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Approximation by Phillips-Szὰsz-Kantorovich Operators

Ali J. Mohammad[1] and Jinan Abd AL Emam Najim[2]

University of Basrah, College of Education for Pure Sciences, Dept. of Maths., Basrah-IRAQ.

[1]Email: alijasmoh@gmail.com, [2]Email: jinan.abd16@gmail.com.

Abstract:

In this paper, we define and study a Philips form of Szὰsz-Kantorovich type operators. We prove that these operators are converge to the function being approximated. Also, we discuss a Voronovaskaja-type asymptotic formula for these operators.

Keywords: Szὰsz-Kantorovich type operators, Phillips operators, Voronovaskaja-type asymptotic formula**.**

1. Introduction

For a function $f \in C[0,\infty)$ and $n \in \mathbb{N} = \{1,2,\dots\}$. The classical Szàsz operators are defined as [1]:

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f(\frac{k}{n}),$$

and the modified of the classical Szàsz operators are defined as [2]:

$$R_n(f(t);x) = n \sum_{k=0}^{\infty} q_{n,k}(x) \int_0^{\infty} q_{n,k}(t) f(t) dt,$$

and the Kantorovich-Szàsz operators are known as [3]:

$$S_n(f(t);x) = n \sum_{k=0}^{\infty} q_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt,$$

where $q_{n,k}(x) = e^{-nx} \frac{(nx)^k}{k!}$, $x \in [0,\infty)$.

In this paper, we introduce a new sequence of linear positive operators $K_n(f(t);x)$ of Phillips Szàsz – Kantorovich operators to approximate a function f belongs to the space

$$C_\gamma[0,\infty) = \{g \in C[0,\infty) : |g(t)| < M(1+t)^\gamma \text{ for some } M > 0, \gamma > 0\},$$

as follows:

$$K_n(f(t);x) = n \sum_{k=1}^{\infty} q_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt + f(0) q_{n,0}(x).$$

Lemma 1.1

For the operator $K_n(f;x)$ and $m \in \mathbb{N}^0 := \{0,1,2,\dots\}$, we have:

(a) $K_n(1;x) = 1$,

(b) $K_n(t;x) = x + \frac{1}{2n}(1 - e^{-nx})$,

(c) $K_n(t^2;x) = x^2 + \frac{2}{n} + \frac{1}{3n^2}(1 - e^{-nx})$,

(d) $K_n(t^m;x) = x^m + \frac{m^2}{2n} x^{m-1} + T.L.P.(x) + \frac{(1-e^{-nx})}{n^m(m+1)}$,

where $T.L.P.(x)$ means Terms in Lower Powers of x.

***Proof*:** By direct computation, we have:

(a) $K_n(1;x) = n \sum_{k=1}^{\infty} q_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} 1\, dt + e^{-nx}$
$= \sum_{k=0}^{\infty} q_{n,k}(x) = 1.$

(b) $K_n(t;x) = n \sum_{k=1}^{\infty} q_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} t\, dt$
$= \frac{n}{2n^2} \sum_{k=1}^{\infty} q_{n,k}(x)(k^2 + 2k + 1 - k^2)$
$= x + \frac{1}{2n}(1 - e^{-nx}).$

(c) $K_n(t^2;x) = n \sum_{k=1}^{\infty} q_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} t^2\, dt$
$= \frac{1}{3n^2} \sum_{k=1}^{\infty} q_{n,k}(x)((k+1)^3 - k^3)$

$$= \frac{1}{3n^2}\{3(n^2x^2 + 3nx + (1 - e^{-nx})\}$$
$$= x^2 + \frac{2}{n}x + \frac{1}{3n^2}(1 - e^{-nx}).$$

(d) $K_n(t^m; x) = n\sum_{k=1}^{\infty} q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} t^m\, dt$

$$= \frac{n}{n^{m+1}(m+1)}\sum_{k=1}^{\infty} q_{n,k}(x)((k+1)^{m+1} - k^{m+1})$$
$$= \frac{1}{n^m(m+1)}\sum_{k=0}^{\infty} q_{n,k}(x)\left\{(m+1)k^m + \frac{m(m+1)}{2}k^{m-1} + \cdots + (m+1)k\right\}$$
$$+ \frac{1}{n^m(m+1)}(1 - e^{-nx})$$
$$= x^m + \frac{m^2}{2n}x^{m-1} + T.L.P.(x) + \frac{(1-e^{-nx})}{n^m(m+1)}$$
$$= x^m + \frac{m^2}{2n}x^{m-1} + T.L.P.(x) + \frac{1-\sum_{k=0}^{\infty}\frac{(-nx)^k}{k!}}{n^m(m+1)}$$

Definition 1.1

For $x \in [0,\infty)$ and $m \in \mathbb{N}^0$, the functions $\emptyset_{n,m}(x)$ are defined as [2]:

$$\emptyset_{n,m}(x) = \sum_{k=0}^{\infty} q_{n,k}(x)k^m .$$

Lemma 1.2 [2]

For the functions $\emptyset_{n,m}(x)$ defined above, we have $\emptyset_{n,0}(x) = 1,\ \emptyset_{n,1}(x) = nx$
$\emptyset_{n,m+1}(x) = x\emptyset'_{n,m}(x) + nx\emptyset_{n,m}(x), m = 2, 3, \ldots$

Definition 1.2

For $m \in \mathbb{N}^0$, we define the m-th order moment $T_{n,m}(x)$ of the operators $K_n(f(t); x)$ as follows:

$$T_{n,m}(x) = K_n((t-x)^m; x) = n\sum_{k=1}^{\infty} q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} (t-x)^m dt + (-x)^m e^{-nx} .$$

Theorem 1.1

For the function $T_{n,m}(x)$, we have the following property:
$T_{n,m}(x) = \sum_{i=0}^{m}\binom{m}{i}(-x)^{m-i}\left\{x^m + \frac{m^2}{2n}x^{m-1} + TLP(x) + \frac{(1-e^{-nx})}{n^m(m+1)}\right\} + (-x)^m e^{-nx},$
for $m \in \mathbb{N}^0$.

Proof: By direct computation, we have:

$$T_{n,m}(x) = n\sum_{k=1}^{\infty} q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} (t-x)^m\, dt + (-x)^m e^{-nx}$$
$$= n\sum_{k=1}^{\infty} q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} \sum_{i=0}^{m}\binom{m}{i}t^i(-x)^{m-i}\, dt + (-x)^m e^{-nx}$$
$$= \sum_{i=0}^{m}\binom{m}{i}(-x)^{m-i}\{n\sum_{k=1}^{\infty} q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} t^i\, dt\,\} + (-x)^m e^{-nx}$$
$$= \sum_{i=0}^{m}\binom{m}{i}(-x)^{m-i}\left\{x^m + \frac{m^2}{2n}x^{m-1} + T.L.P.(x) + \frac{(1-e^{-nx})}{n^m(m+1)}\right\} + (-x)^m e^{-nx}.$$

2. Main Result

Here, we introduce the Voronovaskaja-type asymptotic formula for the operator $K_n(f(t);x)$. Next, we give an error of approximation by the terms of modulus of continuity.

First, we give a theorem of Voronovaskaja-type asymptotic formula as:

Theorem 2.1

Let $f \in C_\gamma[0,\infty)$ for some $\gamma > 0$. If f'' exists at a point $x \in (0,\infty)$, then

$$\lim_{n\to\infty} n\big(K_n(f(t);x) - f(x)\big) = \frac{1}{2}f'(x) + \frac{1}{2}xf''(x)$$

Proof: By Taylor's expansion of $f(t)$, we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{f''(x)}{2}(t-x)^2 + \varepsilon(t,x)(t-x)^2$$

where $\varepsilon(t,x) \to 0$ as $t \to x$. Hence

$$K_n(f(t);x) = f(x)K_n(1;x) + f'(x)K_n\big((t-x);x\big) + \frac{f''(x)}{2}K_n((t-x)^2;x) + K_n(\varepsilon(t,x)(t-x)^2;x)$$

$$\lim_{n\to\infty} n\{K_n(f(t);x) - f(x)\} = \lim_{n\to\infty}\{f'(x)\frac{n}{2n} + \frac{f''(x)}{2}n\left(\frac{1}{n}x + \frac{1}{3n^2}\right) + nK_n(\varepsilon(t,x)(t-x)^2;x)\}$$
$$= \frac{1}{2}f'(x) + \frac{1}{2}xf''(x) + \lim_{n\to\infty} E$$

where $E = nK_n(\varepsilon(t,x)(t-x)^2;x)$.

$$|E| = \left|n^2\sum_{k=1}^{\infty} q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} \varepsilon(t,x)(t-x)^2\,dt + \varepsilon(0,x)e^{-nx}\right|$$

$$\leq n^2\sum_{k=1}^{\infty} q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} |\varepsilon(t,x)|(t-x)^2\,dt + \frac{|\varepsilon(0,x)|}{e^{nx}}$$

$$\leq n^2\sum_{k=1}^{\infty} q_{n,k}(x)\int_{|t-x|<\delta} |\varepsilon(t,x)|(t-x)^2\,dt$$

$$\leq n^2\varepsilon\sum_{k=1}^{\infty} q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} (t-x)^2\,dt$$

$$= \varepsilon n\left\{\frac{x}{n} + \frac{1}{3n^2}\right\} = \varepsilon x = o(1).$$

Now, since $\varepsilon > 0$ is arbitrary, it follows that $E \to 0$ as $n \to \infty$.

Theorem 2.2

Let $f \in C_\gamma[0,\infty)$ for some $\gamma > 0$ and $0 \leq q \leq 2$. If $f^{(q)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, then for sufficiently large n,

$$\|K_n(f(t);x) - f(x)\|_{C[a,b]} \leq C_1 n^{-1}\sum_{i=0}^{q}\left\|f^{(i)}\right\|_{C[a,b]} + C_2\, n^{\frac{-1}{2}}\omega_{f^{(q)}}(n^{\frac{-1}{2}};(a-\eta, b+\eta) + O(n^{-2}),$$

where C_1, C_2 are constants independent on f and n.

Proof: By our hypothesis

$$f(t) = \sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(q)}(\xi) - f^{(q)}(x)}{q!}(t-x)^q\chi(t) + h(t,x)\big(1-\chi(t)\big),$$

where ξ lies between t, x, and $\chi(t)$ is the characteristic function of the interval $(a-\eta\,, b+\eta)$.
For $t \in (a-\eta\,, b+\eta)$ and $x \in [a,b]$, we get

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q\chi(t)\,.$$

For $t \in [0,\infty)/(a-\eta\,, b+\eta)$ and $x \in [a,b]$, we define

$$h(t,x)=f(t)-\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i.$$

Now,

$$K_n(f(t);x)-f(x)=\left\{\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}K_n((t-x)^i;x)-f(x)\right\}$$
$$+K_n(\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^i\chi(t);x)$$
$$+K_n(h(t,x)(1-\chi(t));x)$$

$:=\Sigma_1+\Sigma_2+\Sigma_3.$

By using Lemma 1.1, we get

$$\Sigma_1=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\sum_{m=0}^{j}\binom{j}{m}(-x)^{j-m}\{K_n(t^m;x)-f(x)\}$$
$$=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\sum_{m=0}^{j}\binom{j}{m}(-x)^{j-m}\{x^m+\frac{m^2}{2n}x^{m-1}+T.L.P.(x)+\frac{(1-e^{-nx})}{n^m(m+1)}\}.$$

Consequently,
$\|\Sigma_1\|_{C[a,b]}\le C_1\, n^{-1}(\sum_{i=0}^{q}\|f^{(i)}\|_{C[a,b]})+O(n^{-2})$, uniformly on [a,b].
To estimate Σ_2 we proceed as follows:

$$|\Sigma_2|\le\left|K_n(\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^i\chi(t);x)\right|$$

$$\le\frac{\omega_{f^{(q)}}\left(\delta;(a-\eta,b+\eta)\right)}{q!}\quad K_n(\{1+\frac{|t-x|}{\delta}\}|t-x|^q;x)$$

$$\le\frac{\omega_{f^{(q)}}\left(\delta;(a-\eta,b+\eta)\right)}{q!}\quad\{n\sum_{k=1}^{\infty}|q_{n,k}(x)|\int_{\frac{k}{n}}^{\frac{k+1}{n}}(|t-x|^q+\delta^{-1}|t-x|^{q+1})dt$$
$$+\{|-x|^q+\delta^{-1}|-x|^{q+1}\}e^{-nx}\}.$$

Now, s=0, 1, 2, ..., we have

$$n\sum_{k=1}^{\infty} q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x|^s dt + |-x|^s e^{-nx}$$

$$\leq n\sum_{k=1}^{\infty} q_{n,k}(x)\Big(\int_{\frac{k}{n}}^{\frac{k+1}{n}} dt\Big)^{1/2}\Big(\int_{\frac{k}{n}}^{\frac{k+1}{n}} (t-x)^{2s}\ dt\Big)^{\frac{1}{2}} + o(1)$$

$$\leq \Big(\sum_{k=1}^{\infty} q_{n,k}(x)\Big)^{\frac{1}{2}}\Big(n\sum_{k=1}^{\infty} q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} (t-x)^{2s}\ dt + (-x)^{2s}\, e^{-nx}\Big)^{\frac{1}{2}} + o(1)$$

$$\leq (1-e^{-nx})^{\frac{1}{2}}\,(T_{n,2s}\,(x)\,)^{\frac{1}{2}} + o(1)$$

$\leq O(n^{\frac{-s}{2}})$ uniformly on [a,b].

Choosing $\delta = n^{\frac{-1}{2}}$, then

$$\|\Sigma_2\|_{C[a,b]} \leq \frac{\omega_{f^{(q)}}\left(n^{\frac{-1}{2}};\ (a-\eta, b+\eta)\right)}{q!}\left[O\left(n^{\frac{-q}{2}}\right) + n^{\frac{1}{2}}\ O\left(n^{\frac{-(q+1)}{2}}\right)\right]$$

$$\leq C_2\, n^{\frac{-q}{2}}\,\omega_{f^{(q)}}\left(n^{\frac{-1}{2}};\ (a-\eta, b+\eta)\right).$$

Since $t \in [0,\infty)/(a-\eta\,, b+\eta)$, we can choose $\delta > 0$ in such a way that $|t-x| \geq \delta$ for all $x \in [a,b]$.

$$\|\Sigma_3\|_{C[a,b]} \leq n\sum_{k=1}^{\infty} q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} |\,h(t,x)|\,dt + |\,h(0,x)|e^{-nx}$$

For $|t-x| \geq \delta$, we can find $C > 0$ such that $|h(t,x)| \leq Ce^{\alpha t}$.

$$|\Sigma_3| \leq C\,n\sum_{k=1}^{\infty} q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} e^{\alpha t}\,dt + o(1)$$

$$\leq C\,n\sum_{k=1}^{\infty} q_{n,k}(x)\Big(\int_{\frac{k}{n}}^{\frac{k+1}{n}} dt\Big)^{\frac{1}{2}}\Big(\int_{\frac{k}{n}}^{\frac{k+1}{n}} e^{2\alpha t}\ dt\Big)^{\frac{1}{2}} + o(1)$$

$$\leq C\ \Big(\sum_{k=1}^{\infty} q_{n,k}(x)\Big)^{1/2}\Big(n\sum_{k=1}^{\infty} q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} e^{2\alpha t}\,dt\Big)^{1/2} + o(1)$$

$$\leq C\,(1-e^{-nx})^{1/2}\,O(n^{-s}) + o(1)$$

$\leq O(n^{-s})$ for any $s > 0$, uniformly on [a,b].

Combining the estimates of $\Sigma_1, \Sigma_2, \Sigma_3$ the required result is immediate.

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التقريب بمؤثرات ***Phillips-Szàsz-Kantorovich***

علي جاسم محمد[1] وجنان عبد الامام نجم[2]
قسم الرياضيات، كلية التربية للعلوم الصرفة، جامعة البصرة، البصرة، العراق
[1]Email: alijasmoh@gmail.com, [2]Email: jinan.abd16@gmail.com.

المستخلص:
في هذا البحث، قمنا بتعريف ودراسة صيغة لمؤثرات من نوع Phillips Szàsz-Kantorovich. برهنا تقارب هذه المؤثرات للدالة المراد تقريبها. كذلك، ناقشنا الصيغة المشابهة لـ Voronovaskaja لهذه المؤثرات.