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Almost Kahler manifold

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Abstract:

In this paper we study one of the sixteen classes of almost Hermitian manifold, which is the almost Kahler manifold. We found its structure equation and the components of its Reimannian curvture tensor and then we proved that an almost Kahler manifold is parakahler manifold (of class R_1) if and only if it is Kahler manifold.

Key words: Almost Hermitian manifold, almost Kahler manifold, Riemannian curvature tensor, parakahler manifold, Kahler manifold.

Introduction:

Almost Hermitian manifold (*AH*-manifold) is one of the most important subjects of the differential geometry for the applications of the synthesis of the differential geometrical structure. Conclusions, which were behind the study of *AH*-manifold, were found in many mathematical and theoretical physics aspects, such as ,Kähler manifold which is highly involved for the teaching of differential geometry, algebraic geometry , theory of Lie groups and topology. For that importance, the problem of classification of different kinds of *AH*-manifold according to its manipulations came to the roof.

First attempts were done by Koto [16] in 1960, who found a relationship which was considered as an entry to manifold, and it was almost similar to Kähler manifold. In 1965 Gray [6] found an extreme method to build certain examples from *AH*-manifold and so on, the studies continued until 1980 when the most important research appeared by Gray & Hervela [9]. In this research they found an important way to classify *AH*-manifold, they found the action of unitary group (*U (n)*) on the space of all tensors of type (*3,0*), where this action is irreducible, such that this space decomposes in the direct sum of four irreducible spaces ; therefore , the given action of *U(n)* define invariant subspaces of the space of tensor of type *(3,0)*.

We have noticed most of the studies of this object were done by the language of invariant Koszul [15], but the object will be more suitable if it is studied by the method of adjoint *G*-structure (i.e. the new methodology of exterior forms by Kartan [11]). In this method the geometrical behaviors of *AH*-manifold have not been tested on manifold itself, but tested on *G*-structure space which is linked with this manifold. This method pays attention to the physical geometrical behaviors of this manifold. Then, there was the Russian researcher, V. F. Kirichinko , who did a big change of this study ,when he found two new tensors ,namely, structure and virtual tensors [12], which allowed the researchers to study different kinds of *AH*-manifold . In 1993, Banaru, the Kirichinko's student [1] and [2] succeeded through his Ph.D. thesis to classify the *16* classes of *AH*-manifold using the two tensors of Kirichinko which were named the Kirichinko's tensors.

In this study we will consider one of the basic and important class of *AH*-manifold namely almost Kähler manifold (*AK*- manifold), by using the Kirichinko's tensors.

1- Almost Hermitian manifold

Let M *be* an $2n$- dimensional smooth manifold, $C^{\infty}(M)$ be algebra of smooth functions on *M*, $X(M)$ be Lie algebra of vector fields on *M*. Denote by ∇ to the Riemannian connection of metric g. Let d be the exterior differentiation.

Definition 1.1 [7]

Almost Hermitian structure (in short AH – structure) on M is a pair of tensors $\{ J, g = <.,.> \}$, where J is an almost complex structure and $g = <.,.>$ is a Riemannian metric, such that $< JX, JY > = < X, Y >$, $X, Y \in X(M)$

Definition 1.2 [2], [8]

A smooth manifold *M* with *AH*-structure is called an almost Hermitian manifold (*AH*-manifold).

Remark:

It is known [3], [4] and [5] that the setting of an almost Hermitian structure on *M* equivalent to the setting of a *G*- structure in the principle fiber bundle of all complex frames of manifold *M* which contains *G*- structure, that is the unitary group *U(n)*, and this *U(n)* is called an adjoint *G*- structure.

In the space of the adjoint *G*- structure, the following forms define matrices which give components of tensor fields *g* and *J*:

$$(g_{ij}) = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \quad , \quad (J^i_j) = \begin{pmatrix} \sqrt{-1}I_n & 0 \\ 0 & -\sqrt{-1}I_n \end{pmatrix}, \qquad (1.1)$$

where I_n is a unit matrix of order n.

Since J and g are tensors of type (1,1) and (2,0) respectively, then their components in the space of the fiber bundle of all complex frames satisfy equations [6]:

$$1) dJ^i_j + J^j_k \omega^k_j - J^k_j \omega^i_k = J^i_{j,k} \omega^k$$

$$2) dg_{ij} + g_{kj} \omega^k_i + g_{ik} \omega^k_j = g_{ij,k} \omega_k \qquad (1.2)$$

where $\{\omega^i\}, \{\omega^i_j\}$ are components of mixtures form and Riemannian connection ∇ respectively, $\{J^i_{j,k}\}, \{g_{ij,k}\}$ are components of differential covariant of tensors J and g in this connection respectively.

Since in the Riemannian connection we have: $\nabla g = 0$, then $\nabla g_{ij,k} = 0$

Definition 1.3 [13]

Suppose that *M* is a smooth manifold. The Exterior algebra $\Lambda(X(M))$ denoted by $\Lambda(M)$, which is called Grassman algebra of smooth manifold *M*, and its elements are called differential forms.

Theorem 1.1 [13]

Suppose that *M* is smooth manifold, then there exist a unique *R*-linear mapping: with the following properties: $d : \Lambda(M) \to \Lambda(M)$

1. $d(\Lambda_r(M)) \subset \Lambda_{r+1}(M)$.
2. $df(X) = X(f); f \in C^{\infty}(M), X \in X(M)$
3. $d^2 = d \circ d = 0$.

4. $d(\omega_1 \Lambda \omega_2) = d\omega_1 \Lambda \omega_2 + (-1)^r \omega_1 \Lambda d\omega_2,\ \omega_1 \in \Lambda_r(M),\ \omega_2 \in \Lambda(M)$

Where d is called an operator of exterior differentiation.

Remark:

Assume that the values of indices i,j,k are in the range 1 to $2n$ and the values of indices a,b,c in the range 1 to n. Let $\hat{a} = a + n$.

In the G- structure space, the first and second groups of structure equation of Riemannian connection are given as the following:

1- $d\omega^i = \omega^i_j \wedge \omega^j$

2- $d\omega^i_j = \omega^i_k \Lambda \omega^k_j + \frac{1}{2} R^i_{jkl}\, \omega^k \Lambda \omega^l$

The first group can be written by [13]:

$$d\omega^a = \omega^a_b \wedge \omega^b + B^{ab}_c \omega^c \wedge \omega_b + B^{abc} \omega_b \wedge \omega_c$$
$$d\omega_a = -\omega^b_a \wedge \omega_b + B^c_{ab} \omega_c \wedge \omega^b + B_{abc} \omega^b \wedge \omega^c \qquad (1.3)$$

where $B^{abc} = \frac{\sqrt{-1}}{2} J^a_{[\hat{b},\hat{c}]}$; $B_{abc} = \frac{-\sqrt{-1}}{2} J^{\hat{a}}_{[b,c]}$

$B^{ab}{}_c = \frac{-\sqrt{-1}}{2} J^a_{\hat{b},c}$; $B_{ab}{}^c = \frac{\sqrt{-1}}{2} J^{\hat{a}}_{b,\hat{c}}$

The bracket [ ] refers to the alternative.

The tensors B^{abc}, B_{abc} and $B^{ab}{}_c$, $B_{ab}{}^c$ are called Kirihenko's tensors and are denoted by (*KS*) and (*KV*) respectively [2].

2- Almost Kahler manifold

Definition 2.1 [1]

Let M^{2n} be an almost Hermitian manifold with *AH*-structure $\{J, g = \langle \cdot , \cdot \rangle\}$ and let ∇ be a Riemannian connection of the metric g. *AH*-structure is called almost Kähler structure (*AK*-structure) if the fundamental form $\Omega(X,Y) = \langle X, JY \rangle$ is closed.

That means $d\Omega = 0 \Leftrightarrow d(\omega_a \Lambda \omega^a) = 0$

Theorem 2.1

The total group of structure equation of almost Hermitian manifold is:

1. $d\omega^a = \omega^a_b \Lambda \omega^b + B^{abc}\, \omega_b \Lambda \omega_c$
2. $d\omega_a = -\omega^b_a \Lambda \omega_b + B_{abc}\, \omega^b \Lambda \omega^c$
3. $d\omega^a_b = \omega^a_c \Lambda \omega^c_b + B^{adc}_b\, \omega_c \Lambda \omega_d + B^a_{bcd}\, \omega^c \Lambda \omega^d + (A^{ac}_{bd} + 2B^{ach} B_{hbd})\, \omega^d \Lambda \omega_c$
4. $-\ dB^{abc} = B^{abc}{}_d\, \omega^d + B^{abcd}\, \omega_d + B^{dbc}\, \omega^a_d + B^{adc}\, \omega^b_d + B^{abd}\, \omega^c_d$

Where B^{abc} and B_{abc} are (*KS*) tensors of type (*3,0*) and (*0,3*) respectively and $\{A^{ac}_{bd}\}$ are the system of functions in the adjoint *G*-structure space which are symmetrized by the lower and upper indices.

Proof:

According to the definition 2.1 we have $d\Omega = 0 \Leftrightarrow d(\omega_a \Lambda \omega^a) = 0$

Then we get:

$$d\omega_a \Lambda \omega^a - \omega_a \Lambda d\omega^a = 0 \qquad (2.1)$$

By using equations (1.3) we get:

$$-\omega_a^b\Lambda\omega_b\Lambda\omega^a + B_{ab}^c\ \omega_c\Lambda\omega^b\Lambda\omega^a + B_{[abc]}\ \omega^b\Lambda\omega^c\Lambda\omega^a$$

$$-\omega_a\Lambda\omega_b^a\Lambda\omega^b - B_c^{ab}\ \omega_a\Lambda\omega^c\Lambda\omega_b - B^{[abc]}\ \omega_a\Lambda\omega_b\Lambda\omega_c) = 0$$

According to linearly independent of the basis forms we get:

$$B_c^{ab} = B_{ab}^c = 0\ ,\ B^{[abc]} = B_{[abc]} = 0$$

Therefore the class of almost Kähler manifold has the following properties which are found by Kirichenko [12]:

$$,\ B_c^{ab} = B_{ab}^c = 0\ ,\ B^{[abc]} = B_{[abc]} = 0$$

where B_c^{ab} and B^{abc} are (*KV*) and (*KS*) tensors respectively , then the first group of the structure equation of almost Kähler manifold in the adjoint *G*-structure space has the forms:

1. $d\omega^a = \omega_b^a \Lambda \omega^b + B^{abc}\ \omega_b \Lambda \omega_c$

(2.2) By

2. $d\omega_a = -\omega_a^b\Lambda\omega_b + B_{abc}\omega^b\Lambda\omega^c$

differentiation (2.2:1) we get:

$$d^2\ \omega^a = d\omega_b^a \Lambda \omega^b - \omega_b^a \Lambda d\omega^b + dB^{abc}\ \omega_b \Lambda \omega_c$$
$$+ B^{abc}\ d\,(\omega_b \Lambda \omega_c)$$

According to theorem 1.1 and using relations (2.2) we get:

$$0 = d\omega_b^a\ \Lambda \omega^b - \omega_b^a \Lambda (\omega_c^b \Lambda \omega^c + B^{bdh}\ \omega_d \Lambda \omega_h) + dB^{abc}\ \omega_b \Lambda \omega_c$$
$$+ B^{abc}\,(d\omega_b \Lambda \omega_c + \omega_b \Lambda d\omega_c)$$

then we obtain:

$$(d\omega_b^a - \omega_c^a \Lambda \omega_b^c - 2B^{ach}\ B_{hdb}\ \omega_c \Lambda \omega^d) \Lambda \omega^b +$$
$$(dB^{abc} - B^{abd}\ \omega_d^c - B^{dbc}\ \omega_d^a - B^{adc}\ \omega_d^b)\Lambda\omega_b\Lambda\omega_c = 0 \qquad (2.3)$$

Denote by $\Delta\,\omega_b^a = d\omega_b^a - \omega_c^a \Lambda \omega_b^c - 2B^{ach}\ B_{hdb}\ \omega_c \Lambda \omega^d$

$\Delta\, B^{abc} = dB^{abc} - B^{dbc}\ \omega_d^a - B^{adc}\ \omega_d^b - B^{abd}\ \omega_d^c$ And

then we can write the equation (2.3) as the following form:

$$\Delta\,\omega_b^a \Lambda \omega^b + \Delta\, B^{abc} \Lambda \omega_b \Lambda\ \omega_c = 0 \qquad (2.4)$$

The *2*-form $\Delta\omega_b^a$ can be written by the basis of *2*-form space:

$$\left\{\omega_b^a \Lambda \omega_a^c\ ,\ \omega_b^a \Lambda \omega^c\ , \omega_b^a \Lambda \omega_c\ , \omega^a \Lambda \omega^b\ , \omega^a \Lambda \omega_b , \omega_a \Lambda \omega_b \right\}$$

and the *1*-form ΔB^{abc} can be written by the basis of *1*-form space:

$$\left\{\omega_b^a\ ,\ \omega^a\ , \omega_a\right\}$$

Then we can write $\Delta\omega_b^a$ and ΔB^{abc} as the linear combination of the last two kinds of bases such that:

$$\Delta\omega_b^a = A_{bcg}^{adf}\ \omega_d^c \Lambda \omega_f^g + A_{bcf}^{ad}\ \ \omega_d^c \Lambda \omega^f + A_{bc}^{adf}\ \omega_d^c\ \Lambda \omega_f + A_{bcd}^a\ \omega^c \Lambda \omega^d$$
$$+ A_{bc}^{ad}\ \omega^c \Lambda \omega_d + A^{acd}{}_b\ \omega_c\ \ \Lambda\ \omega_d$$

$$\Delta B^{abc} = B^{abcd}{}_f\ \ \omega_d{}^f + B^{abcd}\ \omega_d + B^{abc}{}_d\omega^d$$

then equation (2.3) will be the following form:

$$A^{adf}_{bcg}\ \omega^{c}_{d}\ \Lambda\,\omega^{g}_{f}\ \Lambda\,\omega^{b}+A^{ad}_{bcf}\ \omega^{c}_{d}\ \Lambda\,\omega^{f}\ \Lambda\,\omega^{b}+A^{adf}_{bc}\ \omega^{c}_{d}\ \Lambda\,\omega_{f}\ \Lambda\,\omega^{b}+$$
$$A^{a}_{bcd}\ \omega^{c}\ \Lambda\,\omega^{d}\ \Lambda\,\omega^{b}+A^{ad}_{bc}\ \omega^{c}\ \Lambda\,\omega_{d}\ \Lambda\,\omega^{b}+A^{acd}_{b}\ \omega_{c}\ \Lambda\,\omega_{d}\ \Lambda\,\omega^{b}+$$
$$B^{abcd}_{f}\ \omega^{f}_{d}\ \Lambda\,\omega_{b}\ \Lambda\,\omega_{c}\ +B^{adc}_{b}\ \omega^{b}\ \Lambda\,\omega_{d}\ \Lambda\,\omega^{c}+B^{abcd}\ \omega_{d}\ \Lambda\,\omega_{b}\ \Lambda\,\omega_{c}=0$$

Since these bases are linearly independent, we get:

$$A^{adf}_{bcg}=0,\ A^{ad}_{[b/c/f]}=0,\ \ A^{adf}_{bc}=0,\ A^{a}_{[bcd]}=0,\ A^{ac}_{[bd]}\ =0$$
$$A^{acd}_{b}=B^{adc}_{b}\ ,B^{abcd}_{f}\ ,=0,B^{a[bcd]}=0$$

This implies

$$\Delta\omega^{a}_{b}=A^{ad}_{bcf}\ \omega^{c}_{d}\ \Lambda\,\omega^{f}\ +A^{a}_{bcd}\ \omega^{c}\ \Lambda\,\omega^{d}\ +A^{ac}_{[bd]}\ \omega^{d}\ \Lambda\,\omega_{c}+A^{acd}_{b}\ \omega_{c}\ \Lambda\,\omega_{d}$$
$$\Delta B^{abc}=B^{abc}_{d}\ \omega^{d}\ +B^{abcd}\ \omega_{d} \tag{2.5}$$

By the same way, by differentiation (2.2: 2) we get:

$$A^{bd}_{acf}\ =\ 0,\ A^{b}_{acd}=\ B^{b}_{acd}\ ,\ B_{a[bcd]}=0,\ B^{d}_{abch}=0,$$
$$A^{[bcd]}_{a}=0,A^{[bc]}_{ad}\ =-2\,(B^{[bc]h}\ \ B_{had}+B^{hcb}\ B_{adh})$$

Then $\Delta\omega^{b}_{a}=B^{bdc}_{a}\ \omega_{c}\ \Lambda\,\omega_{d}\ +B^{b}_{acd}\ \omega^{c}\ \Lambda\,\omega^{d}\ +A^{bc}_{ad}\ \omega_{c}\ \Lambda\,\omega^{d}$

and $\Delta\omega^{a}_{b}=B^{a}_{bcd}\ \omega^{c}\ \ \Lambda\,\omega^{d}\ +B^{adc}_{b}\ \omega_{c}\ \Lambda\,\omega_{d}\ +A^{ac}_{bd}\ \omega^{d}\ \Lambda\,\omega_{c}$

But we have $\Delta\omega^{a}_{b}=d\omega^{a}_{b}-\omega^{a}_{c}\ \Lambda\,\omega^{c}_{b}-2B^{ach}\ B_{hbd}\ \omega^{d}\ \Lambda\,\omega_{c}$

and $\Delta\omega^{b}_{a}=d\omega^{b}_{a}+\omega^{b}_{c}\ \Lambda\,\omega^{c}_{a}-2B^{bch}\ B_{had}\ \omega^{d}\ \Lambda\,\omega_{c}$

Then we get:

$$d\omega^{a}_{b}=\omega^{a}_{c}\ \Lambda\,\omega^{c}_{b}+B^{adc}_{b}\ \omega_{c}\Lambda\omega_{d}\ +B^{a}_{bcd}\ \omega^{c}\ \Lambda\,\omega^{d}\ +(A^{ac}_{bd}\ +2B^{ach}\ B_{hbd})\,\omega^{d}\ \Lambda\omega_{c} \tag{2.6}$$

$$d\omega^{b}_{a}=-\ \omega^{b}_{c}\ \Lambda\,\omega^{c}_{a}+B^{bcd}_{a}\ \omega_{c}\Lambda\ \omega_{d}\ +(A^{bc}_{ad}\ +2B^{bch}\ B_{had})\,\omega^{d}\ \Lambda\omega_{c}\ -\ B^{b}_{acd}\ \omega^{c}\Lambda\omega^{d} \tag{2.7}$$

and also we have $\Delta B_{abc}=B_{abcd}\ \omega^{d}\ +B^{d}_{abc}\ \omega_{d}$

$$\Delta B_{abc}=db_{abc}+B_{dbc}\ \omega^{d}_{a}+B_{adc}\ \omega^{d}_{b}+B_{abd}\ \omega^{d}_{c}$$

Therefore, we obtain:

$$dB_{abc}=B_{abcd}\ \omega^{d}\ +B^{d}_{abc}\ \omega_{d}-B_{dbc}\ \omega^{d}_{a}-B_{adc}\ \omega^{\ d}_{\ b}-B_{abd}\ \omega^{d}_{c} \tag{2.8}$$

and by the same way we obtain:

$$dB^{abc}\ =B^{abc}_{\ d}\ \ \omega^{d}\ +B^{abcd}\ \ \omega_{d}\ +B^{dbc}\ \ \omega^{a}_{d}\ +B^{adc}\ \omega^{b}_{d}\ +B^{abd}\ \omega^{c}_{d} \tag{2.9}$$

Definition 2.2: [5]

The components of the curvature tensor of a Riemannian connection satisfy the following identities:

1. $R^{j}_{ikl}=-R^{j}_{ilk}$
2. $\sum\limits_{(ikl)}R^{j}_{ikl}=0$
3. $R_{ijkl}=-R_{jikl}$
4. $R_{ijkl}=R_{klij}$

Theorem 2.2

The components of Riemannian curvature tensor of almost Kähler manifold in the adjoint *G*- structure space are:

1. $R^a_{bcd} = 2\,B^a_{bcd}$ 2. $R^a_{b\hat{c}d} = 4\,B^{cah}\ B_{dbh} - A^{ac}_{bd} - 2B^{ach}\ B_{hbd}$

3. $R^a_{bc\hat{d}} = A^{ad}_{bc} + 2B^{adh}\ B_{hbc} - 4B^{dah}\ B_{cbh}$ 4. $R^a_{b\hat{c}\hat{d}} = 2B^{adc}_b$

5. $R^{\hat{a}}_{\hat{b}cd} = -2\,B^b_{acd}$ 6. $R^{\hat{a}}_{\hat{b}\hat{c}d} = A^{bc}_{ad} + 2B^{bch}\ B_{had} - 4\,B_{dah}\ B^{cbh}$

7. $R^{\hat{a}}_{\hat{b}c\hat{d}} = 4\,B^{dbh}\ B_{cah} - A^{bd}_{ac} - 2B^{bdh}\ B_{hac}$ 8. $R^{\hat{a}}_{\hat{b}\hat{c}\hat{d}} = -B^{bcd}{}_a$

9. $R^a_{\hat{b}cd} = 4B^{hab}\ B_{hcd}$ 10. $R^a_{\hat{b}\hat{c}d} = -2B^{cab}_d$

11. $R^a_{\hat{b}c\hat{d}} = 2B^{dab}{}_c$ 12. $R^a_{\hat{b}\hat{c}\hat{d}} = -4B^{[c|ab|d]}$

13. $R^{\hat{a}}_{bcd} = -4B_{[c|ab|d]}$ 14. $R^{\hat{a}}_{b\hat{c}d} = 2\,B_{dab}{}^c$

15 $R^{\hat{a}}_{bc\hat{d}} = -2\,B_{cab}{}^d$ 16. $R^{\hat{a}}_{b\hat{c}\hat{d}} = 4\,B^{hcd}\ B_{hab}$

Proof:

Consider the second group of the structure equation of connection in Riemannian manifold:

$$d\omega^i_j = \omega^i_k \wedge \omega^k_j + \frac{1}{2} R^i_{jkl}\ \omega^k \wedge \omega^l$$

In the adjoint *G*-structure space we have: $\omega^a_{\hat{b}} = \omega^{ab} = -\frac{\sqrt{-1}}{2} J^a_{\hat{b},k}\ \omega^k$

$$= -\frac{\sqrt{-1}}{2}(J^a_{\hat{b},h}\ \omega^h + J^a_{\hat{b},\hat{h}}\omega_h)\,;\,B^{ab}_h = 0$$

$$= 2\frac{\sqrt{-1}}{2}\ J^{h[a,b]}\ \omega_h = \sqrt{-1}\,J^{h\ [a,b]}\ \omega_h = \frac{\sqrt{-1}}{2} J^{ab,h}\ \omega_h$$

$$= 2\,B^{hab}\ \omega_h$$

Then we get:

$$\omega^{ab} = 2\,B^{hab}\ \omega_h\,, \qquad \omega^{\hat{b}}_a = \omega_{ab} = 2B_{hab}\ \omega^h \tag{2.10}$$

And $\omega^a_{\hat{c}} \wedge \omega^{\hat{c}}_b = \omega^a_{\hat{h}} \wedge \omega^{\hat{h}}_b$

$$\omega^a_{\hat{c}} \wedge \omega^{\hat{c}}_b = \omega^a_{\hat{h}} \wedge \omega^{\hat{h}}_b$$

$$= 4B^{cah} B_{dbh}\,\omega^d \wedge \omega_c \quad = 2B^{cah} w_c \wedge 2B_{hbd}\,\omega^d$$

Then we can compute the components of Riemannian curvature tensor in the adjoint *G*-structure space:

1. Let $i = a$, $j = b$, then

$$d\omega^a_b = \omega^a_k \wedge \omega^k_b + \frac{1}{2} R^a_{bkl}\ \omega^k \wedge \omega^l$$

$$= \omega^a_c \wedge \omega^c_b + \omega^a_{\hat{c}} \wedge \omega^{\hat{c}}_b + \frac{1}{2} R^a_{bcd}\ \omega^c \wedge \omega^d + \frac{1}{2} R^a_{b\hat{c}\hat{d}}\,\omega_c \wedge \omega_b \ + R^a_{b\hat{c}d}\,\omega_c \wedge \omega^d$$

$$R^a_{b\hat{c}d}\,\omega_c \wedge \omega^d = \frac{1}{2} R^a_{b\hat{c}d}\,\omega_c \wedge \omega^d + \frac{1}{2} R^a_{bc\hat{d}}\,\omega^c \wedge \omega_d$$

$$=\frac{1}{2}R^{a}_{b\hat{c}d}\,\omega_c\Lambda\omega^d+\frac{1}{2}R^{a}_{b\hat{c}d}\,\omega_c\;\Lambda\omega^d$$

By using equation (2.10) we get:

$$d\omega^{a}_{b}=\omega^{a}_{c}\Lambda\,\omega^{c}_{b}+4B^{cah}\,B_{dbh}\,\omega^d\,\Lambda\,\omega_c+\frac{1}{2}R^{a}_{bcd}\,\omega^c\,\Lambda\,\omega^d+R^{a}_{bd\hat{c}}\,\omega^d\,\Lambda\,\omega_c+\frac{1}{2}R^{a}_{b\hat{c}\hat{d}}\,\omega_c\Lambda\omega_d$$

On the other hand in theorem 2.2 we have:

$$d\omega^{a}_{b}=\omega^{a}_{c}\Lambda\,\omega^{c}_{b}+(A^{ac}_{bd}+2B^{ach}\,B_{hbd})\,\omega^d\,\Lambda\,\omega_c+B^{a}_{bcd}\,\omega^c\,\Lambda\,\omega^d+B^{adc}_{b}\,\omega_c\,\Lambda\,\omega_d$$

By comparring these equations we get:

a. $R^{a}_{bcd}=2\,B^{a}_{bcd}$ b. $R^{a}_{b\hat{c}d}=4\,B^{cah}\;B_{dbh}-A^{ac}_{bd}-2B^{ach}\,B_{hbd}$

c. $R^{a}_{bc\hat{d}}=A^{ad}_{bc}+2B^{adh}\,B_{hbc}-4B^{dah}\,B_{cbh}$ d. $R^{a}_{b\hat{c}\hat{d}}=2B^{adc}_{b}$

2. Let $i=\hat{a},\ j=\hat{b}$

then, $d\omega^{\hat{a}}_{\hat{b}}=\omega^{\hat{a}}_{\hat{c}}\;\Lambda\omega^{\hat{c}}_{\hat{b}}+\omega^{\hat{a}}_{c}\;\Lambda\omega^{c}_{\hat{b}}+\frac{1}{2}R^{\hat{a}}_{\hat{b}cd}\;\omega^c\Lambda\omega^d+\;\frac{1}{2}R^{\hat{a}}_{\hat{b}\hat{c}\hat{d}}\omega_c\Lambda\omega_d+R^{\hat{a}}_{\hat{b}c\hat{d}}\omega^c\Lambda\omega_d$

but $d\omega^{\hat{a}}_{\hat{b}}=d\omega^{b}_{a}$,then by using (2.10) we get:

$$d\omega^{b}_{a}=4\,B_{dah}\,B^{cbh}\,\omega^d\,\Lambda\,\omega_c+\frac{1}{2}R^{\hat{a}}_{\hat{b}cd}\,\omega^c\,\Lambda\,\omega^d+R^{\hat{a}}_{\hat{b}\hat{c}d}\,\omega^d\,\Lambda\,\omega_c+\frac{1}{2}R^{\hat{a}}_{\hat{b}\hat{c}\hat{d}}\,\omega_c\,\Lambda\,\omega_d$$

By comparring this equation with equation (2.7) we get:

$$-\omega^{b}_{c}\,\Lambda\,\omega^{c}_{a}+B^{bcd}_{a}\,\omega_c\Lambda\,\omega_d+(A^{bc}_{ad}+2B^{bch}\,B_{had})\,\omega^d\,\Lambda\omega_c-B^{b}_{acd}\,\omega^c\Lambda\omega^d=$$

$$=4\,B_{dah}\,B^{cbh}\,\omega^d\,\Lambda\,\omega_c+\frac{1}{2}R^{\hat{a}}_{\hat{b}cd}\,\omega^c\,\Lambda\,\omega^d+R^{\hat{a}}_{\hat{b}\hat{c}d}\,\omega^d\,\Lambda\,\omega_c+\frac{1}{2}R^{\hat{a}}_{\hat{b}\hat{c}\hat{d}}\,\omega_c\,\Lambda\,\omega_d$$

Then we get:

a. $R^{\hat{a}}_{\hat{b}cd}=-2\,B^{b}_{acd}$ b. $R^{\hat{a}}_{\hat{b}\hat{c}d}=A^{bc}_{ad}+2B^{bch}\;B_{had}-4\,B_{dah}\,B^{cbh}$

c. $R^{\hat{a}}_{\hat{b}c\hat{d}}=4\,B^{dbh}\,B_{cah}-A^{bd}_{ac}-2B^{bdh}\,B_{hac}$ d. $R^{\hat{a}}_{\hat{b}\hat{c}\hat{d}}=-B^{bcd}_{\;\;a}$

3. Let $i=a,\ j=\hat{b}$ then:

$d\omega^{a}_{\hat{b}}=\omega^{a}_{c}\;\Lambda\omega^{c}_{\hat{b}}+\omega^{a}_{\hat{c}}\;\Lambda\omega^{\hat{c}}_{\hat{b}}+\frac{1}{2}R^{a}_{\hat{b}cd}\;\omega^c\Lambda\omega^d+\;\frac{1}{2}R^{a}_{\hat{b}\hat{c}\hat{d}}\omega_c\Lambda\omega_d+R^{a}_{\hat{b}c\hat{d}}\omega^c\Lambda\omega_d$ but $d\omega^{a}_{\hat{b}}=d\omega^{ab}$,

then by using (2.10) we get:

$$d\omega^{ab}=\omega^{a}_{c}\,\Lambda\,2B^{hcb}\,\omega_h+\frac{1}{2}R^{a}_{\hat{b}cd}\,\omega^c\,\Lambda\,\omega^d+\frac{1}{2}R^{a}_{\hat{b}\hat{c}\hat{d}}\,\omega_c\,\Lambda\,\omega_d+R^{a}_{\hat{b}c\hat{d}}\,\omega^c\,\Lambda\,\omega_d$$

On the other hand we have $\omega^{ab}=2B^{cab}\,\omega_c$, then by using the exterior differentiation we get:

$$d\omega^{ab}=2dB^{cab}\,\Lambda\,\omega_c+2B^{cab}\,\Lambda\,d\omega_c$$

$$=2B^{had}\,\omega^{c}_{h}\;\Lambda\,\omega_c+2B^{chb}\,\omega^{a}_{h}\,\Lambda\,\omega_c+2B^{cah}\,\omega^{b}_{h}\,\Lambda\,\omega_c$$

$$+2B^{cab}_{\;\;d}\,\omega^d\,\Lambda\,\omega_c+2B^{[c|ab|d]}\,\omega_d\,\Lambda\,\omega_c-2B^{cab}\,\omega^{f}_{c}\,\Lambda\,\omega_f+2B^{cab}B_{cbh}\omega^d\Lambda\omega^k$$

Then, by comparring the bases we get:

a. $R^{a}_{\hat{b}cd}=4B^{hab}\,B_{hcd}$ b. $R^{a}_{\hat{b}\hat{c}d}=-2B^{cab}_{d}$

c. $R^{a}_{\hat{b}c\hat{d}}=2B^{dab}_{\;\;c}$ d. $R^{a}_{\hat{b}\hat{c}\hat{d}}=-4B^{[c|ab|d]}$

By the same way if $i=\hat{a}$, $j=b$, and by using (2.10), then $d\omega_{ab}$ is the conjugate of $d\omega^{ab}$ and we can find $d\omega_{ab}$ by differentiation of $\omega_{ab}=2B_{cab}\,\omega^{c}$, then we get:

$$
\begin{aligned}
d\omega_{ab} &= 2dB_{cab}\Lambda\omega^{c}+2B_{cab}\Lambda d\omega^{c}\\
&= 2B_{had}\,\omega_{c}^{h}\Lambda\omega^{c}+2B_{chb}\omega_{a}^{h}\Lambda\omega_{c}+2B_{cab}\omega_{b}^{h}\Lambda\omega_{c}\\
&\quad +2B_{cab}^{\ \ \ d}\omega_{d}\Lambda\omega_{c}+2B_{[c|ab|d]}\,\omega^{d}\Lambda\omega_{c}-2B_{cab}\omega_{f}^{c}\Lambda\omega^{f}\\
&\quad +2B_{cab}B^{cbh}\omega_{d}\Lambda\omega_{h}
\end{aligned}
$$

By comparring the bases from the two equations we get:

a. $R_{bcd}^{\hat{a}}=-4B_{[c|ab|d]}$ b. $R_{b\hat{c}d}^{\hat{a}}=2B_{dab}^{\ \ \ c}$

c. $R_{bc\hat{d}}^{\hat{a}}=-2B_{cab}^{\ \ \ d}$ d. $R_{b\hat{c}\hat{d}}^{\hat{a}}=4B^{hcd}B_{hab}$

3- Almost kahler manifold of class R_1

Definition 3.1 [14]

The Riemannian curvature tensor R for M is *4*-covariant tensor:
R: $T_p(M)$ x $T_p(M)$ x $T_p(M)$ x $T_p(M)$ → R which defined by:
$R(X_1, X_2, X_3, X_4) = g(R(X_3, X_4)X_2, X_1)$.
$X_i \in T_p(M)$, $i=1,\dots,4$ and satisfies the following properties:

1. $R(X_1,X_2,X_3,X_4) = -R(X_2, X_1, X_3, X_4)$
2. $R(X_1,X_2,X_3,X_4) = -R(X_1, X_2, X_4, X_3)$
3. $R(X_1,X_2,X_3,X_4) + R(X_1, X_3, X_4, X_2) + R(X_1,X_4,X_2,X_3) = 0$
4. $R(X_1,X_2,X_3,X_4) = R(X_2, X_1, X_4, X_3)$

Where $X_i \in T_p(M)$ $\forall i=1,\dots,4$

A. Gray [9] defined three special classes of *AH*-manifolds, which are defined by the following:

1. Class R_1 if $\langle R(X,Y)Z, W\rangle = \langle R(JX,JY)Z, W\rangle$
2. Class R_2 if $\langle R(X,Y)Z,W\rangle = \langle R(JX,JY)Z,W\rangle + \langle R(JX,Y)JZ,W\rangle$
$+ \langle R(JX,Y)Z,JW\rangle$
3. Class R_3 if $\langle R(X,Y)Z, W\rangle = \langle R(JX,JY)JZ, JW\rangle$ $\forall$ $X,Y,Z \in X(M)$.

A. Gray [8] proved that for random *AH*-manifold, the relation among them is $R_1 \subset R_2 \subset R_3$.

The manifold of class R_1 is called paraKähler manifold [17]. The manifold of class R_3 has been studied by the name *RK*-manifold [18].

The main result in this section is to find the condition by which the almost Kähler manifold is paraKähler manifold (The manifold of class R_1). We need the following propositions:

Proposition 3.1 [10]

An almost Hermition manifold is the manifold of class R_1 if and only if in the adjoint *G*-structure space we have $R_{\hat{a}bcd}=R_{abcd}=R_{\hat{a}\hat{b}cd}=0$.

Proposition 3.2 [2]

Let M be a random AH-manfiold, then AH-structure $\{J, g = <.,>\}$ is:

1. Almost Kähler structure if and only if $B_c^{ab} = B_{ab}^c = 0$, $B^{[abc]} = B_{[abc]} = 0$
2. Kähler structure if and only if $B_c^{ab} = B_{ab}^c = 0$, $B^{abc} = B_{abc} = 0$.

Theorem 3.1

Almost Kähler manifold M is paraKähler manifold if and only M is Kähler manifold.

Proof:

Suppose that M is an arbitrary almost Hermitian manifold.
By the proposition (3.1) we have an almost Hermition manifold is paraKähler manifold if and only if in the adjoint G-structure space we have:

$$R_{\hat{a}bcd} = R_{abcd} = R_{\hat{a}\hat{b}cd} = 0$$

Suppose now that M is almost Kähler manifold, then by the theorem(2.2) we get: $4B^{had}B_{hcd} = 0.$ This means $B^{had}B_{hcd} = 0.$

By folding this equation by a and c we get:

$$B^{had}B_{had} = 0$$

and folding the last equation by b and d we get:

$$B^{hab}B_{hab} = 0$$

According to the complex conjugate we obtained:

$$\sum_{h,a,b} |B_{hab}|^2 = 0 \Leftrightarrow B_{hab} = 0$$

Then by the proposition (3.2) we get that M is Kähler manifold.

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الخلاصة:

في هذا البحث درسنا أحد الأصناف الستة عشر لمتعدد الطيات الهرميتي التقريبي وهو متعدد الطيات كوهلر التقريبي. وجدنا معادلته التركيبية ومركبات تنسر الأنحناء الريماني، ثم برهنا بأن متعدد الطيات كوهلر التقريبي يكون متعدد الطيات فوق كوهلر(من الصنف R_1) إذا وفقط إذا كان متعدد الطيات كوهلر.