cdots,x_{d+1})$ of length d. For some small t's and special d's, the values of $P(t,d)$ have been determined. It is easy to verify that $P(1,d)=\left\lfloor\frac{d+1}{2}\right\rfloor$ for $d\geqslant 2$. Ref. [2] determines $P(2,d)=\left\lceil\frac{d+1}{3}\right\rceil$ for $d\geqslant 3$ and $P(3,d)=\left\lceil\frac{d+2}{4}\right\rceil$ for $d\geqslant 5$.

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Ref. [3] determines $P(t,(2k-1)(t+1)+1)=2k$ for any positive integer k,

$$\left\lceil\frac{d}{t+1}\right\rceil \leqslant P(t,d) \leqslant \left\lceil\frac{d}{t+1}\right\rceil+1$$

for $t=4,5$ and $d\geqslant 4$ and in general

$$\left\lceil\frac{d}{t+1}\right\rceil \leqslant P(t,d) \leqslant \left\lfloor\frac{d-2}{t+1}\right\rfloor+3.$$

Also ref. [4] determines

$$P(t,d) \leqslant \begin{cases}\left\lceil\frac{d-2}{t+1}\right\rceil+2 & \text{if } d\in I'(t,k),\\ \left\lceil\frac{d-2}{t+1}\right\rceil+1 & \text{otherwise,}\end{cases}$$

where $I'(t,k)=\{2k(t+1)+1,2k(t+1)+2,2k(t+1)-t+1\}\cup\{2k(t+1)-t+h: h=6,7,\cdots,t\}$ and the integers $t\geqslant 6, d\geqslant 2$ and $k\geqslant 1$.

Let $C(t,d)$ denote the minimum diameter of a graph obtained by adding t extra edges to a circle of length d. It is easy to verify that $C(1,d)=\left\lfloor\frac{d}{2}\right\rfloor$ for $d\geqslant 2$. Ref. [2] determines $C(2,d)=\left\lceil\frac{d+2}{4}\right\rceil$ for $d\geqslant 4$. For general $t\geqslant 3$ and $d\geqslant 5$, ref. [5] obtains $\left\lceil\frac{d}{t+i}\right\rceil-1<C(t,d)\leqslant\left\lceil\frac{d}{t+i}\right\rceil+3$ for i equal 1 if t is odd and 2 if it is even. WU and XU① prove that for $d\geqslant 5$, $\left\lceil\frac{d}{4}\right\rceil-1<C(3,d)\leqslant\left\lceil\frac{d}{4}\right\rceil$ and in general $C(t,d)\leqslant\left\lceil\frac{d-9+i}{t+i}\right\rceil+3$ for i equal 1 if t is odd and 2 if t is even for any integer $t\geqslant 4$.

Let $f(t,d)$ denote the maximum diameter of a connected graph obtained after deleting t edges from a connected graph of diameter d. For $t=1$, ref. [6] determines $f(1,d)=2d$. For small t and d, Schoone, et al[2] prove that $f(2,d)=3d-1$, $f(3,d)=4d-2$ for $d>1$, $f(t,2)$ is equal to $t+3$ for $t=1,2,3,4,6$ and $t+2$ otherwise, $f(t,d)\leqslant(t+1)d$ for any positive integers t and d, and $f(t,d)\geqslant(t+1)d-t$ for d is even. Also WU and XU① prove $f(t,d)\geqslant(t+1)d-2t+4$ for $t\geqslant 4$ and d is odd not less than 3.

In this paper we first prove that $P(t,d)=\left\lceil\frac{d-2}{t+1}\right\rceil+1$, for $t\geqslant 4, t+4\leqslant d\leqslant t+7$ and for $t=4, d=10k+1$ ($k\geqslant 1$). Then we prove that for $t\geqslant 3$, $C(t,d)=3$ for $t+6\leqslant d\leqslant t+8$, and the lower bound $C(t,d)$ is 3 for $d=t+9$ and $2k$ for $d=(2k-1)(t+2)+2(k\geqslant 1)$. These bounds are tight for some t and d.

After that we prove the conjecture of Schoone et al[2] $f(t,d)\leqslant(t+1)d-t+1$, for $t\geqslant 4, t+4\leqslant d\leqslant t+7$, and $d=2k(t+1)-t, k\geqslant 1$, and for $t=4$ and $d=10k+1$. This upper bound is tight for some d.

1 Several lemmas

Lemma 1.1[4] For any integer $k\geqslant 1$, let $I'(t,k)=\{2k(t+1)+1,2k(t+1)+2,2k(t+1)-t+1\}\cup\{2k(t+1)-t+h: h=6,7,\cdots,t\}$. Then

$$P(t,d) \leqslant \begin{cases}\left\lceil\frac{d-2}{t+1}\right\rceil+2 & \text{if } d\in I'(t,k),\\ \left\lceil\frac{d-2}{t+1}\right\rceil+1 & \text{otherwise,}\end{cases}$$

for the integers $t\geqslant 6$, and $d\geqslant 2$.

Lemma 1.2[3] For given positive integers t and $d(\geqslant 2)$, $\left\lceil\frac{d}{t+1}\right\rceil\leqslant P(t,d)\leqslant\left\lfloor\frac{d-2}{t+1}\right\rfloor+3$. In particular, $P(t,(2k-1)(t+1)+1)=2k$ for any positive integer k, $P(t,d)\leqslant\left\lfloor\frac{d}{t}\right\rfloor+1$ if k is large enough, and $\left\lceil\frac{d}{t+1}\right\rceil\leqslant P(t,d)\leqslant\left\lceil\frac{d}{t+1}\right\rceil+1$ for $t=4,5$ and $d\geqslant 4$.

Lemma 1.3[7] Let G be a connected undirected graph, $S\subset E(G)$ with $|S|=t$. If $h=d(G-S)$ is well defined, then $d(G)\geqslant P(t,h)$.

Lemma 1.4① For $d\geqslant 5$ and $t\geqslant 4$,

$$C(t,d)\leqslant\begin{cases}\frac{d-7}{t+2}+3 & \text{if } t \text{ is even,}\\ \left\lceil\frac{d+t-6}{2t+2}\right\rceil+\left\lceil\frac{d+t-1}{2t+2}\right\rceil\leqslant\frac{d-8}{t+2}+3 & \text{if } t \text{ is odd.}\end{cases}$$

Also $\left\lceil\frac{d}{4}\right\rceil-1\leqslant C(3,d)\leqslant\left\lceil\frac{d}{4}\right\rceil$.

① WU Ye-zhou, XU Jun-ming. On diameters of altered graph, to appear in J. Math. Research Exposition (2006).

2 Proof of main results

2.1 Edge addition

Theorem 2.1 For $t \geqslant 4$,

$$P(t,d)=\left\lceil \frac{d-2}{t+1} \right\rceil + 1 = 3,$$

where $t+4 \leqslant d \leqslant t+7$.

Proof Let $P=(x_0,x_1,\cdots,x_d)$ be an (x_0,x_d)-path and G an altered graph obtained from P by adding t extra edges and having diameter $d(G)=P(t,d)$, where $d \geqslant t+4$.

If $d(G)=1$, then the number of the extra edges is equal to

$$t \geqslant (d-1)+(d-2)+\cdots+1 = d(d-1)/2 \geqslant d-3 = t+1,$$

a contradiction.

If $d(G)=2$, then let x_i be the smallest numbered vertex that G has no edge (x_i,x_j) with $j>i+1$. For each $j=0,1,\cdots,i-1$, there exists a $j'(j' \geqslant j+2)$ such that $(x_j,x_{j'}) \in E(G)$ is an extra edge, and so such edges are at least i.

Since $d(G)=2$, we must be able to reach every other vertex in two steps from x_i. Hence we need edges $(x_{j'},x_j)$ with $j \geqslant j'+2$ for all j with $i+3 \leqslant j \leqslant d$, since these vertices could not be reached in two steps from x_i in P. Such edges are at least $d+1-(i+3)-1=d-i-3$, where the appearance of (-1) is due to the possibility that there may be an extra edge (x_{i-1},x_j) in the first (i) extra edges when $j'=i-1$. Note that if $j'<i-1$, then $(x_i, x_{j'})$ and $(x_{j'},x_j)$ are extra edges. Thus

$$t \geqslant i+(d-i-3)=d-3=t+1,$$

a contradiction.

Thus $P(t,t+4)=d(G) \geqslant 3$. Since $P(t,d) \leqslant P(t,d')$ if $d \leqslant d'$, $P(t,d) \geqslant 3$ for $d \geqslant t+4$. On the other hand, since $t+4,t+5,t+6,t+7 \notin I'(t,k)$ for $k=1$, from Lemma 1.1,

$$P(t,d) \leqslant \left\lceil \frac{d-2}{t+1} \right\rceil + 1 = 3 \text{ for } t \geqslant 6.$$

For $t=4,5$, from Lemma 1.2, $P(t,d) \leqslant \left\lceil \frac{d}{t+1} \right\rceil + 1 = 3$ for $d \geqslant 4$. Thus, $P(t,d)=3$ for $t+4 \leqslant d \leqslant t+7$ and $t \geqslant 4$. □

Theorem 2.2 $P(4,d)=\left\lceil \frac{d-2}{5} \right\rceil + 1$ where $d=10k+1$ and $k \geqslant 1$.

Proof First we prove that $P(4,d) \leqslant \left\lceil \frac{d-2}{5} \right\rceil + 1$, where $d=10k+1$ and $k \geqslant 1$. We add four edges

$$e_1=x_1x_{4k}, \quad e_2=x_{4k}x_{8k+2},$$
$$e_3=x_{2k}x_{6k+1}, \quad e_4=x_{6k+1}x_{d+1}$$

to the path $P=x_1x_2\cdots x_{d+1}$ of length d (see Fig. 1 for $k=2$ and $d=21$). Now the end-vertices of these edges divide P into five segments

$$L_1=P(x_1,x_{2k}), \quad L_2=P(x_{2k},x_{4k}),$$
$$L_3=P(x_{4k},x_{6k+1}), \quad L_4=P(x_{6k+},x_{8k+2}),$$
$$L_5=P(x_{8k+2},x_{d+1}).$$

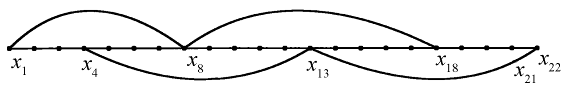

Fig. 1 Construction of Theorem 2.2 for $k=2$ and $d=21$

Define ten cycles as follows

$$C^1=L_1 \cup + L_2+e_1,$$
$$C^2=L_1 \cup + L_3+e_1+e_3,$$
$$C^3=L_1 \cup + L_4+e_1+e_2+e_3,$$
$$C^4=L_1 \cup + L_5+e_1+e_2+e_3+e_4,$$
$$C^5=L_2 \cup + L_3+e_3,$$
$$C^6=L_2 \cup + L_4+e_2+e_3,$$
$$C^7=L_2 \cup + L_5+e_2+e_3+e_4,$$
$$C^8=L_3 \cup + L_4+e_2,$$
$$C^9=L_3 \cup + L_5+e_2+e_5,$$
$$C^{10}=L_4 \cup + L_5+e_4.$$

Their lengths are

$$\varepsilon(C^1)=4k;$$
$$\varepsilon(C^i)=4k+2, \quad \text{for } i=2,5;$$
$$\varepsilon(C^{10}) \leqslant 4k+1;$$
$$\varepsilon(C^i) \leqslant 4k+3; \quad \text{for} \quad i=3,4,6,7,8,9.$$

Then any vertices x and y of G are contained in some cycles C^i defined above. Thus

$$\max\{d(Ci):1 \leqslant i \leqslant 10\} \leqslant \left\lfloor \frac{4k+3}{2} \right\rfloor = 2k+1.$$

This means that for $d=10k+1$ and $k \geqslant 1$,

$$P(4,d) \leqslant d(G) \leqslant 2k+1 = \left\lceil \frac{d-2}{5} \right\rceil + 1.$$

From Lemma 1. 2 we have

$$\left\lceil \frac{10k+1}{5} \right\rceil \leqslant P(4, 10k+1).$$

Since

$$\left\lceil \frac{10k+1}{5} \right\rceil = \left\lceil \frac{10k-1}{5} \right\rceil + 1$$

for $k \geqslant 1$, the theorem follows. □

Theorem 2. 3 For $t \geqslant 3, C(t,d) = 3$ where $t+6 \leqslant d \leqslant t+8$; $3 \leqslant C(t,d) \leqslant 4$ for $d = t+9$; and $2k \leqslant C(t,d) \leqslant 2k+1$ for $d = (2k-1)(t+2)+2$ and t is even and $k \geqslant 1$ or t is odd and $k = 1,2,3$.

Proof It is easy to verify that,

$$C(t, d+1) \geqslant P(t+1, d),$$

since one way of adding $t+1$ edges to a path P_{d+1} is to first add one edge joining two end vertices of P_{d+1} and then to add t edges in an optimal way to result in a cycle C_{d+1}. Then from Theorem 2. 1 we have

$$C(t, d+1) \geqslant 3 \quad \text{for } t+5 \leqslant d \leqslant t+8, t \geqslant 3.$$

Also from Lemma 1. 2 we have

$$C(t, d+1) \geqslant 2k, \text{ for } t \geqslant 3,$$
$$d = (2k-1)(t+2)+1, k \geqslant 1.$$

Now from Lemma 1. 4 we have for $t \geqslant 3$

$$C(t, d+1) \leqslant 3, \quad \text{for } t+5 \leqslant d \leqslant t+7,$$
$$C(t, d+1) \leqslant 4, \quad \text{for } d = t+8,$$

and $C(t, d+1) \leqslant 2k+1$ for $d = (2k-1)(t+2)+1$, and t is even and $k \geqslant 1$ or t is odd and $k = 1,2,3$. So the theorem follows. □

2. 2 Edge deletion

Theorem 2. 4 $f(t,d) \leqslant (t+1)d - t + 1$ for $t \geqslant 4$ and $t+4 \leqslant d \leqslant t+7$, and $d = 2k(t+1) - t$ and $k \geqslant 1$, also for $t = 4$ and $d = 10k+1$.

Proof Let $t \geqslant 4$. From Theorem 2. 1 we have

$$P(t,d) = \left\lceil \frac{d-2}{t+1} \right\rceil + 1, \quad \text{for } t+4 \leqslant d \leqslant t+7.$$

From Theorem 2. 2 we have

$$P(4,d) = \left\lceil \frac{d-2}{5} \right\rceil + 1, \text{for } d = 10k+1 \text{ and } k \geqslant 1.$$

From Lemma 1. 2 we have

$$P(t, (2k-1)(t+1)+1) = 2k.$$

This means

$$P(t,d) = \left\lceil \frac{d-2}{t+1} \right\rceil + 1,$$

for $d = (2k-1)(t+1)+1$, $k \geqslant 1$.

Now let G be an undirected graph with diameter d, $S \subset E(G)$ with $|S| = t$ such that $d(G-S) = h = f(t,d)$. Thus from Lemma 1. 3 and for $t \geqslant 4, d \in I_2(t,1)$, and $d = 2k(t+1) - t + 5$, also for $t = 4$ and $d = 10k+1$ and $k \geqslant 1$, we have

$$\left\lceil \frac{h-2}{t+1} \right\rceil + 1 = P(t,h) \leqslant d.$$

Then

$$\frac{h+t-1}{t+1} \leqslant d.$$

Thus

$$f(t,d) = h \leqslant (t+1)d - t + 1.$$

The theorem follows. □

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