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ROUND-OFF STABILITY OF PICARD ITERATIVE PROCEDURE FOR MULTIVALUED OPERATORS

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ABSTRACT. While solving inclusions numerically by an iterative procedure, usually we follow some theoretical model and deal with an approximate numerical sequence. If the numerical sequence converges to a point anticipated by the theoretical sequence, then we say that the iterative procedure is stable. This kind of study plays a vital role in computational analysis, game theory and computer programming. The purpose of this paper is to discuss stability of the Picard iterative procedure for multivalued operators in metric spaces. Some special cases are discussed as well.

1. Introduction

Let (X, d) be a metric space and T a self-map of X. The solution of a fixed point equation $Tx = x$ for any $x \in X$, is usually approximated by a sequence $\{x_n\}$ in X generated by an iterative procedure $f(T, x_n)$ that converges to a fixed point of T. But, in actual computations, we consider an approximative sequence $\{y_n\}$ in X instead of $\{x_n\}$. The iterative procedure $x_{n+1} = f(T, x_n)$ is considered to be numerically stable if and only if the sequence $\{y_n\}$ converges to the desired solution of the equation $Tx = x$. M. Urabe [16] initiated the study on this kind of problem, and A. M. Ostrowski [11] was the first to obtain the classical stability result on metric spaces (see [8]). Harder & Hicks and Rhoades have obtained stability results for a wider class of contractive type maps (cf. [6, 7, 12, 13]). Singh & Chadha [15] extended Ostrowski's stability theorem (cf. Cor. 3.2) to multivalued operators. In this paper, we discuss the stability of Picard iterative procedure, i.e., $x_{n+1} \in f(T, x_n) = Tx_n$ for multivalued operators using the general contractive condition. Some interesting results as special cases are discussed.

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2. Preliminaries

Consistent with [15], we will use the following notations, where (X, d) is a metric space and $CL(X)$ is the collection of all nonempty closed subsets of X. For A , $B \in CL(X)$ and $\epsilon > 0$,

$$N(\epsilon, A) = \{x \in X : d(x, a) < \epsilon \text{ for some } a \in A\},$$
$$E_{AB} = \{\epsilon > 0 : A \subseteq N(\epsilon, B), B \subseteq N(\epsilon, A)\},$$
$$H(A, B) = \begin{cases} \inf E_{AB}, & \text{if } E_{AB} \neq \emptyset, \\ +\infty, & \text{if } E_{AB} = \emptyset, \end{cases}$$

and for $x \in X$, $D(x, A) = \inf \{d(x, a) : a \in A\}$. H is called the generalized Hausdorff metric for $CL(X)$ induced by the metric d of X.

The following lemma (cf. [14]) will be used.

Lemma. *Let $B \in CL(X)$ and $a \in X$. Then for any $b \in B$, $d(a, b) \leq H(a, B)$.*

Let $T : X \longrightarrow CL(X)$... For a point $x_0 \in X$, let $x_{n+1} \in f(T, x_n)$ denote some iterative procedure. Let $\{x_n\}$ be convergent to a fixed point p of T and $\{y_n\}$ be an approximative sequence in X. Set $\epsilon_n = H(y_{n+1}, f(T, y_n))$, $n = 0, 1, 2, \cdots$. If $\lim_n \epsilon_n = 0$ implies that $\lim_n y_n = p$ then the iterative procedure is said to be T-stable or stable with respect to T (cf. Singh and Chadha [15]). Notice that this definition is essentially due to Harder & Hicks [6] when T is a single-valued self-operator of X. The Picard orbit of a multivalued map $T : X \longrightarrow CL(X)$, at an initial point x_0, is a sequence $\{x_n : x_n \in Tx_{n-1}, n = 1, 2, \cdots\}$ and the space X is T-orbitally complete iff every Cauchy sequence of the form $\{x_{n_i} : x_{n_i} \in Tx_{n_i}\}$ converges in X (cf. Ćirić [4]). Evidently, this means that if the space X is complete then it is T-orbitally complete and the reverse implication is not true. Ćirić [op. cit.] obtained the following result.

Theorem 2.1. *Let X be T-orbitally complete and $T : X \longrightarrow CL(X)$ such that*

$$H(Tx, Ty) \leq q \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{1}{2}(D(x, Ty) + D(y, Tx))\}, \quad (2.1)$$

for all x, $y \in X$ and $0 < q < 1$. Then

(i) for every $x_0 \in X$, there exists an orbit $\{x_n\}$ of T at x_0 and an $u \in X$ such that $\lim_n x_n = u$;

(ii) the point u is fixed under T.

We remark that Theorem 2.1 with (2.1) replaced by

$$H(Tx, Ty) \leq q \max\{d(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\} \quad (2.2)$$

remains an open question. However, it is true when T is single-valued operator on a complete metric space (cf. Ćirić [5]). Also, notice that $T : X \longrightarrow CL(X)$ satisfying (2.3) (see below) need not have a fixed point on a complete metric space (see Osilike[9] and Berinde [1, p.137] when T is single-valued). For a good discussion on the generality, usefulness and importance of the condition (2.3) with T single-valued, one may refer to Berinde [2, 3].

Proposition. *Let $T : X \to CL(X)$ be such that*

$$H(Tx, Ty) \le qd(x, y) + LD(x, Tx), \tag{2.3}$$

for all x, $y \in X$, $0 < q < 1$ and $L \ge 0$. Then
(i) (2.1) $\Rightarrow$ (2.2);
(ii) (2.1) *with* $0 < q < \frac{1}{2} \Rightarrow$ (2.3);
(iii) (2.2) *with* $0 < q < \frac{1}{2} \Rightarrow$ (2.3).

Proof. It is enough to show the last implication. For x, $y \in X$, in view of (2.2), one of the following holds:

$$H(Tx, Ty) \le qd(x, y); H(Tx, Ty) \le qD(x, Tx);$$

$$H(Tx, Ty) \le qD(y, Ty) \le q\{d(y, x) + D(x, Tx) + H(Tx, Ty)\}$$

yielding

$$H(Tx, Ty) \le L\{d(y, x) + D(x, Tx)\}, \text{ where } L = \frac{q}{1-q};$$

$$H(Tx, Ty) \le qD(x, Ty) \le q\{D(x, Tx) + H(Tx, Ty)\}$$

implying

$$H(Tx, Ty) \le LD(x, Tx);$$

$$H(Tx, Ty) \le qD(y, Tx) \le q\{d(y, x) + D(x, Tx)\}.$$

Therefore, in all the cases,

$$H(Tx, Ty) \le qd(x, y) + LD(x, Tx).$$

3. Main results

Theorem 3.1. *Let X be a complete metric space and $T : X \to CL(X)$ satisfying (2.3) for all x, $y \in X$. Let $\{x_n\}_{n=1}^{\infty}$ be an orbit for T at $x_0 \in X$ such that $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point p of T. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in X and set $\epsilon_n = H(y_{n+1}, Ty_n)$, $n = 0, 1, 2, \cdots$. Then*
(I) $d(p, y_{n+1}) = d(p, x_{n+1}) + q^{n+1}d(x_0, y_0) + L\Sigma_{i=0}^{n} q^{n-i} d(x_i, x_{i+1}) + \Sigma_{i=0}^{n} q^{n-i}\epsilon_i$. *Further, if Tp is singleton then*
(II) $\lim_n y_n = p$ *if and only if* $\lim_n \epsilon_n = 0$.

Proof. Let n be a nonnegative integer. Then, by the above lemma and (2.3),

$$\begin{aligned}
&d(x_{n+1}, y_{n+1})\\
&\le H(Tx_n, y_{n+1}) \le H(Tx_n, Ty_n) + H(Ty_n, y_{n+1})\\
&\le qd(x_n, y_n) + LD(x_n, Tx_n) + \epsilon_n\\
&\le q\{qd(x_{n-1}, y_{n-1}) + LD(x_{n-1}, Tx_{n-1}) + \epsilon_{n-1}\} + LD(x_n, Tx_n) + \epsilon_n\\
&\le q^2 d(x_{n-1}, y_{n-1}) + Lqd(x_{n-1}, x_n) + Ld(x_n, x_{n+1}) + q\epsilon_{n-1} + \epsilon_{n+1}.
\end{aligned}$$

Inductively,

$$d(x_{n+1}, y_{n+1}) \le q^{n+1} d(x_0, y_0) + L\Sigma_{i=0}^{n} q^{n-i} d(x_i, x_{i+1}) + \Sigma_{i=0}^{n} q^{n-i}\epsilon_i.$$

So

$$\begin{aligned}
&d(p, y_{n+1})\\
&\le d(p, x_{n+1}) + d(x_{n+1}, y_{n+1})\\
&\le d(p, x_{n+1}) + q^{n+1} d(x_0, y_0) + L\Sigma_{i=0}^{n} q^{n-i} d(x_i, x_{i+1}) + \Sigma_{i=0}^{n} q^{n-i}\epsilon_i.
\end{aligned}$$

This proves (I). To prove (II), first we assume $y_n \to p$ as $n \to \infty$. Note that $H(p, Tp) = 0$ since, by hypothesis, $Tp = \{p\}$. By (2.2),

$$\begin{aligned}
\epsilon_n = H(y_{n+1}, Ty_n) &\le d(y_{n+1}, p) + H(p, Tp) + H(Tp, Ty_n)\\
&\le d(y_{n+1}, p) + qd(p, y_n) + LD(p, Tp).
\end{aligned}$$

Therefore $\lim_n y_n = p$ implies $\lim_n \epsilon_n = 0$.

Now, suppose $\epsilon_n \to 0$ as $n \to \infty$. Since $0 < q < 1$ and $x_n \to p$ as $n \to \infty$, the first two terms on the right hand side of (I) vanish in the limit. Consequently,

$$\lim_n d(p, y_{n+1}) \le \lim_n [L\Sigma_{i=0}^{n} q^{n-i} d(x_i, x_{i+1}) + \Sigma_{i=0}^{n} q^{n-i}\epsilon_i].$$

Let A denote the lower triangular matrix with entries $a_{nj} = q^{n-j}$. Then $\lim_n a_{nj} = 0$ for each j and

$$\lim_n \Sigma_{j=0}^{n} a_{nj} = \lim_n \left(\frac{1 - q^{n+1}}{1 - q}\right) = \frac{1}{1 - q}.$$

Therefore A is multiplicative, i.e., for any convergent sequence $\{s_n\}$, $\lim_n A(s_n) = \frac{1}{1-q} \lim_n s_n$ (cf. Rhoades [13]). Since $\lim_n \epsilon_n = 0$, $\lim_n \left[\Sigma_{j=0}^{n} q^{n-1}\epsilon_j\right] = 0$. Noting that $\lim_n d(x_n, x_{n+1}) = 0$, we get $\lim_n \left[L\Sigma_{i=0}^{n} q^{n-i} d(x_i, x_{i+1})\right] = 0$. This completes the proof.

Corollary 3.1 [15]. *Let X be a complete metric space and $T : X \to CL(X)$ such that $H(Tx, Ty) \le qd(x, y)$ for all x, $y \in X$, where $q < 1$ is a positive number. Let x_0 be an arbitrary point in X and $\{x_n\}_{n=1}^{\infty}$ an orbit for T at x_0 such that $\{x_n\}_{n=1}^{\infty}$ is convergent to a fixed point p of T.*

Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in X and set $\epsilon_n = H(y_{n+1}, Ty_n)$, $n = 0, 1, 2, \cdots$. Then

$$d(p, y_{n+1}) \le d(p, x_{n+1}) + q^{n+1} d(x_0, y_0) + \Sigma_{r=0}^{n} q^{n-r}\epsilon_r.$$

Further, if Tp is singleton then $\lim_n y_n = p$ iff $\lim_n \epsilon_n = 0$.

Corollary 3.2 [11]. *Let (X, d) be a complete metric space and $T : X \to X$ a Banach contraction with contraction constant k. Let $p \in X$ be the fixed point of T. Let $x_0 \in X$ and $x_{n+1} = Tx_n$, $n = 0, 1, 2, \cdots$. Suppose that $\{y_n\}$ is a sequence in X and $\epsilon_n = d(y_{n+1}, Ty_n)$, $n = 0, 1, 2, \cdots$. Then*

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + k^{n+1} d(x_0, y_0) + \Sigma_{r=0}^{n} k^{n-r} \epsilon_r.$$

Moreover, $\lim_n y_n = p$ iff $\lim_n \epsilon_n = 0$.

Corollary 3.3 [9]. *Let (X, d) be a complete metric space and T a selfmap of X such that $d(Tx, Ty) \leq ad(x, y) + Ld(x, Tx)$ for all x, $y \in X$, $0 < a < 1$ and $L \geq 0$. Suppose T has a fixed point p. Let $x_0 \in X$ and let $x_{n+1} = Tx_n$, $n \geq 0$. Let $\{y_n\} \subset X$ and let $\epsilon_n = d(y_{n+1}, Ty_n)$, $n \geq 0$. Then*

$$\begin{aligned} &d(p, y_{n+1}) \\ &\leq d(p, x_{n+1}) + a^{n+1} d(x_0, y_0) + L\Sigma_{i=0}^{n} a^{n-i} d(x_i, Tx_i) + \Sigma_{i=0}^{n} a^{n-i} \epsilon_i. \end{aligned}$$

Also $\lim_n y_n = p$ implies that $\lim_n \epsilon_n = 0$.

Following Singh & Chadha [15], we modify the definition of ϵ_n as

$$\epsilon_n = d(y_{n+1}, p_n),\ p_n \in Ty_n,\ (n = 0, 1, 2, \cdots). \qquad (\star)$$

This facilitates to present another version of Theorem 3.1.

Theorem 3.2. *Let all the hypotheses of Theorem 3.1 hold, wherein the definition of ϵ_n is replaced by $(\star)$. Then*

(III) $d(p, y_{n+1}) \leq d(p, x_{n+1}) + q^{n+1} d(x_0, y_0) + L\Sigma_{i=0}^{n} q^{n-i} d(x_i, x_{i+1}) + \Sigma_{i=0}^{n} q^{n-i}(H_i, +\epsilon_i)$, *where* $H_i := H(x_{i+1}, Tx_i)$.

Further, if Tp is singleton, then

(IV) $\lim_n y_n = p$ *if and only if* $\lim_n \epsilon_n = 0$.

Proof. For any nonnegative integer n,

$$\begin{aligned} &d(x_{n+1}, y_{n+1}) \\ &\leq d(x_{n+1}, p_n) + d(p_n, y_{n+1}) \leq H(x_{n+1}, Ty_n) + \epsilon_n \\ &\leq H(x_{n+1}, Tx_n) + H(Tx_n, Ty_n) + \epsilon_n. \end{aligned}$$

In view of (2.3),

$$\begin{aligned} &d(x_{n+1}, y_{n+1}) \\ &\leq H_n + qd(x_n, y_n) + LD(x_n, Tx_n) + \epsilon_n \\ &\leq H_n + q[H_{n-1} + qd(x_{n-1}, y_{n-1}) + Ld(x_{n-1}, x_n) + \epsilon_{n-1}] + Ld(x_n, x_{n+1}) + \epsilon_n \\ &\leq q^2 d(x_{n-1}, y_{n-1}) + Lqd(x_{n-1}, x_n) + Ld(x_n, x_{n+1}) + q(H_{n-1} + \epsilon_{n-1}) + (H_n + \epsilon_n). \end{aligned}$$

Inductively,

$$d(x_{n+1}, y_{n+1}) \leq q^{n+1} d(x_0, y_0) + L\Sigma_{i=0}^{n} q^{n-i} d(x_i, x_{i+1}) + \Sigma_{i=0}^{n} q^{n-i} d(H_i + \epsilon_i),$$

and the relation (III) follows from,

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + d(x_{n+1}, y_{n+1}).$$

To prove (IV), assume first $y_n \to p$ as $n \to \infty$. Then $\epsilon_n = d(y_{n+1}, p_n) \leq H(y_{n+1}, Ty_n)$. This, as in the proof of Theorem 3.1, gives $\lim_n \epsilon_n = 0$. Now, assume that $\lim_n \epsilon_n = 0$. From (III),

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + q^{n+1} d(x_0, y_0) + L\Sigma_{i=0}^{n} q^{n-i} d(x_i, x_{i+1}) + \Sigma_{i=0}^{n} q^{n-i} t_i,$$

where $t_i = H_i + \epsilon_i$. In view of the proof of Theorem 3.1, it suffices to show that the sequence $\{t_i\}$ is convergent to 0. Since, by assumption, the sequence $\{\epsilon_i\}$ is convergent to 0, it is enough to show that $\{H_i\}$ is also convergent to 0. By (2.2),

$$\begin{aligned} \lim_n H_n = \lim_n H(x_{n+1}, Tx_n) &\leq d(x_{n+1}, p) + D(p, Tp) + H(Tp, Tx_n) \\ &\leq d(x_{n+1}, p) + qd(p, x_n) + LD(p, Tp) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

This completes the proof.

Corollary 3.4. *Let X be a complete metric space and $T : X \to CL(X)$ such that (2.1) holds for all x, $y \in X$. Let x_0 be an arbitrary point in X and $\{x_n\}$ an orbit for T at x_0 such that $\{x_n\}$ is convergent to a fixed point p of T.*

Let $\{y_n\}$ be a sequence in X and set $\epsilon_n = H(y_{n+1}, Ty_n)$, $n = 0, 1, 2, \cdots$. Then the conclusions of Theorem 3.1 and 3.2 follow, wherein $L = \frac{q}{1-q}$ with $q < \frac{1}{2}$.

Proof. It is evident in view of the proposition.

We remark that all the results (theorems and corollaries) stated above remain true when the completeness of the space X is relaxed to the orbital completeness.

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