ADM with Modified Polynomial to improve Newton-Raphson Method for Nonlinear Equations and System of Nonlinear Equations

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Abstract

In this paper, the Adomian Decomposition Method (ADM) with modified polynomial (El-Kalla, 2007) is applied for nonlinear equations and system of nonlinear equations. First we apply it to construct numerical algorithms based on Newton-Raphson method to solve nonlinear equations, also we prove the convergence of this algorithm is cubic. Then we introduce the Revised Adomian Decomposition Method (RADM) to solve a system of nonlinear equations (El-Kalla, 2007). To show the efficiency of the proposed method, we compare the numerical results against different efficient approaches from the literature (Jafari and Daftardar-Gejji, 2006), (Pakdemirli et al., 2008) and (Abbasbandy, 2003). The experimental results show that proposed RADM outperforms the aforementioned methods by providing better accurate results.

Keywords: Nonlinear equations, System of Nonlinear equations, Newton-Raphson method, Adomian Decomposition Method

1. Introduction

Since the early of the 1980s, the ADM has been applied for wide class of functional equations (Adomian, 1994). Adomian gives the solution as infinite series usually converging to an accurate solution.
Many kinds of generalized Newton methods were broadly used and discussed in the literature. Abbasbandy (2003) introduced an improved of Newton-Raphson method by modifying ADM. Jafari and Daftardar-Gejji (2006) introduced the RADM for solving a system of nonlinear equations. In both papers of Abbasbandy (2003) and Jafari and Daftardar-Gejji (2006), Adomian polynomial were used in the applications. Also, another polynomial discussed by El-Kalla (2007) has been applied to the scheme of Abbasbandy (2003) and the RADM. In this paper, we use the RADM to improve the solution of the nonlinear equations and the system of nonlinear equations. Also, we test and discuss different problems for this modified approach. To show the efficiency of our method, the proposed method is compared against the methods that proposed by (Jafari and Daftardar-Gejji, 2006), (Pakdemirli et al., 2008) and

(Abbasbandy, 2003). Maple 13 software is used to implement all the mentioned methods. The reminder of this paper is given as follows; In section 2, the ADM is explained. The application of the ADM for Newton-Raphson method is given in section 3. Also, the Convergence Analysis is introduced in section 4. In section 5, the RADM for systems of nonlinear equations is introduced. Different est Problems ore given in section 6. Finally the conclusions related to this paper are given in section 7.

2. The Adomian Decomposition method

Consider the following equation;

$$Lu + Nu + Ru = f(x) \tag{1}$$

Where L is an invertible linear operator, N represents the nonlinear operator and R is the remaining linear part, from equation (1) we have $Lu = f(x) - Nu - Ru$. Now by applying the inverse operator L^{-1} for both sides of equation (1) then using the initial conditions we find that $u = g(x) - L^{-1}Nu - L^{-1}Ru$, where $L^{-1} = \int_0^x (.)ds$ represents the terms that obtained from integrating the remaining term $f(x)$ and from using the given initial or boundary conditions. The ADM assumes that the nonlinear operator $N(u)$ can be decomposed by an infinite series of polynomials given by:

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, ..., u_n)$$

Where A_n are the Adomians polynomials which are defined by Abbasbandy (2003) as follows:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\Big(\sum_{n=0}^{\infty} \lambda^i h_i\Big)\right]_{\lambda=0}, n = 1, 2, 3, ... \tag{2}$$

So the A_n can be given as:

$$A_0 = N(u_0)$$

$$A_1 = \frac{d}{dx}\Big(N(u_0)\Big)u_1$$

$$A_2 = \frac{1}{2}\frac{d^2}{dx^2}\Big(N(u_0)\Big)u_1^2 + \frac{d}{dx}\Big(N(u_0)\Big)u_2$$

$$A_3 = \frac{1}{6}\frac{d^3}{dx^3}\Big(N(u_0)\Big)u_1^3 + \frac{d^2}{dx^2}\Big(N(u_0)\Big)u_1u_2 + \frac{d}{dx}\Big(N(u_0)\Big)u_3$$

$$A_4 = \frac{1}{24}\frac{d^4}{dx^4}\Big(N(u_0)\Big)u_1^4 + \frac{1}{2}\frac{d^3}{dx^3}\Big(N(u_0)\Big)u_1^2u_2 + \frac{1}{2}\frac{d^2}{dx^2}\Big(N(u_0)\Big)u_2^2 + \frac{d^2}{dx^2}\Big(N(u_0)\Big)u_1u_3 + \frac{d}{dx}\Big(N(u_0)\Big)u_4$$

Also $u(x)$ can be expressed by an infinite series of the form $u(x) = \sum_{n=0}^{\infty} u_n$, identifying u_0, the remaining components for $n = 1, 2, ...$ can be determined by using recurrence relations as follows;

$$u_0(x) = g(x)$$

$$u_n(x) = -L^{-1}(A_n) - L^{-1}[R(u_n)], \quad , \quad n = 1, 2, ...$$

The other polynomials can be generated in a similar way. The solution will be the approximations $\varphi_k = \sum_{n=0}^{k-1} u_k$ with $\lim_{k\to\infty} \varphi_k = u(x)$.
El-Kalla (2007) introduced a new formula for Adomian polynomials, he claimed that the Adomian solution using this new formula converges faster than using Adomian polynomials (2). Kalla polynomial given in the following form:

$$\bar{A}_n = N(S_n) - \sum_{i=0}^{n-1} \bar{A}_i \tag{3}$$

Where $S_n = u_0 + u_1 + ... + u_n$. For example, if $N(u) = u^4$, the first three polynomials using formulas (2) and (3) are computed to be:
Using formula (2):

$$A_0 = u_0^4$$
$$A_1 = 4u_0^3 u_1$$
$$A_2 = 6u_0^2 u_1^2 + 4u_0^3 u_2$$
$$A_3 = 12u_0^2 u_1 u_2 + 4u_0 u_1^3 + 4u_0^3 u_3$$
$$\vdots$$

Using formula (3):

$$\bar{A}_0 = u_0^4$$
$$\bar{A}_1 = 4u_0^3 u_1 + 6u_0^2 u_1^2 + 4u_0 u_1^3 + u_1^4$$
$$\bar{A}_2 = 4u_0^3 u_2 + 6u_0^2 u_2^2 + 4u_0 u_2^3 + 4u_1^3 u_2 + 6u_1^2 u_2^2 + 4u_1 u_2^3 + u_2^4 + 12u_0^2 u_1 u_2 + 12u_0 u_1^2 u_2 + 12u_0 u_1 u_2^2$$
$$\bar{A}_3 = 4u_0^3 u_3 + 6u_0^2 u_3^2 + 4u_0 u_3^3 + 4u_1^3 u_3 + 6u_1^2 u_3^2 + 4u_1 u_3^3 + 4u_2^3 u_3 + 6u_2^2 u_3^2 + 4u_2 u_3^3 + 12u_0 u_2 u_3^2 + 12u_0 u_2^2 u_3$$
$$+ 12u_0 u_1^2 u_3 + 12u_1 u_2 u_3^2 + 12u_0^2 u_1 u_3 + 12u_0^2 u_2 u_3 + 12u_1 u_2^2 u_3 + 12u_0 u_1 u_3^2 + 12u_1^2 u_2 u_3 + 24u_0 u_1 u_2 u_3 + u_3^4$$
$$\vdots$$

3. ADM for Newton-Raphson method

Consider the nonlinear equation of the following form;

$$f(x) = 0 \tag{4}$$

Assume that α is a root for equation (4), f is a two differentiable continuous function in the neighborhood of α. Also, we suppose that $|f(\alpha)| > 0$. To get an approximate solution for equation (4), we may use Newton method, which given as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{5}$$

This method starts an initial guess of the root x_0. Equation (5) will converge to α if the starting point x_0 is close enough to α. This process has the local convergence property (Householder, 1970). By Taylor expansion of $f(x)$ to a higher order we have;
$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) + O(h^3)$ giving $h = \frac{f(x)}{f'(x)} + \frac{h^2}{2}\frac{f''(x)}{f'(x)}$, or

$$h = c + N(h) \tag{6}$$

Where c is constant and $N(h)$ is the nonlinear function, now by applying ADM we have:

$$h = \sum_{n=0}^{\infty} h_n \tag{7}$$

Also, the nonlinear function is decomposed as follows:

$$N(h) = \sum_{n=0}^{\infty} \bar{A}_n \tag{8}$$

In this paper we use the polynomial (3) where $S_n = h_0 + h_1 + ... + h_n$ and $\bar{A}_0 = N(S_0) = N(h_0) = c = \frac{f(x)}{f'(x)}$. Now, substituting (7) and (8) into (6) yield the following equation:

$$\sum_{n=0}^{\infty} h_n = c + \sum_{n=0}^{\infty} \bar{A}_n \tag{9}$$

The convergence of (9) will yield $h_0 = c$, $h_{n+1} = \bar{A}_n, n = 0, 1.2, ...$ and the first few polynomials are given by:

$$\bar{A}_0 = N(h_0)$$

$$\bar{A}_1 = N(h_0 + h_1) - \bar{A}_0$$

$$\bar{A}_2 = N(h_0 + h_1 + h_2) - (\bar{A}_0 + \bar{A}_1)$$

$$\vdots$$

Let $H_m = h_0 + h_1 + ... + h_m = h_0 + \bar{A}_0 + ... + \bar{A}_{m-1}$.
For $m = 0$ we have $h \approx H_0 = h_0 = c = \frac{f(x)}{f'(x)}$, so $\alpha = x - h \approx x - H_0 = x - = \frac{f(x)}{f'(x)}$. This means $x_{n+1} = x_n - \frac{f(x)}{f'(x)}$, which is the Newton-Raphson method.
For $m = 1$ we have $h \approx H_1 = h_0 + h_1 = h_0 + \bar{A}_0$. But $h_0 = c = \frac{f(x)}{f'(x)}$, hence, $\bar{A}_0 = N(h_0) = \frac{h_0^2}{2}\frac{f''(x)}{f'(x)} = \frac{f^2(x)f''(x)}{2f'^3(x)}$ and $\alpha = x - h \approx x - H_1 = x - \frac{f(x)}{f'(x)} - \frac{f^2(x)f''(x)}{2f'^3(x)}$. In other words, $x_{n+1} = x_n - \frac{f(x)}{f'(x)} - \frac{f^2(x)f''(x)}{2f'^3(x)}$, which is the Householder equation (Householder, 1970).
For $m = 2$ and by using Adomian polynomial (2), Abbasbandy (2003) have the following schema:

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)} - \frac{f^2(x)f''(x)}{2f'^3(x)} - \frac{f^3(x)f''^2(x)}{2f'^5(x)}$$

Now by using polynomial (3) (El-Kalla, 2007), we have, $h \approx H_2 = h_0 + h_1 + h_2 = h_0 + \bar{A}_0 + \bar{A}_1$, thus;

$$h_0 = c = \frac{f(x)}{f'(x)}$$

$$h_1 = \bar{A}_0 = N(h_0) = \frac{h_0^2}{2}\frac{f''(x)}{f'(x)} = \frac{f^2(x)f''(x)}{2f'^3(x)}$$

$$\begin{aligned} h_2 = \bar{A}_1 = N(h_0+h_1) - A_0 &= N(h_0+h_1) - N(h_0) \\ &= \frac{(h_0+h_1)^2}{2}\frac{f''(x)}{f'(x)} = \frac{h^2}{2}\frac{f''(x)}{2f'(x)} \\ &= \frac{h_0 h_1 f''(x)}{f'(x)} + \frac{h_1^2 f''(x)}{2f'(x)} \\ &= \frac{f^3(x)f''^2(x)}{2f'^5(x)} + \frac{f^4(x)f''^3(x)}{4f'^7(x)} \end{aligned}$$

Now $\alpha = x - h \approx x - H_2 = x - \left(\frac{f(x)}{f'(x)} + \frac{f^2(x)f''(x)}{2f'^3(x)} + \frac{f^3(x)f''^2(x)}{2f'^5(x)} + \frac{f^4(x)f''^3(x)}{4f'^7(x)}\right)$, Hence;

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)f''(x_n)}{2f'^3(x_n)} - \frac{f^3(x_n)f''^2(x_n)}{2f'^5(x_n)} - \frac{f^4(x_n)f''^3(x_n)}{4f'^7(x_n)} \tag{10}$$

Which is a new schema.

4. Convergence Analysis

In this section we prove the order of convergence of (10) is cubic.

Definition 4.1:

The iterative sequence $\{x_n\}_{n=0}^{\infty}$ converges to the root α with order $P \geq 1$ if the following condition satisfied $|\alpha - x_{n+1}| \leq c|\alpha - x_n|^P$, $n \geq 0$ where c is positive constant.

Theorem 4.1:

The order of convergence of (10) is cubic.

Proof:

To prove the order of convergence of (10) is cubic we put;

$$\phi(x) = x - \frac{f(x)}{f'(x)} - \frac{f^2(x)f''(x)}{2f'^3(x)} - \frac{f^3(x)f''^2(x)}{2f'^5(x)} - \frac{f^4(x)f''^3(x)}{4f'^7(x)}$$

Hence we have, $\phi(\alpha) = 0$.

$$\phi'(x) = -\frac{f^2 f''}{2f'^3} - \frac{f'^3 f'' f'''}{4f'^5} + \frac{3f^3 f''^3}{2f'^6} + \frac{3f^4 f''^2 f'''}{4f'^7} + \frac{7f^4 f''}{4f'^8}$$

This yield $\phi'(x) = 0$. Now;

$$\phi''(x) = \frac{ff''}{f'^2} - \frac{1}{2}\frac{f'^3\left(2ff'f'' + f^2f'''\right) - 3f^2f'^2f'''}{f'^6}$$

$$-\frac{1}{2}\frac{f'^5\left(3f^2f'f''^2 + 2f^3f''f'''\right) - 5f^3f'^4f''^3}{f'^{10}}$$

$$-\frac{1}{4}\frac{f'^7\left(4f^3f'f''^3 + 3f^4f''^2f'''\right) - 7f^4f'^6f''^4}{f'^{14}}$$

Thus, $\phi''(\alpha) = 0$, then we find $\phi'''(x)$ and we put $x = \alpha$, this yield to $\phi'''(\alpha) = -\frac{f''(\alpha)}{f'(\alpha)}$
Now, if we put $x_{n+1} = x_n + e_n$, then we have:

$$\phi(x_{n+1}) = \phi(x_n + e_n) = \phi(x_n) + e_n\phi'(x_n) + \frac{e_n^2}{2}\phi''(x_n) + \frac{e_n^3}{6}\phi'''(x_n) + O(e^4)$$

so, $\phi(x_{n+1}) - \phi(x_n) = e_n\phi'(x_n) + \frac{e_n^2}{2}\phi''(x_n) + \frac{e_n^3}{6}\phi'''(x_n)$
which produce $e_{n+1} = \frac{e_n^3}{6}\phi'''(\alpha)$, this yield $e_{n+1} = -\frac{e_n^3}{6}\frac{f''(\alpha)}{f'(\alpha)}$.
From the last equation we deduce that (10) is cubic convergence

5. RADM for systems of nonlinear equations

Consider the following system of nonlinear equations:

$$f_i(x_1, ..., x_n) = 0, i = 1, ..., n \tag{11}$$

Equation (11) can be written in the following form:

$$x_i = c_i + N_i(x_1, ..., x_n), i = 1, ..., n \tag{12}$$

Where c_i's are constants and N_i's are in general nonlinear functions of their arguments. In the RADM the solution series $\sum_{i=0}^{\infty} x_i$ can be obtained by set,

$$x_{1,0}(x) = c_1,$$

$$x_{1,m+1}(x) = \bar{A}_{1,m},$$

$$x_{k,0}(x) = c_k + N_k^*(x_{1,0}, x_{2,0}, ..., x_{k-1,0}), k = 2, ..., n,$$

$$x_{k,m+1}(x) = \bar{A}_{k,m}^*,$$

Where, $\bar{A}_n = N(S_n) - \sum_{i=0}^{n-1}\bar{A}_i$ and N_k^* are that part of N_k which are independent of $x_k, x_{k+1}, ..., x_n$. Hence, $N_k(x_1, x_2, ..., x_n) = N_k^*(x_{1,0}, x_{2,0}, ..., x_{k-1,0}) + g_k(x_1, x_2, ..., x_n)$, $k = 2, ..., n$ and $\bar{A}_{k,m}^*$ for

$i = 1, ..., n$ is define as:

$$\bar{A}^*_{k,m} = \begin{cases} \bar{A}_{k,m+1} & \text{if} \quad N_k \quad \text{are independent of} \quad x_k, x_{k+1}, ..., x_n \\ \bar{A}^1_{k,m+1} + A^2_{k,m} & \text{if} \quad N_k(x_1, x_2, ..., x_n) = N^*_k(x_1, x_2, ..., x_{k-1}) + g_k(x_1, x_2, ..., x_n) \\ \bar{A}_{k,m} & \text{otherwise} \end{cases}$$

Here $\bar{A}^1_{k,m+1}$ and $\bar{A}^2_{k,m}$ are polynomials in the form (3), corresponding to N^*_k and g_k.

6. Test Problems

In this section we consider examples (3.1) and (3.2) (Abbasbandy, 2003), example (3.3) (Pakdemirli et al., 2008) and examples (3.4) and (3.5) (Jafari and Daftardar-Gejji, 2006) for comparison.

Example 3.1:

Consider the following equation:

$$x^3 + 4x^2 + 8x + 8 = 0$$

With exact solution $\alpha = -2$, $x_0 = -0.9$ the number of iteration in each method (NI) and the obtained solution (OS). This is shown in Table 1.

Example 3.2:

Consider the following equation:

$$e^x - 3x^2 = 0$$

With approximately solution $\alpha = 0.910007$ and $x_0 = 0.51$, see Table 1.

Example 3.3:

Consider the following equation:

$$tan(x) - tanh(x) = 0$$

With the solution $\alpha = 3.9266$ and $x_0 = 3$, see Table 1.

Table 1
Distribution of the types by rank

	Example 3.1		Example 3.2		Example 3.3	
Method	NI	OS	NI	OS	NI	OS
Newton	5	-1.99998	4	0.910006	3	3.9266
(Jafari and Daftardar-Gejji, 2006)	5	-1.99999	4	0.910011	2	3.9266
Eq. (10)	5	-2.00000	4	0.910007	2	3.9266

Example 3.4:

Consider the following nonlinear system:

$$x_1^2 - 10x_1 + x_2^2 + 8 = 0$$

$$x_1 x_2^2 + x_1 - 10x_2 + 8 = 0$$

In view of equation (12) we have the following equations;

$$x_1 = -\frac{1}{10}x_1^2 - \frac{1}{10}x_1^2 - \frac{8}{10}$$

$$x_2 = -\frac{1}{10}x_1x_2^2 - \frac{1}{10}x_1 - \frac{8}{10}$$

The exact solution is $x_1 = 1$, $x_2 = 1$, while RADM with Adomian polynomial (2) produces the solution $x_1 = 0.9983499345$, $x_2 = 0.9983518066$. Whereas the RADM, with polynomial (3) leads to the solution $x_1 = 1.0000000000$, $x_2 = 1.0000000000$.

Example 3.5:
Consider the following nonlinear system:

$$15x_1 + x_2^2 - 4x_3 = 13$$

$$x_1^2 + 10x_2 - e^{-x_3} = 11$$

$$x_2^3 - 25x_3 = -22$$

In view of equation (12) we have the following equations;

$$x_1 = \frac{13}{15} - \frac{1}{15}x_2^2 + \frac{4}{15}x_3$$

$$x_2 = \frac{11}{10} - \frac{1}{10}x_1^2 + \frac{1}{10}e^{-x_3}$$

$$x_3 = \frac{22}{25} + \frac{1}{25}x_2^3$$

The exact solutions is $x_1 = 1.04214966$, $x_2 = 1.03109169$, $x_3 = 0.92384809$. On the other hand, RADM with Adomian polynomial (2) results to the solution $x_1 = 1.04215000$, $x_2 = 1.03109000$, $x_3 = 0.92384800$. While the RADM with polynomial (3) produces the solution $x_1 = 1.042149561$, $x_2 = 1.03109127$, $x_3 = 0.92384815$.

6.1. Conclusions

In this paper we prove that the polynomial of equation (10) is cubic convergence and from the numerical results in section 3, we can see that equation (10) is effective and efficient. Also, from the application of polynomials (2) and (3) for RADM we conclude that polynomial (3) produce more accurate solutions comparing to the Adomian polynomial (2). The main reason for outperforming of (3) to (2) is that all the terms of (3) have deeper series than polynomial (2) as shown in section 2. However, polynomial (3) is simpler than polynomial (2) in calculations. This shows that the ADM with polynomial (3) is more efficient and applicable in application for nonlinear equations and systems of nonlinear equations from wide classes of problems as illustrated in the given test examples.

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