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A Sequence of Linear and Positive Operators for the Functions of Growth 2^x

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Abstract.

The purpose of this paper is to introduce and study a sequence of linear and positive operators to approximate unbounded functions in the interval $[0,\infty)$ of growth2^x.Our aim is to study the convergence of this sequence and introduce some approximation properties which lead us to provide proof that discusses Voronovskaja-type asymptotic formula for this sequence.

Key words:linear Positive Operators, Korovkin Theorem, Voronoviskaja Theorem.

1. Introduction

In [1] Lupas studied this identity

$\frac{1}{(1-a)^{\alpha}} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} a^k$, $|a| < 1$,

where $(\alpha)_k = \begin{cases} \alpha(\alpha+1) \ \dots (\alpha+k-1) & k \in N := \{1,2,\dots\} \\ 1 \quad , & k = 0 \end{cases}$. (1.1)

For more details, see [2],[3],[4]and [5].

Putting $\alpha = nx$ and $x \geq 0$we have the following linear positive operators.

$L_n(f;x) = (1-a)^{nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{k!} a^k f(\frac{k}{n})$.

for $f: \mathbb{R}_0 \to \mathbb{R}$ be a continuous function. Prescribing the condition $L_n(1;x) = 1$, he found that $a = \frac{1}{2}$,([2] and [4])

C_ρ: The set of all continuous functions $f$and unbounded such that ($(t) = O(2^{\rho x})$ for some $\rho > 0$) defined on $\mathbb{R}_0$, satisfying

$\|f\|_\rho = Sup_{x \epsilon \mathbb{R}} \frac{|f(x)|}{\rho(x)}$, where $\rho(x) = 2^x$.

Therefore $L_n(f;x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} \ f(\frac{k}{n})$, $\qquad x \geq 0$ (1.2)

where $x \in \mathbb{R}_0$, $n \in \mathbb{N}$, $\mathbb{R}_0 = [0,\infty)$, observe that these operators have a form very similar with Sz$\tilde{a}$sz-Mirakyan operators.

In this paper we define new operators (2.1) which represented a generalization of the previous operators $L_n(f;x)$ defined in (1.2).

2. Main Results

Now we work on the following operators :

$\tilde{L}_n(f;x) = \frac{1}{G_x} \ 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_{k+r}}{2^{k+r} \ (k+r)!} \ f\left(\frac{k+r}{n}\right)$ (2.1)

where $G_x = \ 2^{-nx} \sum_{k=0}^{r} \frac{(nx)_k}{2^k \ k!}$ or $G_x = \ 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_{k+r}}{2^{k+r} \ (k+r)!}$

$G_x := \sum_{k=0}^{r} d_{n,k}(x)$ or $G_x := \sum_{k=0}^{\infty} d_{n,k+r}(x)$

So, we can write the operator (2.1) as follows:

$\tilde{L}_n(f;x) = \frac{1}{G_x} \sum_{k=0}^{\infty} d_{n,k+r}(x) \ f(\frac{k+r}{n})$ (2.2)

where $x \in \mathbb{R}_0$, $n \in \mathbb{N}$, $\mathbb{R}_0 = [0,\infty)$, $\mathbb{N} := \{1,2,\dots\}$ and $r \in \mathbb{N}$.

Observe that the new operators (2.1)if we put $r = 1$ we get the operators (1.2) also its consequences. So consider the operators $\tilde{L}_n(f;x)$ defined in (2.2) which represent a generalization for the operators $L_n(f;x)$ which defined in (1.2).

In this part, we prove the following lemma before applying Korovkin conditions and then we find Voronoviskija –type formula .

Remark: For $\tilde{L}_n(f(t);x)$, $r \in \mathbb{N}$ and $f \in C_\rho$ we have

suppose $\tilde{L}_j = \sum_{k=0}^{\infty} d_{n,k+r}(x)(k+r)^j$ where $j = 1,2,3,4$

Lemma (2.1): For $\tilde{L}_n(f(t);x)$, $r \in \mathbb{N}$ and $f \in C_\rho$we have

1) $\tilde{L}_1 = 2rd_{n,r}(x) + nxG_x$
2) $\tilde{L}_2 = rd_{n,r}(x)\left[\frac{6nx+2r+6}{3}\right] + n^2x^2G_x + 2nxG_x$
3) $\tilde{L}_3 = \frac{8}{7}rd_{n,r}(x)\left[r^2 + \frac{(nx+r)(3r+nx+5)+21+nx(33+7nx)+4r(nx+3)}{4}\right] + n^3x^3G_x + 6n^2x^2G_x + 6nxG_x$.
4) $\tilde{L}_4 = n^4x^4G_x + 12n^3x^3G_x + 36n^2x^2G_x + 26nxG_x$ $+\frac{16}{15}rd_{n,r}(x)\begin{bmatrix} 1.8214n^3x^3 + 15.3362n^2x^2 + 34.6249nx + 1.4285 + r^3 + 1.6071r^2 \\ +11.5892r + 0.4999r^2nx + 0.75rn^2x^2 + 5.9642rnx \end{bmatrix}$.

Proof:

1) $\tilde{L}_1 = 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r-1)!}$

$= 2^{-nx}\left\{\frac{(nx)_r}{2^r\ (r-1)!} + \sum_{k=1}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r-1)!}\right\}$

$= r\, d_{n,r}(x) + 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r+1}}{2^{k+r+1}\ (k+r)!}$

we have $:= E_1 + E_2$

$E_1 = r\, d_{n,r}(x)$

and $E_2 = 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r+1}}{2^{k+r+1}\ (k+r)!}$

$E_2 = nx\ 2^{-(nx+1)}\sum_{k=0}^{\infty}\frac{(nx+1)_{k+r}}{2^{k+r}(k+r)!}$, (in view equation (1.1))

$= nxI_1$; where $I_1 = 2^{-(nx+1)}\sum_{k=0}^{\infty}\frac{(nx+1)_{k+r}}{2^{k+r}(k+r)!}$

$E_2 = nx\left\{\frac{2^{-nx}}{2}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!} + \frac{2^{-nx}}{2nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}(k+r)!}(k+r)\right\}$

$E_2 = \frac{nxG_x}{2} + \frac{L_1}{2}$,

then $\tilde{L}_1 = 2rd_{n,r}(x) + nxG_x$

2) $\tilde{L}_2 = 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}(k+r-1)!}(k+r)$

$= 2^{-nx}\left\{\frac{r\ (nx)_r}{2^r\ (r-1)!} + \sum_{k=1}^{\infty}\frac{(nx)_{k+r}\ (k+r)}{2^{k+r}\ (k+r-1)!}\right\}$

we have $:= E_3 + E_4$

observe that $E_3 = r^2 d_{n,r}(x)$

$E_4 = nx\ 2^{-(nx+1)}\sum_{k=0}^{\infty}\frac{(nx+1)_{k+r}(k+r+1)}{2^{k+r}(k+r)!}$, (in view equation (1.1))

$= nx\ 2^{-(nx+1)}\sum_{k=0}^{\infty}\frac{(nx+1)_{k+r}}{2^{k+r}(k+r)!}(k+r) + nx\ 2^{-(nx+1)}\sum_{k=0}^{\infty}\frac{(nx+1)_{k+r}}{2^{k+r}(k+r)!}$

$= nx\left\{2^{-(nx+1)}\frac{(nx+1)_r}{2^r(r-1)!} + 2^{-(nx+1)}\sum_{k=0}^{\infty}\frac{(nx+1)_{k+r}}{2^{k+r}(k+r-1)!}\right\} + nxI_1$

$E_4 = nx\frac{rd_{n,r}(x)(nx+r)}{2nx} + nx(nx+1)\ 2^{-(nx+2)}\sum_{k=0}^{\infty}\frac{(nx+2)_{k+r}}{2^{k+r}(k+r)!} + nxI_1$ (using equation (1.1))

let $I_2 = 2^{-(nx+2)}\sum_{k=0}^{\infty}\frac{(nx+2)_{k+r}}{2^{k+r}(k+r)!}$

$E_4 = rd_{n,r}(x)\left[\frac{nx+r+2}{2}\right] + nx(nx+1)I_2 + nxI_1$

note that $I_2 = \frac{2^{-nx}}{4nx(nx+1)}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}(nx+k+r)(nx+k+r+1)}{2^{k+r}(k+r)!}$

By some computations on I_2 and substituting it in E_4, then we get:

$\tilde{L}_2 = rd_{n,r}(x)\left[\frac{6nx+2r+6}{3}\right] + n^2x^2G_x + 2nxG_x$

3) $\tilde{L}_3 = 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}(K+r-1)!}(k+r)^2$

$= 2^{-nx}\left\{\frac{r^2\ (nx)_r}{2^r\ (r-1)!} + \sum_{k=1}^{\infty}\frac{(nx)_{k+r}\ (k+r)^2}{2^{k+r}\ (k+r-1)!}\right\}$

so we have $:= E_5 + E_6$

$E_5 = r^3 d_{n,r}(x)$

$E_6 = nx\ 2^{-nx}\sum_{k=1}^{\infty}\frac{(nx+1)_{k+r-1}}{2^{k+r}(k+r-1)!}(k+r)^2$, (using equation (1.1))

$= nx\ 2^{-(nx+1)}\sum_{k=0}^{\infty}\frac{(nx+1)_{k+r}}{2^{k+r}(k+r)!}(k+r+1)^2$

$= rd_{n,r}(x)\left[(nx+r)(1+\frac{r}{2})\right] + nx(nx+1)2^{-(nx+1)}\sum_{k=1}^{\infty}\frac{(nx+2)_{k+r-1}}{2^{k+r}(k+r-1)!}(k+r)$

$+2nx(nx+1)2^{-(nx+1)}\sum_{k=1}^{\infty}\frac{(nx+2)_{k+r-1}}{2^{k+r}(k+r-1)!} + 2nx(nx+1)I_2 + nxI_1$

$E_6 = rd_{n,r}(x)\left[(nx+r)(\frac{r+2}{2}+\frac{nx+r+1}{4})\right] + nx(nx+1)(nx+2)I_3 + +5nx(nx+1)I_2 + nxI_1$

when $I_3 = 2^{-(nx+3)}\sum_{k=0}^{\infty}\frac{(nx+3)_{k+r}}{2^{k+r}(k+r)!}$

observe that $I_3 = \frac{2^{-nx}}{8nx(nx+1)(nx+2)}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}(nx+k+r)(nx+k+r+1)(nx+k+r+2)}{2^{k+r}(k+r)!}$.

So, by some computations we have

$\tilde{L}_3 = \frac{8}{7}rd_{n,r}(x)\left[r^2+\frac{(nx+r)(3r+nx+5)+21+nx(33+7nx)+4r(nx+3)}{4}\right]+n^3x^3G_x + 6n^2x^2G_x + 6nxG_x$.

4) $\tilde{L}_4 = 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}(k+r-1)!}(k+r)^3$

$= 2^{-nx}\left\{\frac{r^3}{2^r}\frac{(nx)_r}{(r-1)!}+\sum_{k=1}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}}\frac{(k+r)^3}{(k+r-1)!}\right\}$

we have $:= E_7 + E_8$.

$E_7 = r^4 d_{n,r}(x)$

$E_8 = nx\ 2^{-(nx+1)}\sum_{k=0}^{\infty}\frac{(nx+1)_{k+r}}{2^{k+r}(k+r)!}(k+r+1)^3$ (using equation (1.1)).

$E_8 = nx\ 2^{-(nx+1)}\sum_{k=0}^{\infty}\frac{(nx+1)_{k+r}}{2^{k+r}(k+r-1)!}(k+r)^2 + 3nx\ 2^{-(nx+1)}\sum_{k=0}^{\infty}\frac{(nx+1)_{k+r}}{2^{k+r}(k+r-1)!}(k+r)$

$+3nx\ 2^{-(nx+1)}\sum_{k=0}^{\infty}\frac{(nx+1)_{k+r}}{2^{k+r}(k+r-1)!} + nx\ 2^{-(nx+1)}\sum_{k=0}^{\infty}\frac{(nx+1)_{k+r}}{2^{k+r}(k+r)!}$, (by using equation1.1))

$= rd_{n,r}(x)\left[(nx+r)\left(\frac{r^2+3r+3}{2}\right)+(nx+r+1)\left(\frac{5+r}{4}+\frac{(nx+r)(nx+r+2)}{8}\right)\right]$

$+nxI_1 + 7nx(nx+1)I_2 + 6nx(nx+1)(nx+2)I_3 + nx(nx+1)(nx+2)(nx+3)I_4$

when $I_4 = 2^{-(nx+4)}\sum_{k=0}^{\infty}\frac{(nx+4)_{k+r}}{2^{k+r}(k+r)!}$.

observe that $I_4 = \frac{2^{-nx}}{16\,nx(nx+1)(nx+2)(nx+3)}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}(nx+k+r)(nx+k+r+1)(nx+k+r+2)(nx+k+r+3)}{2^{k+r}(k+r)!}$

Therefore, by some computations we get

$$\tilde{L}_4 = \frac{16}{15}rd_{n,r}(x)\begin{bmatrix}1.8214n^3x^3 + 15.3362n^2x^2 + 34.6249nx + 1.4285 + r^3 + 1.6071r^2\\ +11.5892r + 0.4999r^2nx + 0.75rn^2x^2 + 5.9642rnx\end{bmatrix}$$

$+n^4x^4G_x + 12n^3x^3G_x + 36n^2x^2G_x + 26nxG_x$. ■

Theorem (2.2) (Korovkin Theorem):

For $x \in \mathbb{R}_0$, $f \in C_\rho$ and by applying Korovkin Theorem on the operator $\tilde{L}_n(f;x)$, we have:

1) $\tilde{L}_n(1;x) = 1$

2) $\tilde{L}_n(t;x) = x + \frac{2r}{n\ G_x}d_{n,r}(x)$.

3) $\tilde{L_n}(t^2;x) = x^2 + \frac{2x}{n} + rd_{n,r}(x)\left[\frac{6nx+2r+6}{3n^2G_x}\right]$.

4) $\tilde{L}_n(t^3;x) = x^3 + \frac{6x^2}{n} + \frac{6x}{n^2}$

$+\frac{8}{7n^3G_x}rd_{n,r}(x)\left[r^2+\frac{(nx+r)(3r+nx+5)+21+nx(33+7nx)+4r(nx+3)}{4}\right]$.

5) $\tilde{L}_n(t^4;x) = x^4 + \frac{28x^3}{5\,n} + \frac{3511x^2}{105\,n^2} + \frac{158x}{7\,n^3}$

$$+\frac{16}{15\,n^4\ G_x}rd_{n,r}(x)\begin{bmatrix}1.8214n^3x^3 + 15.3362n^2x^2 + 34.6249nx + 1.4285 + r^3 + 1.6071r^2\\ +11.5892r + 0.4999r^2nx + 0.75rn^2x^2 + 5.9642rnx\end{bmatrix}$$

Proof:
Using direct computations, we get the consequence (1)and(2).
Then, to prove (3)
$\tilde{L}_n(t^2;x)=\frac{L_2}{n^2G_x}=x^2+\frac{2x}{n}+rd_{n,r}(x)\left[\frac{6nx+2r+6}{3n^2G_x}\right].$
So, by the same proceedings we get (4)
$\tilde{L}_n(t^3;x)=\frac{L_3}{n^3G_x}$
$=\frac{8}{7n^3G_x}rd_{n,r}(x)\left[r^2+\frac{(nx+r)(3r+nx+5)+21+nx(33+7nx)+4r(nx+3)}{4}\right]+x^3+\frac{6x^2}{n}+\frac{6x}{n^2}.$
Finally, we get (5)
$\tilde{L}_n(t^4;x)=\frac{L_4}{n^4G_x}=x^4+\frac{12x^3}{n}+\frac{36x^2}{n^2}+\frac{268x}{n^3}$
$+\frac{16}{15\,n^4\;G_x}rd_{n,r}(x)\begin{bmatrix}1.8214n^3x^3+15.3362n^2x^2+34.6249nx+1.4285+r^3+1.6071r^2\\+11.5892r+0.4999r^2nx+0.75rn^2x^2+5.9642rnx\end{bmatrix}.$ ■

The next Lemma(2.3) explains some properties of the m-th order moment $\tilde{T}_{n,m}(x)$ to the operators $\tilde{L}_n(f(t);x)$ where $\tilde{T}_{n,m}(x)=\tilde{L}_n((t-x)^m;x)$.

Lemma (2.3): Let $r\in\mathbb{N}$, then for all $x\in\mathbb{R}_0$ and $n\in\mathbb{N}$, we have

1) $\tilde{T}_{n,0}(x)=1.$
2) $\tilde{T}_{n,1}(x)=\frac{2r}{n\;G_x}d_{n,r}(x).$
3) $\tilde{T}_{n,2}(x)=rd_{n,r}(x)\left(\frac{6nx+2r+6}{3n^2G_x}+\frac{4x}{nG_x}\right)+\frac{2x}{n}.$
4) $\tilde{T}_{n,3}(x)=rd_{n,r}(x)\left(\frac{8r^2+2(nx+r)(3r+nx+5)+42+2nx(33+7nx)+8r(nx+3)}{7n^3G_x}-\frac{2rx+6}{n^2G_x}\right)+\frac{6x}{n^2}.$
5) $\tilde{T}_{n,4}(x)=rd_{n,r}(x)\Big[5.9428\frac{x^3}{nG_x}+28.358\frac{x^2}{n^2G_x}+36.9332\frac{x}{n^3G_x}+4.8\frac{rx^2}{n^2G_x}+0.3618\frac{rx}{n^3G_x}+$
$+\frac{1.5237}{n^4G_x}+1.0666\frac{r^3}{n^4G_x}+1.7142\frac{r^2}{n^4G_x}-7.9999\frac{r^2}{n^3G_x}+12.3618\frac{r}{n^4G_x}+0.5332\frac{xr^2}{n^3G_x}-$
$9.1428\frac{rx}{n^2G_x}-9.1428\frac{x^2}{nG_x}-9.4285\frac{x}{n^2G_x}-19.4285\frac{r}{n^3G_x}-\frac{24}{n^3G_x}\Big]+\frac{12x^2}{n^2}+\frac{36x}{n^3}.$

Theorem (2.4): *(Voronovskaja theorem)*
Let $f\in C_\rho$, be a continuous in every subinterval $[a,b]\subseteq(0,\infty)$. Suppose that $f''(x)$ exists at a some point $x>0$, then
$\lim_{n\to\infty}n\{\tilde{L}_n(f;x)-f(x)\}=\frac{2}{G_x}rd_{n,r}(x)f'(x)+f''(x)\left\{rd_{n,r}(x)\left(\frac{3nx+r+3}{3nG_x}\right)+x-\frac{2x}{G_x}\right\}.$
Proof: By using Taylor's expansion for $f(t)$ about x, we have;[6]

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+\varepsilon(t,x)(t-x)^2$$

where $t\in(0,\infty)$ and $\varepsilon(t,x)$ is a function belonging to $C_\rho\in[0,\infty)$ and $\varepsilon(t,x)=0$ when $t\to x$ for $n\in\mathbb{N}$.
Hence by linearity of $\tilde{L}_n(f;x)$, we have
$\tilde{L}_n(f(t);x)=f(x)+f'(x)\tilde{L}_n((t-x);x)+\frac{1}{2}f''(x)\tilde{L}_n((t-x)^2;x)+\tilde{L}_n(\varepsilon(t;x)(t-x)^2;x).$
By Lemma (2.3), we get
$\tilde{L}_n(f(t);x)=f(x)+f'(x)\left\{\frac{2rd_{n,r}(x)}{nG_x}\right\}+\frac{1}{2}f''(x)\left\{\frac{2x}{n}+rd_{n,r}(x)\left(\frac{6nx+2r+6}{3n^2G_x}-\frac{4x}{nG_x}\right)\right\}+\tilde{L}_n(\varepsilon(t;x)(t-x)^2;x).$

$\lim_{n\to\infty} n\{\tilde{L}_n(f;x) - f(x)\} = f'(x)\left\{\frac{2}{G_x} r d_{n,r}(x)\right\} + f''(x)\left\{x + r d_{n,r}(x)\left(\frac{3nx+r+3}{3nG_x} - \frac{2x}{G_x}\right)\right\} +$
$\lim\limits_{n\to\infty} n\tilde{L}_n(\varepsilon(t;x)(t-x)^2;x)$.
By Cauchy-Schwartz inequality, we have:

$\left|\tilde{L}_n(\varepsilon(t;x)(t -\right.$

$\leq \varepsilon^2(t;x).\left(\tilde{L}_n((t-x)^4;x)\right)^{1/2} \to 0$ as $n \to \infty$. (in view of lemma (3.3))
$\lim\limits_{n\to\infty} n\tilde{L}_n(\varepsilon(t;x)(t-x)^2;x) = 0.$

So $\lim_{n\to\infty} n\{\tilde{L}_n(f;x) - f(x)\} = \frac{2}{G_x} r d_{n,r}(x) f'(x) + f''(x)\left\{r d_{n,r}(x)\left(\frac{3nx+r+3}{3nG_x}\right) + x - \frac{2x}{G_x}\right\}$.■

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متتابعة من المؤثرات الخطية الموجبة للدوال ذات النمو 2^x

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المستخلص

الغرض من هذا البحث هو تقديم متتابعة من المؤثرات الخطية الموجبة ودراستها لتقريب دوال غير مقيدة في الفترة $[0,\infty)$ للنمو 2^x. هدفنا هو دراسة تقارب هذه المتتابعة وتقديم بعض خواص هذا التقريب والتي تقودنا إلى تقديم برهان الصيغة المشابه لـVoronovskaja لهذه المتتابعة.

كلمات مفتاحية : مؤثرات خطية موجبة ، مبرهنة كورفكن ، مبرهنة فرونوفسكجيا