



ISSN: 0067-2904
GIF: 0.851

On Truncated of General Family of Baskakov –Type Operators

Ali J. Mohammad^{1*}, A.K. Hassan²

¹Mathematics Department, College of Education for Pure Sciences, University of Basrah, Basrah, Iraq

²Department of Mathematics, College of Science, University of Basrah, Basrah, Iraq

Abstract

Recently, in 2014 [1] the authors introduced a general family of summation integral Baskakov-type operators $B_{n,k,r}(f;x)$. In this paper, we investigate approximation properties of partial sums for this general family.

Keywords: Baskakov operator, Beta operator, truncated operator, order of approximation, modulus of continuity.

حول قطع تعميم عائلة من نوع مؤثر باسكاكوف

علي جاسم محمد^{1*}، امل خليل حسن²

¹ قسم الرياضيات، كلية التربية للعلوم الصرفة، جامعة البصرة، بصرة، العراق

² قسم الرياضيات، كلية العلوم، جامعة البصرة، بصرة، العراق

الخلاصة

مؤخراً في عام 2014 قام الباحثون في [1] بدراسة تعميم لعائلة المجموع -تكاملي من نوع مؤثر

باسكاكوف $B_{n,k,r}(f;x)$. في بحثنا هذا ناقشنا خصائص التقريب للمجموع الجزئي لتلك العائلة.

1.Introduction

The study of truncated for a sequence of linear positive operators were began by some publications as [2-8]. In [7, 2012], Lehnoff discussed and study the Truncated of Baskakov operators which is defined as:

$$L_{n,N}(f;x) = \sum_{k=0}^N p_{n,k}(x) f\left(\frac{k}{n}\right)$$

$$\text{Where } p_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}, \quad (1)$$

for $f \in C_\alpha[0,\infty) = \{f \in C[0,\infty): f(t) = O((1+t)^\alpha) \text{ for some } \alpha > 0\}$, $n \in \mathbb{N} = \{1,2, \dots\}$, where $N = N(n,x)$ are positive integers. The space $C_\alpha[0,\infty)$ is normed by the norm

$$\|f(t)\|_{C_\alpha} = \sup_{t \in [0,\infty)} \omega_\alpha(t) |f(t)|$$

where $\omega_\alpha(x) = (1+x^\alpha)^{-1}$, if $\alpha \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$, $\omega_0(x) = 1$, for $x \in \mathbb{R}^0 = [0,\infty)$ such that $\omega_\alpha f$ is uniformly continuous and bounded on $\mathbb{R}^0$ for $f \in C_\alpha[0,\infty)$.

The theorem presented in [7] shows that:

If $\alpha \in \mathbb{N}^0$, $n \in \mathbb{N}$ and $x > 0$ be fixed and $N = N(n,x)$ be an integer such that $N > nx$. then the inequality:

$$\omega_\alpha(x) |L_{n,N}(f(t); x) - f(x)| \leq K_1(\alpha) \omega\left(f; C_\alpha; \sqrt{\frac{(x^2+x)}{n}}\right) + \|f\|_\alpha \frac{(x+1)^{\frac{3}{2}}}{\sqrt{2\pi nx}} \frac{N}{N-nx}$$

*Email: alijasmoh@gmail.com

holds for every $f \in C_\alpha[0, \infty)$ where $K_1(\alpha) = \text{constant} > 0$ and $\omega(f; C_\alpha)$ is the modulus of continuity of f ,

i.e. $\omega(f; C_\alpha; t) = \sup_{0 \leq h \leq t} \|\Delta_h f(\cdot)\|_\alpha$ for $t \in \mathbb{R}^0$ and $\Delta_h f(x) = f(x+h) - f(x)$ and $\lim_{n \rightarrow \infty} \frac{N-nx}{\sqrt{n}} = \infty$ then the convergence,

$$\lim_{n \rightarrow \infty} L_{n,N}(f; x_0) = f(x_0) \quad \text{Holds for every } x_0 \in (0, \infty).$$

In [1], a general form of classical Baskakov operators was introduced as follows:

$$B_{n,k,r}(f; x) = \frac{1}{(n)_r} \sum_{k=0}^{\infty} \beta_{n,k,r}(x) f\left(\frac{k}{n}\right) \quad (2)$$

$$\text{where} \quad \beta_{n,k,r}(x) = \frac{(n+k)_r}{(1+x)^r} p_{n,x}(x) \quad (3)$$

and

$$(x)_k = \begin{cases} x(x+1)(x+2) \dots (x+k-1), & k = 1, 2, 3, \dots \\ 1 & k = 0 \end{cases}$$

for $x \in \mathbb{R}^0, r \in \mathbb{N}^0$ and $n \in \mathbb{N}$.

From [9], we can find that for $f \in C_\alpha$ drive the estimation

$$\omega_\alpha(x) B_{n,k,r}(|f(t) - f(x)|; x) \leq M_1(\alpha) \omega\left(f; C_\alpha; \sqrt{\frac{x(1+x)}{n} + \frac{rx+2rx^2}{n^2}}\right) \quad (4)$$

where $M_1(\alpha) = \text{constant} > 0$.

The purpose of this paper is to derive correspond results for the truncated of family general Baskakov operators:

$$B_{n,N,r}(f(t); x) = \frac{1}{(n)_r} \sum_{k=0}^N \beta_{n,k,r}(x) f\left(\frac{k}{n}\right) \quad (5)$$

for $f \in C_\alpha$ and $n, N \in \mathbb{N}, N = N(n, x)$

2. The properties of $B_{n,N,r}(f)$.

From (3), (4) and (5) we get

$$\sum_{k=0}^{\infty} \beta_{n,k,r}(x) = 1 \quad x \in \mathbb{R}^0, n \in \mathbb{N} \quad (6)$$

$$\text{and } B_{n,N,r}(f(t); 0) = f(0) = B_{n,k,r}(f(t); 0), \text{ where } f \in C_\alpha \text{ and } n, N \in \mathbb{N} \quad (7)$$

Theorem.

Let $\alpha \in \mathbb{N}^0, n \in \mathbb{N}$ and $x > 0$ be fixed and let $N = N(n, x)$ be an integers such that: $N > (n+r)x$ and $r \in \mathbb{N}^0$. Then

$$\begin{aligned} \omega_\alpha |B_{n,N,r}(f(t); x) - f(x)| & \leq M_2(\alpha) \omega\left(f; C_\alpha; \sqrt{\frac{x(1+x)}{n} + \frac{rx+2rx^2}{n^2}}\right) \\ & + \|f\|_\alpha \frac{(n+r)!}{(n-1)!(N+n+r)} \frac{N(1+x)^{\frac{3}{2}}}{\sqrt{2\pi(n+r)x} (N-(n+r)x)} \end{aligned} \quad (8)$$

Proof.

From (3), (5) and (6) we can find:

$$B_{n,N,r}((f(x); x) - f(x)) = B_{n,N,r}(f(t) - f(x); x) - f(x) \sum_{k=1}^{\infty} \beta_{n,k+N,r}(x) \quad (9)$$

Then by (4), for $f \in C_\alpha$

$$\begin{aligned} \omega_\alpha(x) |B_{n,N,r}(f(t)) - f(x); x| & \leq \omega_\alpha(x) B_{n,N,r}(|f(t) - f(x)|; x) \\ & \leq \omega_\alpha(x) B_{n,N,r}(|f(t) - f(x)|; x) \leq M_1(\alpha) \omega\left(f; C_\alpha; \sqrt{\frac{x(1+x)}{n} + \frac{rx+2rx^2}{n^2}}\right) \end{aligned} \quad (10)$$

Since,

$$\sum_{k=1}^{\infty} \beta_{n,N+k,r}(x) = \sum_{k=1}^{\infty} \frac{(n+k+N)_r}{(1+x)^r} \binom{n+k+N-1}{k+N} x^{k+N} (1+x)^{-n-k-N}$$

$$\begin{aligned}
 &= \sum_{k=1}^{\infty} \frac{(n+k+N)(n+k+N+1) \dots (n+k+N+r-1)}{(1+x)^r} \\
 &\quad \times \frac{(n+k+N-1)!}{(k+N)!(n-1)!} x^{k+n} (1+x)^{-n-k-N} \\
 &= \beta_{n,N,r}(x) \sum_{k=1}^{\infty} \frac{(n+r+N)(n+r+N+1) \dots (n+r+N+k-1)}{(N+1)(N+2) \dots (N+k)} \left(\frac{x}{1+x}\right)^k \\
 &< \beta_{n,N,r}(x) \sum_{k=1}^{\infty} \left[\left(1 + \frac{n+r-1}{N+1}\right) \left(\frac{x}{1+x}\right) \right]^k \\
 &< \beta_{n,N,r}(x) \sum_{k=1}^{\infty} \left[\left(1 + \frac{n+r}{N}\right) \left(\frac{x}{1+x}\right) \right]^k = \beta_{n,N,r}(x) \frac{(N+n+r)x}{N-(n+r)x}
 \end{aligned}$$

If $N > (n+r)x, x > 0$

$$\begin{aligned}
 \beta_{n,N,r}(x) &< \beta_{n,N,r}(N/(n+r)) = \frac{(n+N)_r}{\left(1 + \frac{N}{n+r}\right)^r} \binom{n+N-1}{N} \left(\frac{N}{n+r}\right)^N \left(1 + \frac{N}{n+r}\right)^{-n-N} \\
 &= \frac{(n+N)_r}{\left(1 + \frac{N}{n+r}\right)^r} \frac{(n+N-1)!}{N!(n-1)!} \left(\frac{N}{n+r}\right)^N \left(1 + \frac{N}{n+r}\right)^{-n-N} \\
 &= \frac{(n+r)!}{(n-1)!(N+n+r)} \frac{(n+r)^{n+r} e^{-(n+r)}}{(n+r)!} \frac{N^N e^{-N}}{N!} \left[\frac{(N+n+r)^{N+n+r} e^{-(N+n+r)}}{(N+n+r)!} \right]^{-1} \\
 &< \frac{(n+r)!}{(n-1)!(N+n+r)} \frac{\sqrt{(N+n+r)}}{\sqrt{2\pi N(n+r)}}, \text{ (by the stirling formula)}
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \omega_{\alpha}(x) \left| f(x) \sum_{k=N+1}^{\infty} \beta_{n,k,r}(x) \right| &\leq \|f\|_{\alpha} \frac{(n+r)!}{(n-1)!(N+n+r)} \frac{\sqrt{N+n+r}}{\sqrt{2\pi N(n+r)}} \frac{N + (n+r)x}{N - (n+r)x} \\
 &\leq \|f\|_{\alpha} \frac{(n+r)!}{(n-1)!(N+n+r)} \frac{N(1+x)^{\frac{3}{2}}}{\sqrt{2\pi(n+r)x(N-(n+r)x)}}. \quad (11)
 \end{aligned}$$

Then from (10) and (11) we get (8).

From the theorem above, we can derive some corollaries by take $x > 0$ as follows:

Corollary 1. Let $x_0 \in (0, \infty), r \in \mathbb{N}^0$ and $N = N(n, x_0)$ positive integers such that for all $n \in \mathbb{N}$.

(i) $N > (n+r)x_0$

(ii) $\left(\frac{N}{n+r}\right)_{n=1}^{\infty}$ is bounded sequence

(iii) $\lim_{n \rightarrow \infty} \frac{N-(n+r)x}{\sqrt{n}} = \infty$. Then the convergence

$$\lim_{n \rightarrow \infty} B_{n,N,r}(f; x_0) = f(x_0) \quad \text{for every } f \in C_{\alpha}, \alpha \in \mathbb{N}^0 \quad (12)$$

Corollary 2.

For a fixed $x_0 > 0$ let $N = [(n+r)(x_0+1)]$ for $n \in \mathbb{N}, r \in \mathbb{N}^0$. Then the condition (i)-(iii) are satisfied, and hence (11) holds for $f \in C_{\alpha}$. Also, we have:

$$\begin{aligned}
 \omega_{\alpha}(x_0) |B_{n,N,r}(x)(f; x_0) - f(x_0)| &\leq M_1(\alpha, x_0) \left[\omega \left(f; C_{\alpha}; \sqrt{\frac{x_0(1+x_0)}{n} + \frac{rx_0 + 2rx_0^2}{n^2}} \right) \right] \\
 &\quad + \|f\|_{\alpha} \frac{1}{\sqrt{n}}.
 \end{aligned}$$

For every $f \in C_{\alpha}$ and $n \in \mathbb{N}$, where $M(\alpha, x_0)$ be appositve constant.

Corollary 3. Let $x_0 \in (0, \infty)$ and let $N = [(n+r)x_0 + \sqrt{n+r}\delta_n]$ for $n \in \mathbb{N}$, $r \in \mathbb{N}^0$.

Where (α_n) is given non- bounded sequence of numbers $\delta_n \geq 1$ such that $\left(\frac{\delta_n}{\sqrt{n}}\right)$ is bounded. Then (i)-(iii) are satisfied and hence (11) holds for every $f \in C_{\alpha}$, and we have;

$$\omega_{\alpha}(x)|B_{n,N,r}(f; x_0) - f(x_0)| \leq M_2(\alpha, x_0) \left(\omega \left(f; C_{\alpha}; \frac{x_0(1+x_0)}{n} + \frac{rx_0 + 2rx_0^2}{n^2} \right) + \|f\|_{\alpha} \frac{1}{\delta_n} \right)$$

for $n \in \mathbb{N}$, where $M_2(\alpha, x_0)$ be a positive constant.

References

1. Mohammad A.J., Hassan A.K. and Abdul-Hammed S.A. **2014**. A general family of summation Integral –Type operators. *J. of Basrah Reserches (Sciences)*, 40(3) A, pp:65-75.
2. Grof, J. **1980**. Uber approximation durch polynomi mit belegfunktionen, *Acta Math. Aced. Sci. Hungar.*, 35, pp:109-116.
3. Lehnoff, H.G. **1984**. On a modified of Szasz-Mirakyan operator, *J. Approx. Theory*, 42, pp:278-282.
4. Omei E. **1986**. Note on operators of Szasz-Mirakyan type, *J. Approx. Theory*, 47, 246-254.
5. Walczk Z. **2007**. Convergence of Szasz-Mirakyan type operators, *Rev. Anal. Number. Theor. Approx.*, 36 (1), pp:107-113.
6. Xie L. and T. Xie T. **2008**. Approximation theorems for localized Szasz- Mirakjan operators, *J. Approx.Theory*, 152, pp:125-134.
7. Rempulska L. **2012**. On Truncated MKZ and Baskakov Operators, *J. of Math. Analysis*, 6(8), pp:377-384.
8. Mohammad A.J., Abdul Satar H.A. and Sadeg H.J. Ontruncated of classical Beta-type operators. *J. of Basrah Reserches (Sciences)*, 41(3) A, pp:15-23.
9. Rempulska L. and Skorupka M. **2004**, On strong approximation of function by linear operators, *Math. J. Okayama Univ.*, 46, pp:153-161.