On a fixed point theorems for multivalued maps in b-metric space

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Abstract:

In this paper, we prove the existence of the fixed points for ciric's contractive condition combing with Berinde condition the class of so called ciric strong almost contraction in b-metric spaces. The ciric strong almost contraction appear to be one of the most general metrical condition for which the set of fixed points is not a singleton.

Our results extend and unify a multitude of fixed point theorems for multivalued maps.

Mathematics Subject Classifications (2000): 54H25**, 47**H10.

Keywords: fixed point, b-metric space, multivalued map

1-Introduction:

The concept of b-metric space was introduced by Czerwik((1998), since then several papers deal with fixed point theory for singlevalued and multivalued operators in b-metric see Singh et al.(2005),Boriceanu (2009) and Hashim (2011) . Our purpose is to show that some well-known fixed point theorems are valid in b-metric spaces.

Let (X,d) be a metric space and $CL(X)$ denotes the class of all nonempty closed subset of X and $CB(X)$ denotes the class of all nonempty closed bounded subset of X. For A,B,C we consider $d(x,B)=\inf\{d(x,y);y\in B\}$, the distance between x and B. For any $A,B\in CL(X)$, define a function $H:CL(X)\times CL(X)\to[0,\infty]$

$$H(A,B)=max\ \{\sup_{x\in A}d(x,B),\sup_{y\in B}d(y,A)\},$$

then, H is said to be the generalize Housdorffmetric on $CL(X)$ induced by the metric d, see for instance Czerwik S.

***Theorem (1.1)*:**Czerwik(1998)

Let (X,d) be a complete b-metric space. If $T:X\to CL(X)$, satisfies the inequality $\mathrm{H(Tx,Ty)}\le\alpha \mathrm{d(x,y)}$, $\mathrm{x,y}\in \mathrm{X}$, where $0\le\alpha\le b^{-1}$, *Then*

(i) for every $x_o\in X$, there exist a sequence $\{x_n\}\subset X$ *and* $u\in X$ *such that* $x_{n+1}\in Tx_n$, $n=0,1,2,3...$, and $\lim_{n\to\infty}x_n=u$,

(ii) the point u is a fixed point of T, *i.e* $u\in Tu$.

Theorem (1.2): **Berinde et al.(2007)**

Let (X,d) be a complete metric space, $T:X\to CB(X)$ be a multivalued map and $L\ge 0$. Assume that $H(Tx,Ty)\le\alpha(d(x,y))d(x,y)+L\,d(y,Tx)$, for all $x,y\in X$, where α is a function from $[0,\infty)$ into $[0,\infty)$ satisfying $\limsup_{s\to t^+}\alpha(s)<1$ for all $t\in[0,\infty)$. Then $F(T)\neq\emptyset$ where $F(T)$, the set of fixed point of T. As known, a mapping $\varphi:R_+\to R_+$ is called a comparison function if it is increasing and $\varphi^n(t)\to 0, n\to\infty$, for any $t\in R_+$

Lemma (1.3): **Rus(2001)**

If $\varphi: R^+ \to R^+$ is a comparison function, then;

1) each iterate φ^k of φ, $k \geq 1$, is also a comparison function:
2) φ is continuous at zero;
3) $\varphi(t) < t$, for any $t > 0$.

Let $T: X \to CL(X)$ and consider the following condition for all $x, y \in X$:

1) $H(Tx, Ty) \leq \alpha(d(x,y)) + Ld(y,Tx)$, where $L \geq 0$ and $0 < \alpha < 1$

2) $H(Tx, Ty) \leq \alpha M(x,y)$,

where $M(x,y) = \left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\left[d(x,Ty) + d(y,Tx)\right] \right\}$,

3) $H(Tx, Ty) \leq \alpha M(x,y) + Ld(y,Tx)$,

4) $H(Tx, Ty) \leq \varphi M_1(x,y) + Ld(y,Tx)$.

Where $M_1(x,y) = \{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2b}\left[d(x,Ty) + d(y,Tx)\right]\}$, where φ is b-comparison function with $\varphi(z) = \frac{z}{b}$, $\varphi(0) = 0, b \geq 1$.

Remark (1.4):

Condition (1) is a very general contractive condition that allows the operator T to have more than one unique fixed point and it includes many contractive conditions from Rhoade's classification Rhoade(1977) and it is called a multivalued weak (almost) contraction, see Berinde et al.(2007) .Condition (1) and (2) are independents.

Condition (3) introduced by Berinde(2009) and by replacing the term $\alpha d(x,y)$ in (1) by $\varphi M_1(x,y)$ we get contractive condition (1). It is obvious that condition (4) is more general contractive condition than (1) ,(2) and (3).

2. **Preliminaries**

Consistent with Hashim(2011) and Czerwik(1998) we use the following notations and definitions.

Definition (2.1): Czerwik**(**1998)**.**

Let X be a nonempty set and $b \geq 1$ a given real number. Afunction $d : X \times X \to R^{+}$(nonnegative real numbers) is called a b-metric space provided that, for all $x, y, z \in X$,

1) $d(x,y) = 0 \ iff \ x = y$,

2) $d(x,y) = d(x,y)$,

3) $d(x,z) \leq b[d(x,y) + d(y,z)]$.

The pair (X,d) is called a b-metric space. We remark that a metric space is evidently a b-metric space. However, Czerwik [1] has shown a b-metric on X need not be a metric space on X.

The following example show b-metric on X not be a metric on X.

Example (2.2):Czerwik**(**1998)**.**

Let $X = \{x_1, x_2, x_3\}$ and $d : X \times X \to R^{+}$ such that $d(x_1,x_3) = a \geq 2$, $d(x_1,x_3) = d(x_2,x_3) = 1$and $d(x_n,x_n) = 0$, $d(x_n,x_k) = d(x_k,x_n)$for $n,k = 1,2,3...,$. Then,$d(x_n,x_k) \leq \frac{a}{2}[d(x_n,x_i) + d(x_i,x_k)]$, $n,k,i = 1,2,3$. Then (X,d) is a b-metric space. And if $a > 2$, the ordinary triangle inequalitydoes not hold.

Definition (2.3):Boriceanu (2009)

Let (X,d) be a b-metric space. Then a sequence $\{x_n\}_{n \in N}$ in X is called

a) Cauchy if and only if for every $\varepsilon > 0$ there exists $n(\varepsilon) \in N$ such that for each $n, m \ge n(\varepsilon)$ we have $d(x_n, x_m) < \varepsilon$.

b) Convergent if and only if there exists $x \in X$ such that for all $\varepsilon > 0$ there exists $n(\varepsilon) \in N$ such that for all $n \ge n(\varepsilon)$ we have $d(x_n, x) < \varepsilon$.

c) The b-metric space is complete if every Cauchy sequence converges.

Lemma (2.4): Czerwik**(**1998)**.**

For any $A, B, C \in CL(X)$.

(i) $d(x, B) \le d(x, y)$, for any $y \in B$.

(ii) $d(A, B) \le H(A, B)$,

(iii) $d(x, B) \le H(A, B)$, $x \in A$

(iii) $H(A, C) \le s[H(A, B) + H(B, C)]$,

(iv) $d(x, A) \le sd(x, y) + sd(y, A)$, $x, y \in X$.

Lemma (2.5)] Czerwik**(**1998)***.:***

Let (X, d) be a b-metric space and $A, B \in CL(X)$. Then for each $\alpha > 0$ and for all $b \in B$ there exists $a \in A$ such that $d(a, b) \le H(A, B) + \alpha$.

Lemma (2.6): Singh et al. (2008).

Let (X, d) be a b-metric space and $\{y_n\}$ is a sequence in X such that $d(y_{n+1}, y_{n+2}) \le qd(y_n, y_{n+1})$, $n = 0, 1, 2, \ldots$

Then the sequence in X $\{y_n\}$ is a Cauchy sequence in X, provided that $sq < 1$ where $q \in (0,1)$ and $s \ge 1$

Lemma 2.7: Pacurar(*2010)*

Any b-comparison function is a comparison function.

3. Main Results

Theorem (3.1):

Let (X,d) be a complete b-metric space with constant $b \geq 1$ and $T: X \to CL(X)$ a multivalued operator. Suppose that there exists a continuous $\varphi: R^+ \to R^+$ with $\varphi(z) = \frac{z}{b}$, $\varphi(0) = 0$, and for all $x, y \in X$. We have $H(Tx, Ty) \leq \varphi M_1(x, y) + Ld(y, Tx)$, with strict inequality if $M_1(x, y) \neq 0$. Where

$M_1(x,y) = Max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2b}\left[d(x,Ty) + d(y,Tx)\right]\right\}$. Then T has a fixed point.

Proof:

Let $x_0 \in X$ and $x_1 \in Tx_0$. If $H(Tx_0, Tx_1) = 0$. Then, $Tx_0 = Tx_1$; that is $x_1 \in T_1$, which implies that $Fix(T) \neq \emptyset$.

Let $H(Tx_0, Tx_1) \neq 0$. Since $H(Tx_0, Tx_1) < \varphi(M_1(x_0, x_1) + Ld(x_1, Tx_1)$, by Lemma (2.5) we may choose $\varepsilon > 0$ with

$$H(Tx_0, Tx_1) + \varepsilon \leq \varphi(M_1(x_0, x_1)) + Ld(x_1, Tx_0) \leq \varphi(M_1(x_0, x_1),$$

Next we choose $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq H(Tx_0, Tx_1) + \varepsilon \leq \varphi(M_1(x_0, x_1)) \leq \varphi \max\{d(x_0, x_1),$$
$$d(x_0, Tx_0), d(x_1, Tx_1), \frac{1}{2b}\left[d(x_0, Tx_1) + d(x_1, Tx_0)\right]\}$$
.

There is 4 cases,

1) If $M_1(x_0, x_1) = d(x_0, x_1)$. Then ,

$$d(x_1, x_2) \leq \varphi d(x_0, x_1), \qquad (3.1)$$

2) If $M_1(x_1,x_0)=d(x_0,Tx_0)$, then we have,

$d(x_1,x_2)\le H(Tx_0,Tx_1)+\delta\le\varphi M(x_0,x_1)+Ld(x_1,Tx_0)\le\varphi d(x_0,x_1)<d(x_0,x_1)$,

3) If $M_1(x_0,x_1)=d(x_1,Tx_1)$, then,

$d(x_1,x_2)\le\varphi d(x_1,Tx_1)+Ld(x_1,Tx_0)\le\varphi d(x_1,Tx_1)<d(x_1,x_2)$,

4) If $M_1(x_0,x_1)=\frac{1}{2b}\left[d(x_0,Tx_1)+d(x_1,Tx_0)\right]=\frac{1}{2b}d(x_0,Tx_1)$, then,

$d(x_1,x_2)\le\varphi[\frac{1}{2b}d(x_0,Tx_1)]<\frac{1}{2b}d(x_0,Tx_1)\le\frac{1}{2b}d(x_0,x_2)\le\frac{1}{2}[d(x_0,x_1)+d(x_1,x_2)$,

Hence $d(x_1,x_2)<\frac{1}{2}d(x_0,x_1)$.

Thus (3.1)is true in all cases, $d(x_1,x_2)\le\varphi d(x_0,x_1)<d(x_0,x_1)$. Next we assume $M_1(x_1,x_2)\ne0$ choose $\delta>0$ with $H(Tx_1,Tx_2)+\delta\le\varphi(M_1(x_1,x_2)+Ld(x_2,Tx_1)$

Let $x_3\in Tx_2$ such that, $d(x_2,x_3)\le H(Tx_1,Tx_2)+\delta\le\varphi(M_1(x_1,x_2)$.

If $M_1(x_2,x_3)=0$, then $Fix(T)\ne\emptyset$

If $M_1(x_2,x_3)>0$. Then, we will show that

$$d(x_2,x_3)\le\varphi(d(x_1,x_2)\le\varphi^2d(x_0,x_1), \qquad (3.2)$$

Now if $M_1(x_1,x_2)=d(x_1,x_2)$, then (3.2) is true.

If $M_1(x_1,x_2)=d(x_1,Tx_1)$, then we have

$d(x_2,x_3)\le\varphi d(x_1,Tx_1)+Ld(x_3,Tx_2)\le\varphi d(x_1,x_2)$, thus (3.2) is true.

If $M_1(x_1,x_2)=d(x_2,Tx_2))$, then

$d(x_2,x_3)\le\varphi(d(x_2,Tx_2)<d(x_2,Tx_2)\le d(x_2,x_3)$, which is a contraction.

If $M_1(x_1,x_2)=\frac{1}{2b}d(x_1,Tx_2)$, then, we have

$$d(x_2,x_3)\le\varphi\left(\frac{1}{2b}d(x_1,Tx_2)\right)<\frac{1}{2}d(x_1,Tx_2)\le\frac{1}{2b}d(x_1,x_3)\le\frac{1}{2}\left[d(x_1,x_2)+d(x_2,x_3)\right],$$

Thus $d(x_2,x_3)<d(x_1,x_2)$, which is a contraction. Thus (3.2) is true.

Hence in all cases (3.2) is true.

By an inductively procedure we will obtain $x_{n+1}\in Tx_n$, for $n=3,4,\ldots$ with

$$d(x_{n+1},x_n)\le\varphi M_1(x_{n-1},x_n),$$

The argument above guarantees that

$$d(x_n,x_{n+1})\le\varphi(d(x_{n-1},x_n)\le\ldots\le\varphi^n(d(x_0,x_1).$$

Next we will prove that $\{x_n\}$ is a Cauchy sequence

$$d(x_n,x_{n+p})\le bd(x_n,x_{n+1})+b^2d(x_{n+1},x_{n+2})+\ldots+b^pd(x_{n+p-1},x_{n+p}),$$

$$d(x_n,x_{n+p})\le b\varphi^n d(x_0,x_1)+b^2\varphi^{n+1}d(x_0,x_1)+\ldots+b^p\varphi^{n+p-1}d(x_0,x_1),$$

Which can also be written as

$$d(x_n,x_{n+p})\le\frac{1}{b^{n-1}}\left[b^n\varphi^n d(x_0,x_1)+b^{n+1}\varphi^{n+1}d(x_0,x_1)+b^{n+p-1}\varphi^{n+p-1}d(x_0,x_1)\right]$$

$$\le\frac{1}{b^{n-1}}\sum_{J=n}^{\infty}b^j\varphi^j d(x_0,x_1)\to 0, as\ n\to\infty.$$

Thus $\{x_n\}$ is Cauchy sequence in the complete b-metric space (X,d). So there is $x^*\in X$ such that $x^*=\lim_{n\to\infty}x_n$.

In the following we prove that x^* is a fixed point of Ti.e. $x^* \in Tx^*$.

$$d(x^*,Tx^*) \le b[d(x^*,x_n) + d(x_n,Tx^*)] \le b[d(x^*,x_n) + H(Tx_{n-1},Tx^*)] \le b[d(x^*,x_n) + \varphi(M_1(x_{n-1},x^*))] + Ld(x^*,Tx_{n-1}) = bd(x^*,x_n) + b(\varphi M_1(x^*,x_{n-1})).$$

For $n \to \infty$,

$$d(x^*,Tx^*) \le b\varphi(\max\{0,0,d(x^*,Tx^*),\frac{1}{2b}(d(x^*,Tx^*)\} \le b\varphi d(x^*,Tx^*) < d(x^*,Tx^*),$$

Which implies that $d(x^*,Tx^*)=0,$ so that $x \in \bar{T}x = Tx$, ($\bar{T}$ is the closure of T).

Remark (3.2):

(1) If $L=0$ in condition then we obtain theorem (3.1) of Boriceanu(2009).

(2) If $\varphi(z)=\alpha z$ in condition (4) then we obtain corollary(3.1) of Singh et al.(2008) with $Y = X$ and $f = I$.

(3) Theorem (3.1) generalize the main results ofPacurar (2010),Pacurar (2013)and Singh *et al.(*2012).

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حول مبرهنات النقطة الصامدة للدوال المتعددة القيم في الفضاء المتري b

امل محمدهاشم البطاط دعاء لفتة باقر

قسم الرياضيات- كلية العلوم –جامعة البصرة

الخلاصة

في هذا البحث برهن على وجود النقاط الصامده باستخدام شرط Ciricوشرط Berindeمعا في الفضاء المتريb . ان شرطCiricالمسمى بالانكماش القوي تقريبا هو من اكثر الشروط تعميما والذي تكون فية النقاط الصامدة ليست وحيدة . في هذاالبحث توحيد وتوسيع بعض مبرهنات النقطة الصامدة للدوال متعددة القيم.