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Comparison of Some Sequences of the Different Proofs for the First Weierstrass Approximation Theorem

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Abstract
In this papers, we compare tends speed of some sequences of linear positive operators, introduced in the different proofs of Weierstrass approximation theorem, and apply these sequences to approximate a test function that we choose. Afterword's, we explain tends speed of the approximation by figures and the CPU time by tables. It turns out that the best tends speed and CPU time is occur by using the Bernstein polynomials, and introduced three modifications for the sequences (Weierstrass, Landau, Jackson). In addition, we compared tends speed of these three modifications with their original sequences. We found that, all of these modifications are better than their original sequences and the Bernstein polynomials.

Keywords
Weierstrass approximation theorem, Sequence of linear positive operators, Weierstrass polynomials, Landau polynomials, Bernstein polynomials, Jackson polynomials.
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1. Introduction

Weierstrass 1885, introduced his fundamental approximation theorem stated as follows:
If f is a continuous real-valued function on $[a, b]$ and if any $\epsilon > 0$ is given, then there exists a sequence of polynomials $\{p_n\}$ on $[a, b]$ such that $|f(x) - p_n(x)| < \epsilon$ for all $x \in [a, b]$ as n sufficiently large. In other words, any continuous function on a closed and bounded interval can be uniformly approximated on that interval by polynomials [13]. The sequence of Weierstrass has slower tends of speed in applications.

There are many proofs of this theorem which depend on the definition of sequence which is used in the proof of the theorem. Some of these proofs include the technique of the proof given by Weierstrass itself. For more reprints in this field, we refer to [2, 5 and 8].
Landau 1908, introduced a simpler proof of Weierstrass theorem by using another sequence of polynomials [7]. In addition, tends of speed that occur by the application of this sequence is very slow. Bernstein 1912, introduced the simplest proof of Weierstrass approximation theorem by giving a new sequence of polynomials called Bernstein polynomials [1]. This sequence depends on the binomial expansion of the term $\left(x + (1 - x)\right)^n = 1$ and $x \in [0,1]$. Then, the Bernstein polynomials need little mathematical calculations in comparison with the sequences of Weierstrass and Landau. Hence, it has fast tends speed in applications.
Jackson 1934, introduced a proof of Weierstrass approximation theorem by using a new sequence of polynomials called Jackson polynomials depending on the integral of the polynomials [3]. In addition, tends speed which occurs by the application of this sequence is very slow. Stone 1937 has generalized Weierstrass approximation theorem to compact subset of the real numbers, then it has been called "Stone-Weierstrass Approximation Theorem" [10]. For some researches on this theorem we refer to [9, 11].
Kumar and Pathan 2016, introduced a generalization of Weierstrass approximation theorem for a general class of polynomials [6].

In our work, we apply these sequences, Weierstrass, Landau, Bernstein and Jackson by taking a test function $f(t) = \sin(10t)\exp(-3t) + 0.3$. And, we compare tends speed and CPU time of these sequences. Tends speed of these sequences are explained by figures (4.1-

4.3) and the CPU time are given by tables (4.1-4.3). We found that any convergence sequence of linear positive operators that has polynomial as a limit point forms a proof to Weierstrass approximation theorem. In addition, we introduce modifications for the sequences Weierstrass, Landau, Bernstein and Jackson, respectively and compare tends speed and CPU time of these modifications. Also, we explain tends speed of these sequences by figures (4.4-4.6) and. CPU time by table (4.4).

Definition 1.1. (Weierstrass Polynomials) [13]

For $f(x) \in C[a,b]$, the n-th order Weierstrass polynomial to the function f is define as:

$$W_n(f;x) = \frac{1}{I_n}\int_{a1}^{b1} f(t)\, e^{-n(t-x)^2} dt$$

where

$$I_n = \int_{-c}^{c} e^{-n\,t^2}\, dt$$

$a1 = a - \delta, b1 = b + \delta, \delta > 0$, and $c = b1 - a1$.

In [13], shows that the Weierstrass sequence has the following property:

$$\lim_{n\to\infty} W_n(f;x) = f(x),$$

this approximation is uniform in the interval $[a,b]$.

Definition 1.2 (Landau polynomials)

For $f(x) \in C[a,b]$, the n-th order Landau polynomial to the function f is defined as:

$$L_n(f;x) = \frac{1}{d_n}\int_{a1}^{b1} f(t)\left(\frac{c^2-(t-x)^2}{c^2}\right)^n dt$$

where

$$d_n = \int_{-c}^{c}\left(\frac{c^2-t^2}{c^2}\right)^n dt$$

$a1 = a - \delta, b1 = b + \delta, \delta > 0$, and $c = b1 - a1$.

The Landau sequence has the following property:[7]

$$\lim_{n\to\infty} L_n(f;x) = f(x),$$

this approximation is uniformly in the interval $[a,b]$.

Definition 1.3 (Bernstein polynomials)

For $f(x) \in [0,1]$, the n-th order Bernstein polynomial to the function f is defined as:

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k},$$

where $x \in [0,1]$.

We have the following fact: [1]

$$\lim_{n\to\infty} B_n(f;x) = f(x),$$

this approximation is uniformly in the interval [0,1].

Definition 1.4 (Jackson polynomials)

Suppose that $f(x)$ be a continuous and bounded function in the interval $[a,b]$. The n-th order of Jackson polynomial for $f(x)$ is defined as:

$$P_n(f;x) = \frac{1}{J_n}\int_{-1}^{1} f(x+u)\,(1-u^2)^n du$$

where

$$J_n = \int_{-1}^{1} (1-u^2)^n du$$

and

$$f(x) = \begin{cases} 1 & ; x \le 0 \\ \frac{x}{a} f(a) & ; 0 < x < a \\ \frac{1-x}{1-b} f(b) & ; b < x < 1 \\ 0 & ; x \ge 1 \end{cases}.$$

The Jackson sequence has the following property:[3]

$$\lim_{n\to\infty} P_n(f;x) = f(x),$$

Lemma 1.1 [4]

Let the function $Q(x)$ satisfies the conditions

1. $Q(x)$ is continuous the interval $-c \le x \le c, c > 0$;
2. $Q(0) = 1, 0 \le Q(x) < 1$, if $x \ne 0, -c \le x \le c$, and if we put $Z_n = \int_{-c}^{c} Q^n(x)dx$, $Z_n(\delta) = \int_{-\delta}^{\delta} Q^n(x)dx$, $0 < \delta \le c$, then $\lim_{n\to\infty} \frac{Z_n(\delta)}{Z_n} = 1$.

Theorem 1.1 [4]

If a function $Q(x)$ satisfies the conditions of Lemma above and $Z_n = \int_{-c}^{c} Q^n(x)dx$ then the sequence of operators

$$L_n(f;x) = \frac{1}{Z_n}\int_a^{b`} f(t)\, Q^n(t-x)dt,\ 0 < b-a \le c$$

converges uniformly to the function $f(x)$ in $[a+\delta, b-\delta], \delta > 0$, if $f(x)$ is continuous in $[a,b]$.

2. **Our Modifications**

We introduce three modifications for sequences Weierstrass, Landau and Jackson denote them by $\widetilde{W}_n(f;x)$, $\widetilde{L}_n(f;x)$ and $\widetilde{P}_n(f;x)$, respectively. In addition, we apply these modifications by using the same test function. The modifications of Weierstrass, Landau and Jackson sequences are defined as follows:

2.1 Modification of Weierstrass polynomials

We define the modification of Weierstrass polynomials as:

$$\widetilde{W}_n(f;x) = \frac{1}{\bar{I}_n}\int_{a1}^{b1} f(t).\, e^{-n^4 (t-x)^4}\, dt$$

where

$$\bar{I}_n = \int_{-c}^{c} e^{-n^4 (t)^4}\, dt$$

$a1 = a-\delta, b1 = b+\delta, \delta > 0$ and $c = b1 - a1$.

2.2 Modification of Landau polynomials

We define the modification of Landau polynomials as:

$$\widetilde{L}_n(f;x) = \frac{1}{\bar{d}_n}\int_{a1}^{b1} f(t)\left[1-\left(\frac{t-x}{c}\right)^{\frac{1}{r}}\right]^n dt$$

where

$$\bar{d}_n = \int_{-c}^{c}\left[1-\left(\frac{t}{c}\right)^{\frac{1}{r}}\right]^n dt,$$

$a1 = a-\delta, b1 = b+\delta, \delta > 0, c = b1 - a1$ and $r = 1,2,\ldots$.

4.3 Modification of Jackson polynomials

$$\widetilde{P}_n(f;x) = \frac{1}{\widetilde{J}_n}\int_{-1}^{1} f(x+u)\left(1-(|u|)^{\frac{1}{r}}\right)^n du,$$

where

$$\widetilde{J}_n = \int_{-1}^{1}\left(1-(|u|)^{\frac{1}{r}}\right)^n du$$

and $a \le x \le b$

$$f(x) = \begin{cases} 0 & ; x \le 0 \\ \frac{x}{a} f(a) & ; 0 < x < a \\ \frac{1-x}{1-b} f(b) & ; b < x < 1 \\ 0 & ; x \ge 1 \end{cases}.$$

3. The convergence of Modifications polynomials

The convergence of these modifications polynomials is explained below.

3.1 The Convergence of Weierstrass Modification.

$f(x)$ be continuous function on the interval $[a, b]$, the $Q^n(t-x) = e^{-n^4(t-x)^4}$ and $Q(x) = e^{-x^4}$ be continuous in every interval.

So the first condition of Lemma 1.1 holds.

now $Q(0) = 1$, $0 < Q(x) < 1, x \neq 0$, from this the second condition of Lemma 1.1 holds.

$Q(x)$ Satisfies the condition of Theorem 1.1, the sequence of polynomials $\widetilde{W}_n(f;x)$ converges uniformly with the function $f(x)$. So, the modification of Weierstrass sequence has the following property:

$$\lim_{n\to\infty}\widetilde{W}_n(f;x) = f(x),$$

this approximation is uniform in the interval $[a, b]$.

3.2 The Convergence of Landau modification

$f(x)$ be continuous function on the interval $[a, b]$, and $Q(t-x) = 1-\left(\frac{t-x}{c}\right)^{\frac{1}{r}}$ and $Q(x) = 1-\left(\left(\frac{x}{c}\right)\right)^{\frac{1}{r}}$ and this polynomials be continuous in every interval. So, the first condition of Lemma 1.1 holds.

now $Q(0) = 1$, $0 < Q(x) < 1, x \neq 0$, from this the second condition of Lemma 1.1 hold.

$Q(x)$ Satisfies the condition of Theorem 1.1, the sequence of polynomials $\widetilde{L}_n(f;x)$ converge uniformly to the function $f(x)$. So, the modification of Landau sequence has the following property:

$$\lim_{n\to\infty}\widetilde{L}_n(f;x) = f(x),$$

this approximation is uniform in the interval $[a, b]$.

3.3 The Convergence of Jackson modification

$f(x)$ be continuous function on the interval $[a, b]$, $Q(t-x) = 1-(|u|)^{\frac{1}{r}}$ and $Q(x) = 1-\left((|-x|)\right)^{\frac{1}{r}}$ and this polynomials be continuous in every interval where $u = t-x$. So, the first condition of Lemma 1.1 holds.

now $Q(0) = 1$, $0 < Q(x) < 1, x \neq 0$, from this the second condition of Lemma 1.1 holds.

Q(x) Satisfies the condition of Theorem 1.1, the sequence of polynomials $\widetilde{P}_n(f;x)$ converges uniformly with the function f(x). So, the modification of Jackson sequence has the following property:

$$\lim_{n\to\infty}\widetilde{P}_n(f;x) = f(x),$$

this approximation is uniform in the interval $[a,b]$.

4. Application

We apply the sequences $W_n(f;x)$, $L_n(f;x)$, $B_n(f;x)$, $P_n(f;x)$ and $\widetilde{W}_n(f;x)$, $\widetilde{L}_n(f;x)$, $\widetilde{P}_n(f;x)$ by taking the test function $f(t) = \sin(10t)\exp(-3t) + 0.3$. We compare tends speed of the approximation of these polynomials by some graphics, we explain the differences in tends speed of these sequences in the figures (4.1-4.6) and the differences in CPU time in the tables (4.1-4.4). It turns out that the best sequence in application is the Bernstein polynomials; this occurs because this sequence has no integral and its summation is finite. So, it needs few mathematical calculations in the application. We compare tends speed and CPU time of each modification with its original sequence and with the others sequences (Weierstrass, Landau, Bernstein and Jackson). We found that these three modifications have tends speed better than the four original sequences.

The figure (4.1), explains tends speed of all original sequences $W_n(f;x)$, $L_n(f;x)$, $B_n(f;x)$ and $P_n(f;x)$, we approximate the test function by applying them. The best tends speed and CPU time occurs by Bernstein polynomials to approximating the test function for $n = 10$.

The figure (4.2), explains tends speed of $W_n(f;x)$, $L_n(f;x)$, $B_n(f;x)$ and $P_n(f;x)$ sequences for $n = 40$, and explains if n increases tends speed of Jackson polynomials will fail in application and remain best tends speed and CPU time occurs by Bernstein polynomials and also the figure (4.3) explains tends speed of these sequences for $n = 60$.

The figure (4.4), explains tends speed of the modification of Weierstrass polynomials better than all of the original polynomials to approximate the same test function.

The figure (4.5), explains tends speed of the modification of Landau polynomials better than all of the original polynomials to approximate the test function.

The figure (4.6), shows tends speed of the modification of Jackson polynomials better than all of the original polynomials to approximate the test function.

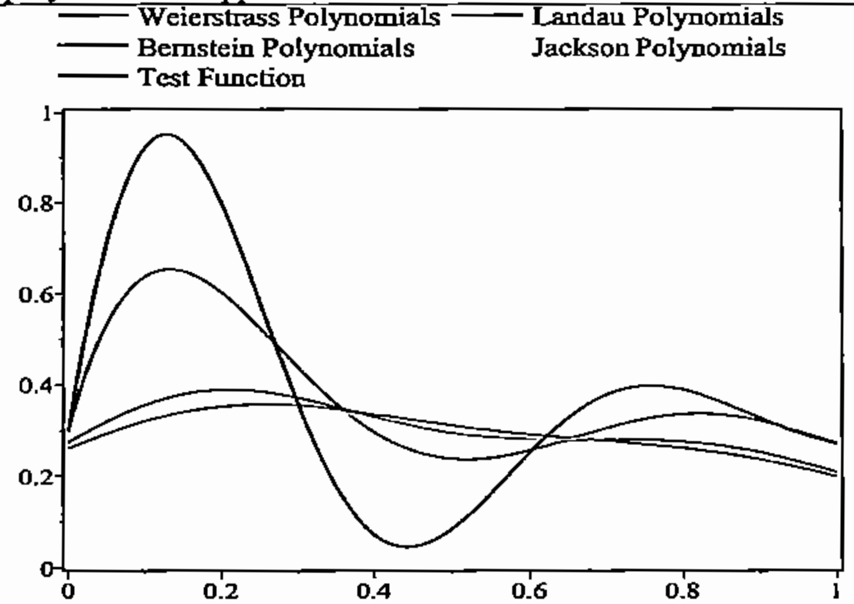

Figure 4.1: Approximation test function f by $W_n(f;x)$, $L_n(f;x)$, $B_n(f;x)$ and $P_n(f;x)$ for $n = 10$

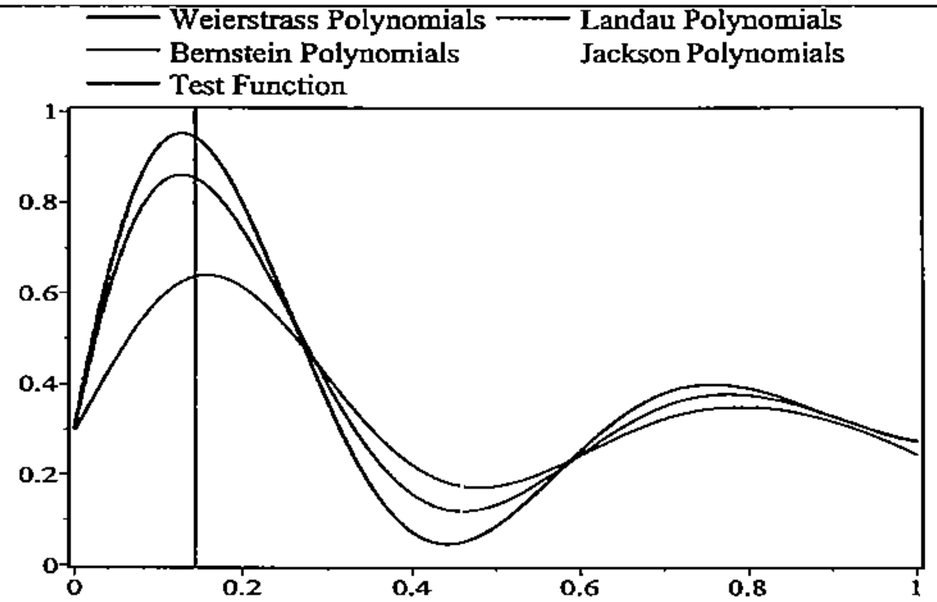

Figure 4.2: Approximation test function f
by $W_n(f;x)$, $L_n(f;x)$, $B_n(f;x)$ and $P_n(f;x)$ for n=40

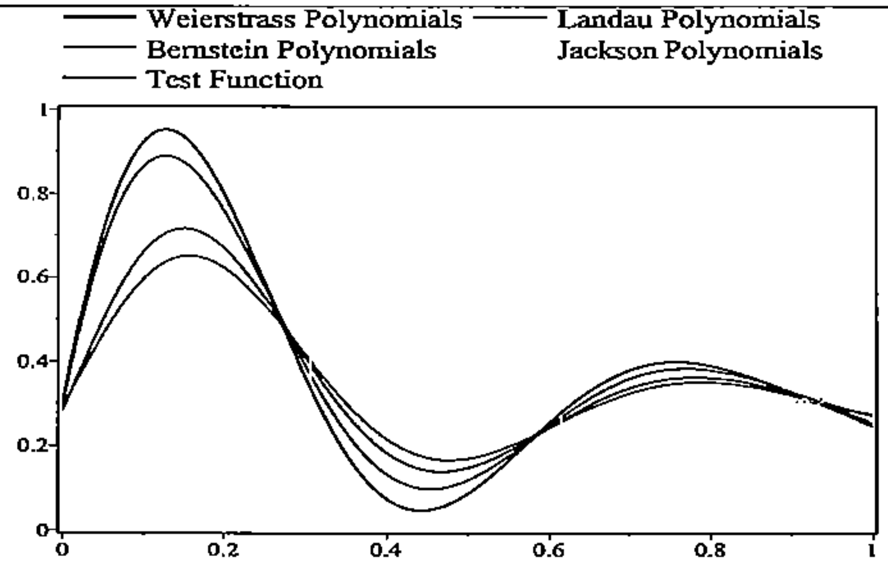

Figure 4.3: Approximation test function f
by $W_n(f;x)$, $L_n(f;x)$, $B_n(f;x)$ and $P_n(f;x)$ for $n = 60$

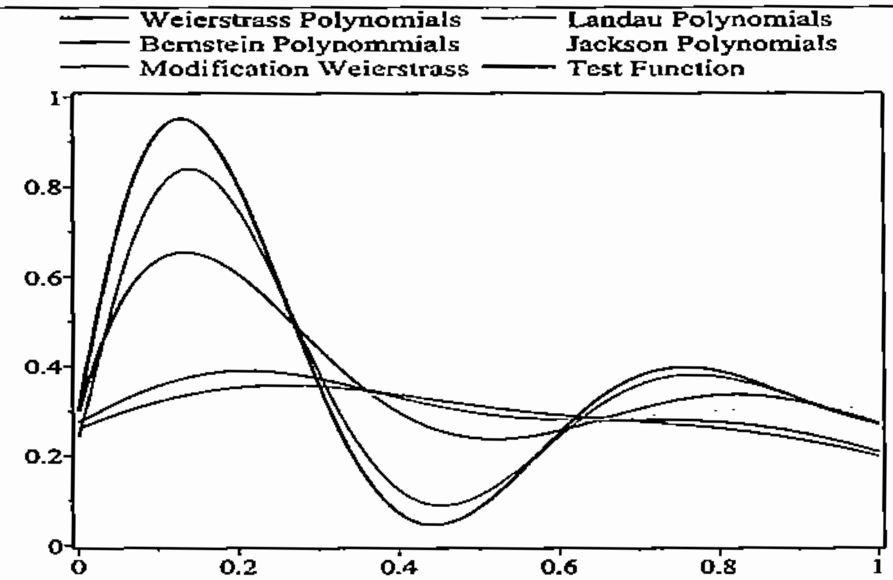

Figure 4.4: Approximation f
by the modification of Weierstrass and all original polynomials

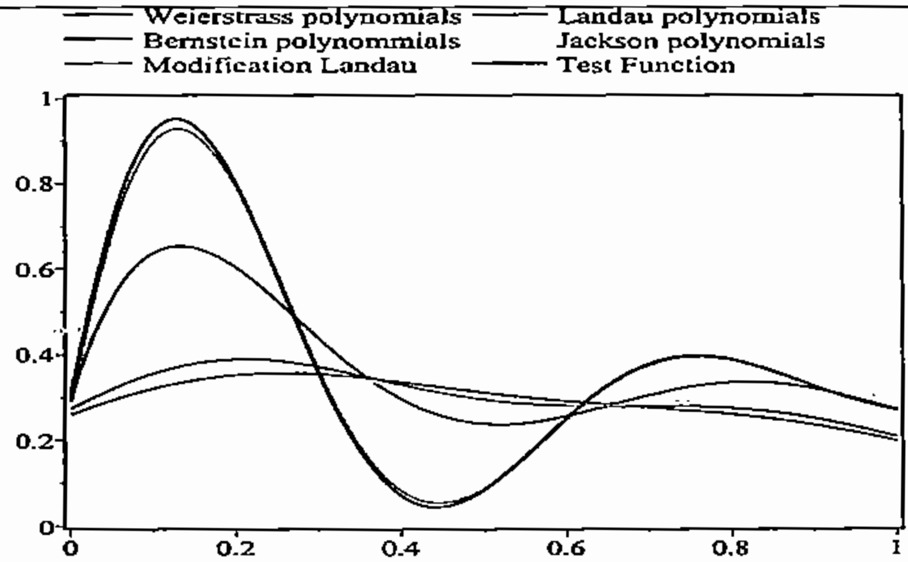

Figure 4.5: Approximation f
by the modification of Landau and all original polynomials

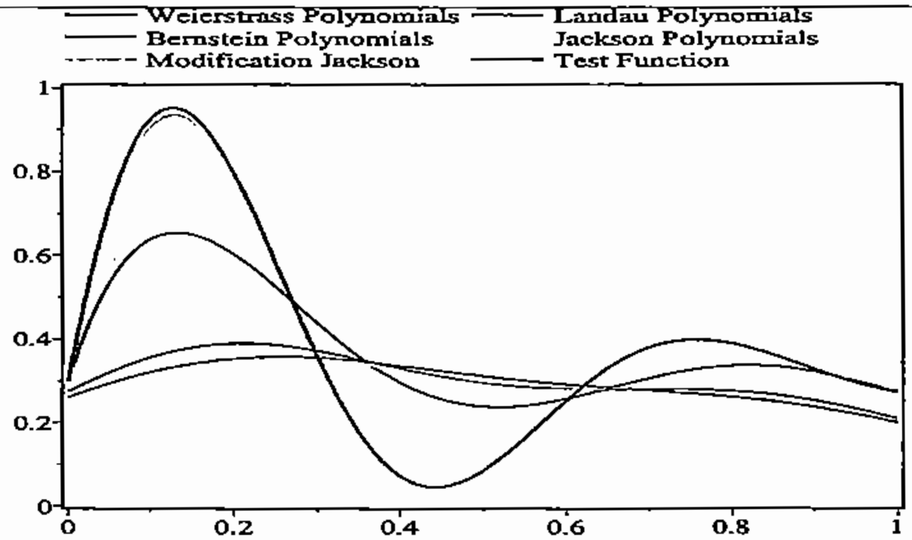

Figure 4.6: Approximation f
by the modification of Jackson and all original polynomials

4.1 The CPU Time

We introduced three tables which explain the CPU time for these sequences $W_n(f;x)$, $L_n(f;x)$, $B_n(f;x)$ and $P_n(f;x)$ as (n =10, 40, 60) respectively. We found the best CPU time introduced by Bernstein polynomials by using the same test function and the same Computer.

Table (4.1): Explains the CPU time for n=10.

The sequence	CPU Time
$W_n(f;x)$	2.15 s
$L_n(f;x)$	7.8 s
$B_n(f;x)$	0.09 s
$P_n(f;x)$	17.78 s

Table (4.2): Explains the CPU time for n = 40.

The sequence	CPU time
$W_n(f;x)$	2.29 s
$L_n(f;x)$	780.02 s
$B_n(f;x)$	0.06 s
$P_n(f;x)$	427.45 s

Table (4.3): Explains the CPU time for n=60

The sequence	CPU Time
$W_n(f;x)$	2.90 s
$L_n(f;x)$	1582.70 s
$B_n(f;x)$	0.62 s
$P_n(f;x)$	1130.41 s

4.2 The CPU Time

We introduce the following table which explains the CPU time for all of the modifications of polynomials for n=10 and best CPU time introduced by Landau polynomials

Table (4.4): Explains the CPU time for n=10

The Modification sequence	CPU Time
$\widetilde{W}_n(f;x)$	251.41 s
$\widetilde{L}_n(f;x)$	10.99 s
$\widetilde{P}_n(f;x)$	110.88 s

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Numerical Solution of Multi- Fractional Order Optimal Control Problems

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Abstract:
This paper aims to apply the Laplace transform for solving Multi-fractional order optimal control problems (FOCPs) with a quadratic performance index. The Laplace transform is a powerful tool in applied mathematics and engineering. It will allow us to transform fractional differential equations into algebraic equations and then by solving this algebraic equations, we can obtain the unknown function by using the Inverse Laplace Transform. The control function **u(t)** is relies on the approximation of the necessary optimality conditions in terms of the associated Hamiltonian. An illustrative example demonstrates the simplicity and efficiency of proposed method.
Keywords: Fractional derivative, Fractional optimal control problem, Caputo Fractional derivative, Laplace transform and Inverse Laplace Transform, pproximation solution.

1-Introduction

Fractional optimal control problems (FOCPs) are optimal control problems associated with fractional dynamic systems. The fractional optimal control theory is a very new topic in mathematics. FOCPs may be defined in terms of different types of fractional derivatives. But the most important types of fractional derivatives are the Riemann-Liouville and the Caputo fractional derivatives. In Agrawal [3], Agrawal and Baleanu [4] the authors obtained necessary conditions for FOCPs with the Riemann-Liouville derivative and were able to solve the problem numerically. Agrawal [1] presented a quadratic numerical scheme for a class of fractional optimal control problems (FOCPs). In Agrawal [5], the FOCPs are formulated for a class of distributed systems where the fractional derivative is defined in the Caputo sense,and a numerical technique for FOCPs presented. Baleanu et al. [7] used a direct numerical scheme to find a solution of the FOCPs. In Biswas and Sen [10], FOCPs with fixed final time are considered and a transversely condition is obtained. scheme for FOCPs based on integer order optimal controls problem. In Youse et al. [8]the usage of Legendre multi wavelet basis and collocation method was proposed for obtaining the approximate solution of FOCPs. Tricaud and Chen[9] proposed a rational approximation based on the Hankel data matrix of the impulse response to obtain a solution for the general time-optimal problem

In this paper we solving a multi-fractional order optimal control problems by Approximate Laplace transform.

This paper is organized as follows: In **section 2**, we present some basic definitions of fractional calculus .In **section 3**, contains the necessary optimality conditions of the FOCP model. **section 4**,present the Laplace transform and inverse Laplace transform for some functions .In **section 5**,the proposed method is applied to several examples. Also a conclusion is given in the last section.

2- Basic definitions of fractional calculus.

2-1 Fractional Derivatives and Integrals.

Definition 2.1. Let x : [a, b] →R be a function,$\alpha > 0$ a real number, and n=[α],where [α] denotes the smallest integer greater than or equal to α. The left (left RLFI) and right (right RLFI) Riemann−Liouville fractional integrals are defined by:

$$a\, I_t^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\mathrm{T})^{\alpha-1} x(\mathrm{T})d\mathrm{T} \quad (left\ RLFI\ ), \qquad ...(1)$$