Journal of Basrah Researches ((Sciences)) Vol. 36, No. 3, 15 June ((2010))

MULTI-VALUED FIXED POINT THEOREM IN METRIC SPACES BY ALTERING DISTANCE BETWEEN THE POINTS

ISSN 1817 – 2695

AMAL . M. HASHIM

Department of Mathematics, College of Science, University of Basrah, Basrah, Iraq
E-mail: amalmhashim@yahoo.com

((Received 8/10/2009 , Accepted 19/5/2010))

Abstract.

In this paper we established some results on fixed point theorems for multi-valued mappings on a complete metric spaces by altering the distance between the points. We extend and generalized some well-known results in [2], [5], [6], [8] and [9].

Key Words: Multi-valued maps, fixed point, compact metric space, complete metric space.

INTRODUCTION

Delbosco [1] and Skof [12] have established fixed point theorems for self-maps of complete metric spaces by altering the distances between the points with the use of map $\phi:[0,\infty)\to[0,\infty)$ satisfying the following properties:

(i) ϕ is continuous and strictly increasing in R^+;

(ii) $\phi(t)=0$ iff $t=0$;

(iii) $\phi(t)\ge Mt^u$ for every $t>0, u>0$ are constants.

Khan et al. [5] gave the following theorem,

Theorem K: Let T be a self-map of a complete metric space (X,d) and ϕ satisfying (i) and (ii). Furthermore, Let a, b, c be three decreasing maps from R^+ into [0,1) such that

$$a(t)+2b(t)+c(t)<1$$

For every $t>0$.

Suppose that T satisfies the following condition :

$$(I)\quad \phi(d(Tx,Ty))\le a(d(x,y))\phi(d(x,y))+b(d(x,y))\left[\phi(d(x,Tx))+\phi(d(y,Ty))\right]$$
$$+c(d(x,y))\min\{\phi(d(y,Tx)),\phi(d(x,Ty))\}$$

Where x,y in X and $x\ne y$. Then T has a unique fixed point.

In this paper we shall establish a common fixed point theorem for a pair of multi-valued mappings T and S and a single valued mapping which satisfies the contractive definition corresponding to (I).

PDF Created with deskPDF PDF Writer - Trial :: http://www.docudesk.com

2.Basic Preliminaries

We generally follow the definitions and notations used in [3], [10] and [11]. Given a metric space *(X, d)*, let *(CL(X), H)* and *(C(X), H)* denote respectively the hyperspaces of non-empty closed and non-empty compact subsets of *X*, wherein *H* is the Hausdorff metric inducted by *d*. Throughout, *d*(*A*, *B*) will denote the ordinary distance between subsets *A* and *B* of *X* and *d(x, B)* will stand for *d(A, B)* when *A* is singleton {*x*}. Further, let *Y* be an arbitrary non-empty set and $C(f,\ T)=\{u:\ fu\in Tu\}$, the collection of coincidence points of the maps $f\ :Y\rightarrow X$ and $T:Y\rightarrow CL(X)$.

We shall use the following lemma and definitions.

Lemma 2.1 [6]. Let A, B are two compact subset of a metric space, then for all $x\in A$ there exists $y\in B$ such that $d\left(x,y\right)\le H\left(A,B\right)$.

Definition 2.2 [2]. A map $\psi:[0,\infty)\rightarrow[0,1)$ will be said to have property (p) if for every $t>0$, there exists $n(t)>0$ such that

$$0\le S-t<\eta\left(t\right)\rightarrow\psi\left(s\right)\le\gamma(t)<1.$$

Definition 2.3 [Itoh and Takahashi [7]. The maps $T:X\rightarrow CL(X)$ collection of all closed subsets of X and $f:X\rightarrow X$ are IT-commuting at a point $v\in X$ if $fTv\subset Tfv$.

Definition 2.4 [3] Let $T:X\rightarrow CB(X)$. The map $f:X\rightarrow X$ is said to be T-weakly commuting at $x\in X$ if $ffx\in Tfx$

Here we remark that for hybrid pairs (f,T) (IT)-commuting at the coincidence points implies that f is T-weakly commuting but the converse is not true in general.

Example 2.5[3]Let $X=[1,\infty)$ with the usual metric, Define $f:X\rightarrow X$, $T:X\rightarrow CB(X)$ by $fx=2x$ and $Tx=[1,2x+1]$ for all $x\in X$ then for all $x\in X$, $fx\in Tx$, $ffx=4x\in[1,4x+1]=Tfx$, $fTx=[2,4x+2]\not\subset Tfx$ therefore f is $T-$weakly commuting but not $IT-$commuting.

3. Our Main Results

Theorem 3.1 Let (X,d) be a compact metric space, $T,S:Y\rightarrow CL(X)$, $f:Y\rightarrow X$ such that $TY\subset fY$ and $SY\subset fY$, $\phi:R^{+}\rightarrow R^{+}$ is continuous and strictly increasing such that $\phi(0)=0$ and Let a, b and c be three maps from R^{+} into [0,1) such that $a+2b$ satisfies the property (P), and,

$$(II)\quad \phi\left(H\left(Tx,Sy\right)\right)\le a\left(d\left(fx,fy\right)\right)\phi\left(d\left(fx,fy\right)\right)+b\left(d\left(fx,fy\right)\right)\left[\phi\left(d\left(fx,Tx\right)+\phi\left(d\left(fy,Sy\right)\right)\right)\right]$$
$$+c\left(d\left(fx,fy\right)\right)\min\left\{\phi\left(d\left(fx,Sy\right),\phi\left(d\left(fy,Tx\right)\right)\right)\right\},$$

Then $C(f,T)$ and $C(f,S)$ are nonempty.

Further, if $Y=X$, z is a coincidence point of f,T and S and fz is a fixed point of f then

(i) fz is also a fixed point of T (resp S) provided f is T-weakly commuting at z (resp f is S-weakly commuting at z),

PDF Created with deskPDF PDF Writer - Trial :: http://www.docudesk.com

(ii) fz is a common fixed point of T and S provided f is T-weakly commuting at z and f is S-weakly commuting at z.

Proof. Pick $x_0 \in Y$. Construct two sequences $\{x_n\} \subset Y$ and $\{y_n\} \subset X$ in the following manner. Since $x_0 \in Y$ and $TY \subset fY$, there exists a point $x_1 \in Y$ such that

$$y_1 = fx_1 \in Tx_o.$$ Since $SY \subset fY$, in view of the remark following Lemma 2.1, we can choose $x_1 \in Y$ such that $y_2 = fx_2 \in Sx_1$ and

$$d(y_1, y_2) = d(fx_1, fx_2) \le H(Tx_0, Sx_1)$$

So by using (II)

$$\begin{aligned}
\phi\big(d(fx_1, fx_2)\big) &\le \phi\big(H(Tx_0, Sx_1)\big) \\
&\le a\big(d(fx_0, fx_1)\big)\phi\big(d(fx_0, fx_1)\big) + b\big(d(fx_0, fx_1)\big)\big[\phi\big(d(fx_0, Tx_0)\big) + \phi\big(d(fx_1, Sx_1)\big)\big] \\
&\quad + c\big(d(fx_0, fx_1)\big)\min\big\{\phi\big(d(fx_0, Sx_1)\big), \phi\big(d(fx_1, Tx_0)\big)\big\} \\
&\le a\big(d(fx_0, fx_1)\big)\phi\big(d(fx_0, fx_1)\big) + b\big(d(fx_0, fx_1)\big)\big[\phi\big(d(fx_0, fx_1) + \phi\big(d(fx_1, fx_2)\big)\big] \\
&\quad + c\big(d(fx_0, fx_1)\big)\min\big\{\phi\big(d(fx_0, fx_2)\big), \phi\big(d(fx_1, fx_1)\big)\big\} \\
&\le a\big(d(fx_0, fx_1)\big)\phi\big(d(fx_0, fx_1)\big) + b\big(d(fx_0, fx_1)\big)\big[\phi\big(d(fx_0, fx_1) + \phi\big(d(fx_1, fx_2)\big)\big],
\end{aligned}$$

Hence $\phi\big(d(fx_1, fx_2)\big) \le q\big(d(fx_0, fx_1)\big)\phi\big(d(fx_0, fx_1)\big)$ where

$$q\big(d(fx_0, fx_1)\big) = \frac{a\big(d(fx_0, fx_1)\big) + b\big(d(fx_0, fx_1)\big)}{1 - b\big(d(fx_0, fx_1)\big)}$$

By induction there exists x_n such that

$$fx_{2n+1} \in Tx_{2n}, n = 0,1,2,......$$

$$fx_{2n+2} \in Sx_{2n+1}, n = 0,1,2,....$$ and

$$\phi\big(d(fx_n, fx_{n+1})\big) \le q\big(d(fx_{n-1}, fx_n)\big)\phi\big(d(fx_{n-1}, fx_n)\big) < \phi\big(d(fx_{n-1}, fx_n)\big) \ (*).$$

Put $t_n = d(fx_n, fx_{n+1})$, then we have $\phi(t_n) < \phi(t_{n-1})$, but ϕ is increasing, so $\{t_n\}$ is decreasing, hence $\lim_n t_n = t$ exists. Now we show $t = 0$, suppose that $t > 0$, then since q satisfies the property (P)

$\exists \eta(t) > 0$ such that $0 \le s - t < \eta(t) \Rightarrow q(s) \le \gamma(t) < 1$, but $t_n \ge t$ and $\lim_n t_n = t$, so there exists n_o such that

$$n \ge n_0 \Rightarrow 0 \le t_n - t < n(t),$$

Consequently $q(t_n) \le \gamma(t) < 1$. from the inequality $(*)$ we deduce that

$$\phi(t_n) \le \gamma(t)\phi(t_{n-1}) \qquad \forall n \ge n_0,$$

Hence when $n \to \infty$ we obtain

$$\phi(t) \le \gamma(t)\ \phi(t) < \phi(t),$$

this is a contradiction, so $\lim_n d\ (fx_n, fx_{n+1}) = 0$, since, X is compact, so $\{fx_n\}$ has a convergent subsequence which we denote by $\{fx_k\}$, Let

PDF Created with deskPDF PDF Writer - Trial :: http://www.docudesk.com

$fx = \lim_k fx_k$, we have to show that fx is a coincidence point of T and S. By condition (II).

$$\phi\left(d\left(fx_{k+1}, Sx\right)\right) \le \phi\left(H\left(Tx_k, Sx\right)\right)$$
$$\le a\left(d\left(fx_k, fx\right)\right)\ \phi\left(d\left(fx_k, fx\right)\right) + b\left(d\left(fx_k, fx\right)\right)\left[\phi\left(d\left(fx_k, Tx_k\right)\right) + \phi\left(d\left(fx, Sx\right)\right)\right]$$
$$+ c\left(d\left(fx_k, fx\right)\right)\min\left\{\phi\left(d\left(fx_k, Sx\right)\right), \phi\left(d\left(fx, Tx_k\right)\right)\right\}$$
$$< \phi\left(d\left(fx_k, fx\right)\right) + \frac{1}{2}\left[\phi\left(d\left(fx_k, fx_{k+1}\right)\right) + \phi\left(d\left(fx, Sx\right)\right)\right]$$
$$+ \phi\left(d\left(fx_k, fx_{k+1}\right)\right),$$

therefore taking $k \to \infty$, we have

$$\phi\left(d\left(fx, Sx\right)\right) \le \frac{1}{2}\phi\left(d\left(fx, Sx\right)\right),$$ hence $\phi\left(d\left(fx, Sx\right)\right) = 0,$

Consequently $d\left(fx, Sx\right) = 0,$ and $fx \in Sx$.
Similarly, we can show that $fx \in Tx$.

Hence $C(f,T)$ and $C(f,S)$ are non-empty.

Further, $fz = ffz$ and f is T-weakly commuting at $z \in C(f,T)$ which implies that $fz \in Tfz$. So fz is common fixed point of f that T. The proof of fz is common fixed point of f and S is similar. Now (ii) is immediate.

Corollary 3.2 [9]. Theorem 3.1 with $S = T$, and f = identity map.

Theorem 3.3: Let T and S be self-maps of a complete metric space (X,d) and ϕ satisfying (i) and (ii). Furthermore, let a, b and c be three functions from (0, ∞) into [0, 1) such that $a + 2b$ and $a + c$ have both property (P). Suppose that T and S satisfy the following condition.

$$(III)\quad \phi\left(d\left(Tx, Sy\right)\right) \le a\left(d\left(x, y\right)\right)\phi\left(d\left(x, y\right)\right)$$
$$+ b\left(d\left(x, y\right)\right)\left[\phi\left(d\left(x, Tx\right)\right) + \phi\left(d\left(y, Sy\right)\right)\right]$$
$$+ c\left(d\left(x, y\right)\right)\min\left\{\phi\left(d\left(y, Tx\right)\right), \phi\left(d\left(x, Sy\right)\right)\right\}.$$

Where x, y in X and $x \ne y$. Then
T and *S* have a unique fixed point.

Proof: Let x_o be some point in *X*. We define

$$x_{2n+1} = Tx_{2n}, \quad n = 0,\ 1,\ 2,\ ...,$$
$$x_{2n+2} = Sx_{2n+1}, \quad n = 0,1,2,...,$$

We put $t_n := d\left(x_n, x_{n+1}\right)$ for all integer n. without loss of generality; we may assume that $t_n > 0$ for all integer n.
Then by using (III)

$$\phi(t_n) = \phi\left(d\left(Tx_{n-1}, Sx_n\right)\right)$$
$$\le a\left(d\left(x_{n-1}, x_n\right)\right)\phi\left(d\left(x_{n-1}, x_n\right)\right) + b\left(d\left(x_{n-1}, x_n\right)\right)\left[\phi\left(d\left(x_{n-1}, Tx_{n-1}\right)\right) + \phi\left(d\left(x_n, Sx_n\right)\right)\right]$$
$$+ c\left(d\left(x_{n-1}, x_n\right)\right)\min\left\{\phi\left(d\left(x_n, Tx_{n-1}\right)\right), \phi\left(d\left(x_{n-1}, Sx_n\right)\right)\right\}$$

So

$$\phi(t_n) \le a(t_{n-1})\phi(t_{n-1}) + b(t_{n-1})\left[\phi(t_{n-1}) + \phi(t_n)\right].$$

Hence we obtain :

$$\phi(t_n) \le q(t_{n-1})\phi(t_{n-1}) < \phi(t_{n-1}). \quad (*)$$

PDF Created with deskPDF PDF Writer - Trial :: http://www.docudesk.com

Where $q(t_{n-1}) = \dfrac{a(t_{n-1}) + b(t_{n-1})}{1 - b(t_{n-1})}$

Since ϕ is increasing, $\{t_n\}$ is decreasing sequence.

Let $\underset{n}{Lim}\, t_n = t$, assume that *t >0*. Since *q* satisfies the property *(P)*, then for $0 \le s - t < \eta(t)$ we have $q(s) \le \gamma(t) < 1$, but $t_n \ge t$, $\underset{n}{Lim}\, t_n = t$, so there exists n_0 such that $n \ge n_0 \Rightarrow 0 \le t_n - t < \eta(t)$, Consequently $q(t_n) \le \gamma(t) < 1$. From equality (*), we deduce that $\phi(t_n) \le \gamma(t)\phi(t_{n-1}) \quad \forall n \ge n_0$, hence when $n \to \infty$ we obtain

$$\phi(t) \le \gamma(t)\phi(t) < \phi(t),$$

This is a contraction, consequently t = 0, i.e.

$$\underset{n}{lim}\, d(x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ is not Cauchy sequence there exists $\varepsilon > 0$, and two sequences $\{m(k)\}, \{n(k)\}, n(k) > m(k), m(k) \to \infty$ as $k \to \infty$ such that

$$d\left(x_{n(k)}, x_{m(k)}\right) > \varepsilon \quad \text{while} \quad d\left(x_{n(k)-1}, x_{m(k)}\right) \le \varepsilon .$$

Then we have

$$\varepsilon < d_k = d\left(x_{n(k)}, x_{m(k)}\right) \le d\left(x_{n(k)}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{m(k)}\right)$$
$$\le t_{n(k)} + \varepsilon,$$

since $\{t_n\}$ converge to 0, $d_k \to \varepsilon$. Furthermore, by triangular inequality, it follows that

$$d_k - t_{n(k)} - t_{m(k)} \le d\left(x_{n(k)+1}, x_{m(k)+1}\right) \le d_k + t_{n(k)} + t_{m(k)}$$

and therefore the sequence $\left\{d\left(x_{n(k)+1}, x_{m(k)+1}\right)\right\}$ converges to ε.

From (III), we also deduce.

$$\phi\left(d(x_{n(k)+1}, x_{m(k)+1}\right) = \phi\left(d(Tx_{n(k)}, Sx_{m(k)}\right)$$
$$\le a\left(d(x_{n(k)}, x_{m(k)}\right)\phi\left(d(x_{n(k)}, x_{m(k)}\right)$$
$$+b\left(d\left(x_{n(k)}, x_{m(k)}\right)\right)\left[\phi\left(d(x_{n(k)}, Tx_{n(k)}\right) + \phi\left(d(x_{m(k)}, Sx_{m(k)}\right)\right]$$
$$+c\left(d\left(x_{n(k)}, x_{m(k)}\right)\right)\min\left\{\phi\left(d\left(x_{m(k)}, Tx_{n(k)}\right)\right), \phi\left(d\left(x_{n(k)}, Sx_{m(k)}\right)\right)\right\}$$
$$\le a(d_k)\phi(d_k) + b(d_k)\left[\phi\left(t_{n(k)} + \phi\left(t_{m(k)}\right)\right)\right]$$
$$+ c(d_k)\phi\left(d_k + t_{n(k)}\right) \qquad (1)$$

Since $d_k > \varepsilon$, $d_k \to \in$ and $a + c$ have property *(P)*, there exists an integer k_o such that

$$k \ge k_o \to 0 \le d_k - \varepsilon < n(\varepsilon) \Rightarrow a(d_k) + c(d_k) \le \gamma(\varepsilon) < 1 \qquad (2)$$

From (1) and (2) we have

PDF Created with deskPDF PDF Writer - Trial :: http://www.docudesk.com

$$\phi\left(d\left(x_{n(k)+1}, x_{m(k)+1}\right)\right) \leq \gamma(\varepsilon)\phi\left(d_k + t_{n(k)} + \phi\left(t_{n(k)}\right) + \phi\left(T_{m(k)}\right)\right),$$

For all $k \geq k_o$,

as $k \to \infty$, we obtain

$$\phi(\varepsilon) \leq \gamma(\varepsilon)\ \phi\ (\varepsilon) < \phi(\varepsilon).$$

this is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in a complete metric space, then there exists a point z in X such that $x_n \to z$ as $n \to \infty$.

Now we shall show that $Sz = z$.

From (III), we have

$$\begin{aligned}\phi\left(d\ (x_n, Sz\ )\right) &= \phi(Tx_{n-1}, Sz\ )\\ &\leq a\left(d\ (x_{n-1}, z\ )\ \phi\left(d\ (x_{n-1}, z\ )\right)\right.\\ &\quad + b\left(d\ (x_{n-1}, z\ )\right)\left[\phi\left(d\ (x_{n-1}, Tx_{n-1})\right) + \phi\left(d\ (z, Sz\ )\right)\right]\\ &\quad + c\left(d\ (x_{n-1}, z\ )\right)\min\left\{\phi(d\ (x_{n-1}, Sz\ ), \phi\left(d\ (z\ , Tx_{n-1})\right)\right\}.\end{aligned} \tag{3}$$

Using $b < \dfrac{1}{2}$ and letting $n \to \infty$ in (3),

We have $\phi\left(\ d\ (z\ , Sz\ )\right) = 0$ and therefore $d(z, Sz) = 0$, i.e. Sz=z

Similarly, it can be shown that $Tz = z$. Hence S and T have a common fixed point z in X. We claim that z is the unique common fixed point of T and S. For this, we suppose that $w(w \neq z)$ is another fixed point of S and T. Then

$$\begin{aligned}&\phi\left(d\ (z\ , w\ )\right) = \phi\left(d\ (Tz\ , Sw\ )\right)\\ &\leq a\left(d\ (z\ , w\ )\right)\ \phi\left(d\ (z\ , w\ )\right)\\ &+ b\left(d\ (z\ , w\ )\right)\left[\phi\left(d\ (z\ , Tz\ )\right) + \phi\left(d\ (w\ , Sw\ )\right)\right]\\ &+ c\left(\ d\ (z\ , w\ )\right) min\left\{\ \phi\left(\ d\ (z\ , Sw\ )\right), \phi\left(d\ (w\ , Tz\ )\right)\right\}\\ &< \phi(d(z, w))\end{aligned}$$

This is a contradiction. Hence $w = z$

Corollary 3.4 [8]. Let T and S be self-maps on a complete metric space (X, d) and ϕ satisfying the condition (i) and (ii). Furthermore, let a, b and c be three decreasing maps from R^+ into [0,1) such that

$$a(t) + 2b(t) + c(t) < 1$$

For every $t > 0$.

Suppose that T and S satisfy condition (III). Then T and S have a unique fixed point in X.

REFERENCES

PDF Created with deskPDF PDF Writer - Trial :: http://www.docudesk.com

1. B. Delbosco, Un estensione diun teoreme sul punto fisso di s-Reich, Rend-sem. Mat. Unvies. Politecn. Torino 35, 233-239, (1976-77).

2. M. EL Amrani and A. B. Mbarki, Fixed point theorem by altering distance between the points, Southwest Journal of pure and Applied Mathematics, Vol. 1, 16-21, (2000).

3. T. Karmran, Coincidence and fixed points for hybrid strict contraction, J. Math. Anal. Appl. 299, 235-241, (2004).

4. J. Matkowski, Fixed point theorem for mapping with a contractive iterate a point, Proc. Amer. Math. Soc. 62, no 2, 344-348, (1977).

5. M. S. Khan.,M. Swaleh And S. Sessa , Fixed point theorems by altering distance between the points, Bull-Austral-Math. Soc. 30, 1-9. (1984).

6. B. Marzouki and A. B. Mbarki, Multivalued fixed point theorems by altering distance between the points, south west J. Pure and Appl. Math., Issue 1, , 126-134, July (2002).

7. S. Itoh and W. Takahashi, Single-valued mappings, multivalued mappings and fixed point theorems J. Math. Anal. Appl. 59, 514-521, (1977).

8. R. A. Rashwan and A. M. Sadeek, A common fixed point theorem in complete metric spaces, southwest. J. Pure and Appl. Math. Vol. No. 01, 6-10, (1996).

9. R. A. Rashwan and A. M. Sadeek, Multivalued fixed point theorems by altering distance between the points, Southwest J. Pure and Appl. Math., Issue 1, 126-134, (2002).

10. S. L. Singh and Amal M. Hashim, New coincidence and fixed point theorems for strictly contractive hybrid maps, J. Austral. Math. Anal. And Appl. 2(1), 1-7, (2005).

11. S. L. Singh and Amal M. Hashim, New coincidence and fixed point reciprocally continous and compatible hybrid maps.phs.sci,18,no.2, 97-104, (2004).

12. S. Skof, Teorema di punti fisso per application ineglispazi metrici, Atti, Accad. Aci. Torino 111, 323-329, (1977).

مبرهنات النقطة الصامدة للدوال متعددة القيم بواسطة تغيير المسافات بين النقاط في الفضاء المتري

PDF Created with deskPDF PDF Writer - Trial :: http://www.docudesk.com

أمل محمد هاشم البطاط

قسم الرياضيات – كلية العلوم – جامعة البصرة – البصرة – العراق

الملخص:

في هذا البحث حصلنا على بعض النتائج حول مبرهنات النقاط الثابتة للدوال الاحادية ومتعددة القيم في الفضـــاء المتـــري بواسطة تغيير المسافات بين النقاط . لقد تم توسيع وتعميم بعض النتائج المعروفة في [2] ,[5] ,[6], [8] و [9].

PDF Created with deskPDF PDF Writer - Trial :: http://www.docudesk.com