On Weak Stability Of Iteration Procedures For Some Multi-valued Contractive Maps In Metric space

Amal M. Hashim
amalmhashim@yhoo.com
Shereen J. Abbas
Department of Mathematics, College of Science,
University of Basrah, Basrah- Iraq

Abstract :

In this paper we prove w^2–stability for certain class of multi-valued maps that satisfy some general contractive condition in generalized Hausdorff metric space. Some well known results are also derived as a special cases.

Keywords: Coincidence and Fixed points ,Weak Stable iteration, w^2–stability.

حول الاستقرار الضعيف للعمليات المكررة لبعض دوال الانكماش ذات القيم المتعددة في الفضاء المتري

أمل محمد هاشم البطاط
شيرين عباس
قسم الرياضيات / كلية العلوم / جامعة البصرة

الملخص

في هذا البحث برهنا w^2-stability لفئة معينةمن الدوال ذات القيم المتعددة التي تحقق شرط التقليص العام في فضاء هاوزدورف المعمم المتري اشتقت بعض النتائج المعروفة كنتائج خاصة

1- Introduction

Brined [1] introduced a weaker concept of stability, called weak stability. Timis [10] established weaker notion, named w^2–stability because of the restriction of an approximation sequence. Some fixed point iteration are not weakly stable so it is used a weaker type sequence named equivalent sequence and gave weak stability results of Picard iteration for various contractive maps.

However, a formal definition of stability of general iterative procedures has been studied by Harder in her Ph. D. Thesis [5] and published in the papers [3] and [4], but the concept of stability is not very precise because of the sequence $\{y_n\}$ which is arbitrary taken.

In this paper, we give w^2–stability results of picard iteration for multi-valued maps with coincidence points satisfying some contractive type mappings. Since the stability results for multi-valued contractions have been found useful in the area of generalized differential equations and other contexts (see, for instance [2], [6], [11] and [12]).

2- Preliminaries

Throughout the paper, let (X,d) be a metric space. We shall follow the following notions and definitions.

$CL(X)=\{A : A$ is non empty closed subset of $X\}$. We consider

$$H(A,B)=\max\{\sup d(a,B)\ ;\ a\in A,$$
$$\sup d(A,b)\ ;\ b\in B\},$$

For all $A,B\in CL(X)$ and $d(a,B)=\inf\{d(a,b)\ ,\ b\in B\}.$

H is called the generalized Hausdorff metric for $CL(X)$ induced by d.

In [9] Timis presented some mappings $T:X\to X$ satisfying various contractive conditions for which the associated Picard iteration is w^2–stability.

Their corresponding condition in case of a pair (S,T) mapping, where $S:Y\to X$ single-valued map and $T:Y\to CL(X)$ multi-valued map with $x,y\in Y$ and $x\neq y$ are in the following form.

(1.1) $H(Tx,Ty)<m(x,y)$

Where

$$m(x,y)=\max\{d(Sx,Sy),$$
$$\left[d(Sx,Tx)+d(Sy,Ty)\right]/2,$$
$$\left[d(Sx,Ty)+d(Sy,Tx)\right]/2\}$$

(1.2) $H(Tx,Ty)<M(x,y)$

Where:

$$M(x,y)=\max\{d(Sx,Sy),$$
$$d(Sx,Tx),d(Sy,Ty),$$
$$\left[d(Sx,Ty)+d(Sy,Tx)\right]/2\},$$

(1.3) $H(Tx,Ty)<N(x,y)$

Where:

$$N(x,y)=\max\{d(Sx,Sy),$$
$$d(Sx,Tx),d(Sy,Ty)$$
$$,d(Sx,Ty),d(Sy,Tx)\},$$

$$H(Tx,Ty)<M(x,y)+Ld(Sx,Ty) \qquad \text{Where } L\geq 0.$$

We remarked that in case of single-valued maps with $S=$ identity map in the metric space (X,d).

i. Condition (1.1) implies (1.2) that is any mapping which satisfies condition (1.1) also satisfies condition (1.2).

ii. (1.2) implies (1.3) and (1.3), (1.4) are independent, and (1.2) implies (1.4) for more details (see [7] for instance).

Definition 2.1 [1]

Let (X,d) be a metric space and $\{x_n\}_{n=1}^{\infty}\subset X$ be given sequence. We shall say that $\{y_n\}_{n=1}^{\infty}\subset X$ is an approximate sequence of $\{x_n\}$ if, for any $k\in N$ (Natural numbers), there exists $\eta=\eta(k)$ such that

$d(x_n,y_n)\leq\eta,$ for all $n\geq k$.

Lemma 2.2 [1]

The sequence $\{y_n\}$ is an approximate sequence of $\{x_n\}$ if and only if, there exists a decreasing sequence of positive numbers $\{\eta_n\}$ converging to $\eta \geq 0$ such that

$d(x_n, y_n) \leq \eta_n$, for all $n \geq k$.

Definition 2.3 [9]

Let (X, d) be a metric space and let $S : X \to X$. two sequence $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are called S-equivalent sequences if $d(Sx_n, Sy_n) \to 0$, as $n \to \infty$.

Lemma 2.4 [8]

Let $B \in CL(X)$ and $a \in X$, then for any $b \in B$, $d(a,b) \leq H(a,B)$.

We remarked that any equivalent sequence is an approximate sequence but the reverse is not true, as it shown in the next example.

Example 2.5 [10]

Let $\{x_n\}_{n=0}^{\infty}$ to be the sequence with $x_n = n$. first we take an equivalent sequence of $\{x_n\}_{n=0}^{\infty}$ to be $\{y_n\}_{n=0}^{\infty}$, $y_n = n + \frac{1}{n}$. in this case, we have that

$d(y_n, x_n) = d(n, n + \frac{1}{n}) = \frac{1}{n} \to 0$, as $n \to \infty$.

Now we take an approximate sequence of $\{x_n\}_{n=0}^{\infty}$ to be $\{y_n\}_{n=0}^{\infty}$, $y_n = n + \frac{n}{2n+1}$, Then,

$$d(y_n, x_n) = d\left(n, \frac{n}{2n+1}\right)$$

$$= \frac{n}{2n+1} \to \frac{1}{2} > 0$$

as $n \to \infty$.

Definition 2.6 [13]

Let (X, d) be a metric space and $Y \subseteq X$ let $S : Y \to X$, $T : Y \to CL(X)$ be such that $TY \subseteq TX$ and z is a coincidence point of S and T, that is $u = Sz \in Tz$. for any $x_0 \in Y$, let the sequence $\{Sx_n\}$ be generated by the general procedure,

$$Sx_{n+1} \in f(T, x_n), \qquad n = 0,1,.... \qquad (1.5)$$

Converges to an element $u \in X$. let $\{Sy_n\}$ be an approximate sequence of $\{Sx_n\}$, we have that $H(Sy_{n+1}, f(T, y_n)) = 0$ implies $\lim_{n\to\infty} Sy_n = u$. Then (1.5) is called weakly (S,T)-stable or weakly stable with respect to (S,T).

3- Main Results

We shall introduce the following definition.

Definition 3.1

Let (X,d) be a metric space. Let $S:Y \to X$ and $T:Y \to CL(X)$ be such as $TY \subseteq SY$ and z is a coincidence point of S and T, that is, $u = Sz \in Tz$.

For any $x_0 \in Y$, let the sequence $\{Sx_n\}$ be generated by the general iteration procedure, (1.5) converges to an element $u \in X$. let $\{Sy_n\}_{n=0}^{\infty} \subset X$ be an equivalent sequence of $\{Sx_n\}$, and we have that $H(Sy_{n+1}, f(T, y_n)) = 0$ implies $\lim_{n\to\infty} Sy_n = u$. then (1.5) is called w^2–stable with respect to (S,T).

Theorem 3.2

Let (X,d) be a metric space and $Y \subseteq X$. Let $T:Y \to CL(X)$, $S:Y \to X$ such as $TY \subseteq SY$ and one of SY or TY is a complete subspace of X. let z be a coincidence point of T and S, that is, $u = Sz \in Tz$.

For any $x_0 \in Y$, let the sequence $\{Sx_n\}$ generated by Picard iteration $Sx_{n+1} \in Tx_n$ converges to u.

Let $\{Sy_n\} \subseteq X$ be an equivalent sequence of $\{Sx_n\}$ and define $\varepsilon_n = H(Sy_{n+1}, Ty_n), \quad n = 0,1,\ldots$.

If the pair (S,T) satisfy condition (1.2), and if Tz is singleton, then the Picard iteration is w^2–stable with respect to (S,T).

Proof :

Consider $\{Sy_n\}$ to be equivalent sequence of $\{Sx_n\}$. Then according to definition3.1, if $\lim_{n\to\infty} H(Sy_{n+1}, Ty_n) = 0$ implies that $\lim_{n\to\infty} Sy_n = u$, then the Picard iteration is w^2- (S,T) stable.

In order to prove this, we suppose that $\lim_{n\to\infty} H(Sy_{n+1}, Ty_n) = 0$,

therefore, $\forall \varepsilon > 0$, $\exists\, n_0 = n(\varepsilon)$ such that $H(Sy_{n+1}, Ty_n) < \varepsilon \quad \forall\, n \ge n_0$

$$
\begin{aligned}
d(Sy_{n+1}, u) &\le d(Sy_{n+1}, Sx_{n+1}) + d(Sx_{n+1}, u) \\
&\le d(Sx_{n+1}, Ty_n) + H(Ty_n, Sy_{n+1}) \\
&\quad + d(Sx_{n+1}, u). \\
&\le H(Tx_n, Ty_n) + H(Ty_n, Sy_{n+1}) \\
&\quad + d(Sx_{n+1}, u) \\
&\le \max\{d(Sx_n, Sy_n), d(Sx_n, Tx_n), \\
&\quad d(Sy_n, Ty_n), \frac{1}{2}[d(Sx_n, Ty_n) \\
&\quad + d(Tx_n, Sy_n)]\} + H(Ty_n, Sy_{n+1}) \\
&\quad + d(Sx_{n+1}, u).
\end{aligned}
$$

From the hypothesis, $Sx_n \to u$ and $Sx_{n+1} \in Tx_n$ we have that

$$d(Sx_n, Tx_n) \le d(Sx_n, Sx_{n+1}) + d(Sx_{n+1}, Tx_n)$$

$\leq d(Sx_n,u)+d(u,Sx_{n+1})+d(Sx_{n+1},Tx_n)\rightarrow 0$

Now if $M(x,y)=d(Sx_n,Tx_n)$ by taking limit we obtain $d(Sy_{n+1},u)\rightarrow 0.$

If $M(x,y)=d(Sy_n,Ty_n),$ we have

$$d(Sy_n,Ty_n)\leq d(Sy_n,Sx_n)+d(Sx_n,Sx_{n+1})$$
$$+d(Sx_{n+1},Sy_{n+1})+d(Sy_{n+1}+Ty_n).$$

From definition 2.3, we have that $d(Sx_n,Sy_n)\rightarrow 0$ and by taking limit, we obtain $d(Sy_{n+1},u)\rightarrow 0$

If $M(x,y)=d(Sx_n,Sy_n),$ from definition 2.3, we have $d(Sx_n,Sy_n)\rightarrow 0$ and by taking limit, we obtain $d(Sy_{n+1},u)\rightarrow 0$ .If

$$M(x,y)=\frac{1}{2}[d(Sx_n,Ty_n)+d(Sy_n,Tx_n)]$$
$$\leq\frac{1}{2}[d(Sx_n,Sy_n)+d(Sy_n,Ty_n)$$
$$+d(Sy_n,Tx_n)]$$

Taking limit, we obtain $d(Sy_{n+1},u)\rightarrow 0$

Hence $\lim_{n\to\infty}Sy_n=0$

This complete the proof of the theorem.

Theorem 3.3

Let (X,d) be a metric space and $Y\subseteq X$. Let $T:Y\rightarrow CL(X)$, $S:Y\rightarrow X$ such as $TY\subseteq SY$ and one of SY or TY is a complete subspace of X. Let z be a coincidence point of T and S, that is, $u=Sz\in Tz$.

For any $x_0\in Y$, let the sequence $\{Sx_n\}$ generated by Picard iteration $Sx_{n+1}\in Tx_n$ converges to u.

Let $\{Sy_n\}\subseteq X$ be an equivalent sequence of $\{Sx_n\}$ and define $\varepsilon_n=H(Sy_{n+1},Ty_n),\quad n=0,1,...$.

If the pair (S,T) satisfy condition (1.4), and if Tz is singleton, then the Picard iteration is w^2–stable with respect to (S,T).

The following example show that (S,T) is not stable but weakly stable and hence w^2–stable with respect to (S,T).

Example 3.4

Let $X=[0,1]$ and $T:X\rightarrow X$, $S:X\rightarrow X$ such as $TX\subseteq SX$ and

$$TX=\begin{cases}\{0\}, & if\quad x\in\left[0,\frac{1}{2}\right]\\ \{\frac{1}{2}\}, & if\quad x\in\left(\frac{1}{2},1\right]\end{cases}$$

$$TX=\{0\}\cup\{\frac{1}{2}\}=\{0,\frac{1}{2}\}\subseteq SX=X=[0,1]$$

$Sx=x$

Where $[0,1]$ endowed with the usual metric T is continuous at each point of $[0,1]$ except at $\frac{1}{2}$. T has a unique fixed point at 0, i. e, $0 \in T(0) = \{0\}$.

T satisfies condition (1.3).

If $0 \le x \le \frac{1}{2}$, $0 \le y \le \frac{1}{2}$ and $x \ne y$. Then

$$H(Tx,Ty) = 0 < |x-y|$$
$$= \max\{|x-y|,|x-Tx|,|y-Ty|,$$
$$\frac{1}{2}[|x-Ty|+|y-Tx|]\}$$

If $\frac{1}{2} < x \le 1$ and $\frac{1}{2} < y \le 1$ and $x \ne y$. then

$$H(Tx,Ty) = 0 < |x-y| = \max\{|x-y|,|x-Tx|,$$
$$|y-Ty|,\frac{1}{2}[|x-Ty|+|y-Tx|]\}$$

If $0 \le x \le \frac{1}{2}$ and $\frac{1}{2} < y \le 1$ and $x \ne y$. then

$$H(Tx,Ty) = \frac{1}{2} < y = \max\left\{\left|\frac{1}{2}-x\right|, y\right\}$$
$$= \max\{|x-Tx|,|y-Ty|\},$$

Thus

$$H(Tx,Ty) < \max\{|x-y|,|x-Tx|,$$
$$|y-Ty|,\frac{1}{2}[|x-Ty|+|y-Tx|]\}$$

In order to study the (S,T) -stability, let $x_0 \in [0,1]$, $Sx_{n+1} = x_{n+1} \in Tx_n$, for $n = 0,1,...$

Then,
$$x_1 \in Tx_0 = \begin{cases} \{0\} & if \ x_0 \in \left[0,\frac{1}{2}\right] \\ \left\{\frac{1}{2}\right\} & if \ x_0 \in \left(\frac{1}{2},1\right] \end{cases}$$

In each case, $x_2 \in Tx_1 = \{0\}$ and $x_n = 0, \quad \forall n \ge 2$.

so, $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} x_n = 0 \in T\{0\}$

To show that the picard iteration is not (S,T) -stable,

Let $Sy_n = y_n = \frac{n^2+1}{2n^2}$, $\quad n \ge 1$.

Then

$$\varepsilon_n = H(Sy_{n+1},Ty_n) = |y_{n+1} - Ty_n|$$
$$= \left|\frac{(n+1)^2+1}{2(n+1)^2} - \frac{1}{2}\right|,$$

because of $y_n \ge \frac{1}{2}$, for $n \ge 1$.

Therefore, $\lim_{n\to\infty} \varepsilon_n = 0$ but $\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} y_n = \frac{1}{2}$, so the Picard iteration is not (S,T)-stable.

In order to show that (S,T) weak stability. We take an approximate sequence $\{Sy_n\}$ of $\{Sx_n\}$, from lemma (2.3)

$|Sx_n - Sy_n| = |x_n - y_n| \le \eta_n, \quad n \ge k.$

$-\eta_n \le x_n - y_n \le \eta_n$

$0 \le y_n \le x_n + \eta_n, \ \ n \ge k$

Since $x_n = 0$ for $n \ge 2$,

$0 \le y_n \le \eta_n \qquad , n \ge k_1 = \max\{2, k\}$

We can choose $\{\eta_n\}$ such that $\eta_n \le \frac{1}{2}$, $n \ge k_1$ and therefore, $0 \le y_n \le \frac{1}{2}, \quad \forall n \ge k_1$

So $Ty_n = \{0\}$ and the results that $\varepsilon_n = H(Sy_{n+1}, Ty_n) = Sy_{n+1} = y_{n+1}$

Now $\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} y_{n+1} = 0,$

So the iteration procedure is weakly stable with respect to (S,T).

Hence, it is w^2–stable with respect to (S,T).

Corollary 3.5 [9]

Let (X,d) be a complete metric space and $S,T : X \to X$ such that $TX \subseteq SX$ satisfying the following condition :

$$d(Tx,Ty) < \max\{d(Sx,Tx), d(Sy,Ty), d(Sx,Sy), \frac{1}{2}[d(Sx,Ty) + d(Sy,Tx)]\}, \quad \text{For all } x, y \in X \text{ and } x \ne y.$$

Let $\{Sy_n\}_{n=0}^{\infty}$ an iteration procedure defined by $x_0 \in X$ and $Sx_{n+1} = Tx_n$, for all $n \ge 0$ and the sequence $\{Sx_n\}$ converge to u, where u is a coincidence point of S and T, then the Picard iteration is w^2–stable with respect to (S,T).

The following example explains the stability, weakly stability and w^2–stability for some contraction condition in case of single-valued map.

Example 3.5

Let $X = [0,1]$ and let $T : X \to X$ be such that $Tx = x$, where X has the usual metric, Evidently, every point of X is a fixed point of T. let $x_0 = \frac{1}{2}$.

Then $x_{n+1} = Tx_n = T^{n+1}x_0 = \frac{1}{2}, \quad n = 0,1,2,\ldots$,

thus $\lim_{n\to\infty} x_n = \frac{1}{2}$. let $\{y_n\}_{n=o}^{\infty}$ be an arbitrary sequence in X such that $y_0 = x_0$ and $y_0 = \frac{1}{n}, \ \ n = 1,2,3,\ldots$.

Thus $\lim |y_{n+1} - Ty_n| = \left|\frac{1}{n+1} - \frac{1}{n}\right| = \frac{1}{n(n+1)} \to 0$ as $n \to \infty$, however, $\lim_{n\to\infty} y_n = 0 \ne \lim_{n\to\infty} x_n = \frac{1}{2}$.

Therefore, the iterative procedure $x_{n+1} = Tx_n$ is not stable.

Now, if we choose $\{y_n\}$ an approximate sequence of $\{x_n\}$ such that $y_0 = x_0$ and $y_n = \frac{2n+1}{2n}$. In this case, we have $d(y_n, x_n) = \frac{n+1}{2n} \to \frac{1}{2} > 0$ as $n \to \infty$, however, $\lim_{n\to\infty} y_n = \frac{1}{2} = \lim_{n\to\infty} x_n$. Therefore, the iterative procedure $x_{n+1} = Tx_n$ is weakly stable.

Finally, let $\{y_n\}$ be an equivalent sequence of $\{x_n\}$ such that $y_0 = x_n$ and $y_n = \frac{n+1}{2}$. in this case, we have $d(y_n, x_n) = \frac{1}{2n} \to 0$ as $n \to \infty$.

However, $\lim_{n\to\infty} y_n = \frac{1}{2} = \lim_{n\to\infty} x_n = T\left(\frac{1}{2}\right)$.

Therefore, the iterative procedures $x_{n+1} = Tx_n, \quad n = 0,1,2,\ldots$. is w^2–stable.

Note that T is non expansive, that is, $d(Tx, Ty) \le d(x, y)$, for all $x, y \in X$.

Remark 3.7

i. If S is the identity mapping in corollary 3.4 we have theorem 2.4 [10].

ii. Every stable iteration is weakly stable but the reverse may not true (see [11]).

iii. Every weakly stable iteration is w^2–stable but the reverse may not true (see [9]).

iv. There is some mappings that satisfy contraction condition and for which the Picard iteration is not (S,T) -stable, it is not (S,T) -weakly stable but it is (S,T) -w^2stable (see [9]).

References:

[1]Berinde, V. Iterative approximation of fixed points. Springer Veriag, lectures notes in Mathematics, 2007.

[2]Cardinali. T. and Rubbioni. P, A generalization of the caristi fixed point theorem in metric space, Fixed point theory. 11(2010), 3-10.

[3]Harder, A. M. and Hicks, T. L. A stable iteration procedure for non expansive mapping. Math. Japon. 33(1998), 687-692.

[4]Harder, A. M. and Hicks, T. L. Stability results for fixed point iteration procedures. Math. Japon. 33(1988), 693-706.

[5]Harder, A. M. Fixed point theory and stability results for fixed point iteration procedures. Ph. D. thesis. University of Missouri-Rolla. Missouri. (1987).

[6]Lim, T. C. , Fixed point stability for set-valued contractive mappings with applications to generalized differential equations. J. Math. Anal. Appl. 110 (1985), 436-441.

[7]Rhoades, B. E. , A comparison of various definition of contractive mappings, Tran. Amer. Math. Soc. 226 (1977), 257-290.

[8]Singh S. L. , Chary Bhatnagar and Hashim A. M. Round of stability of picard iterative procedure for multi-valued operators.Non linear Anal. Forum 10(1) (2005), 13-19.

[9]Timis, I. and Berinde, V. Weak stability of iterative procedures for some coincidence theorems. Creative Math. & Inf. 19 (2010), 85-95.

[10]Timis, I. On the weak stability of picard iteration for some contractive type mappings and coincidence theorems. Int. J. Com. Appl. 37 (2012), 9-13.

[11]Timis, I. On the weak stability of picard iteration for some contractive type mappings. Annals of university of Craiova – Mathematics, and computer science series.37 (2) (2010), 106-114.

[12]Tolostonogo V. A. Differential inclusions in Banach space kluwer Academic publishers, 2000 (2008), 2942-2949.

[13]Zaineb Sami Almousawy; Fixed point theorems and Multi-valued operators. M. SC. Thesis. University of Basrah (supervised by Dr. Amal M. Hashim) 2012.