Available online at: www.basra-science-journal.org

ISSN –1817 –2695

A General Family of Summation Integral Baskakov-Type Operators

A.J. Mohammad[1] A.K.Hassan[2] S.A.Abdul-Hammed[3]

[1,3]*Mathematics Department, Education College, Basrah University, Basrah, Iraq.*

[2] *Mathematics Department, Science College, Basrah University, Basrah, Iraq.*

[1.] E-mail:alijasmoh@gmail.com

[3.]E-mail: safaaalrar@yahoo.com

Received 29-12-2013 , Accepted 15-9-2014

Abstract

In this paper, we introduce and study a general family of summation integralBaskakov-type operators. Firstly, we prove that thisfamily converges to the functions beingapproximated by applying Korovkin's Theorem for a sequence of algebraic linear positive operators. Secondly, we establish a Voronovaskaja–type asymptotic formula for this family. Finally, we obtain an estimation of the error for this family in terms of the modulus of continuity.

Keywords: Linear positive operators,Voronovaskaja-typeasymptotic formula, Simultaneous approximation,Modulus of continuity.

1. Introduction

In 1985,Sahai and Prasad [6] defined and studied the summation- integral Baskakov-type operatorsdefined as:

$$P_n(f;x) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t)f(t)dt, \qquad x \in [0,\infty)$$

where$p_{n,k}(x) = \binom{n+k-1}{k} x^k(1+x)^{-n-k}$.

In 1998, Agrawal and Thamer[1] introduced and studied the summation integral Phillips Baskakov-type operators defined as:

$$A_n(f;x) = (n-1)\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t)f(t)dt + f(0)p_{n,0}(x).$$

Next, in 2003Srivatava and Gupta [7] introduced the summation integral Beta-type operators defined as:

$$U_n(f;x) = \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} b_{n,k}(t)f(t)dt, \quad x \in [0,\infty)$$

where $b_{n,k}(x) = \frac{(n+k)!}{k!(n-1)!} x^k(1+x)^{-n-k-1} = \frac{(n+k)}{(1+x)} p_{n,k}(x)$.

After that, in 2005 Gupta and Doğru [3] defined the Phillips summation integral Beta-type operators as:

$$B_n(f;x)=\frac{1}{n}\sum_{k=1}^{\infty} b_{n,k}(x)\int_0^{\infty} b_{n,k-1}(t)f(t)dt+(1+x)^{-n-1}f(0).$$

Recently, in 2012 Mohammad and Hassan [4] were defined and studied a generalization form of $A_n(f,x)$as:

$$S_{n,v}(f;x)=(n-1)\sum_{k=v}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k-v}(x)f(t)dt+f(0)\sum_{k=0}^{v-1} p_{n,k}(x),$$

where $v\in N^0=\{0,1,2,\dots\}$.

In this paper, we define and study a general family of the Phillips summation integralBaskakov-type operators. The study of eachsequence in the above isa special case of our study here. Our family is restrict to all these sequences and more by a suitable choice of the values $r_1,r_2\in N^0$.

Suppose that $C[0,\infty)$ denotes the space of all continuous real-valued functions on the interval $[0,\infty)$.The subspace $C_\propto[0,\infty)$of the space $C[0,\infty)$ is defined as
$C_\propto[0,\infty)=\{f\in C[0,\infty): f(t)=O((1+t)^\propto),$ for some $\propto>0\}$.The subspace $C_\propto[0,\infty)$ is normed by the norm

$$\|f\|_{C_\propto}=\sup_{t\epsilon[0,\infty)}|f(t)|(1+t)^{-\propto}.$$

First, for $f\in C_\propto[0,\infty), r\in N^0$, we define the operators $B_{n,k,r}(f;x)$ as:

$$B_{n,k,r}(f;x)=\frac{1}{(n)_r}\sum_{k=0}^{\infty}\beta_{n,k,r}(x)f\left(\frac{k}{n}\right). \tag{1.1}$$

Where

$$\beta_{n,k,r}(x)=\frac{(n+k)_r}{(1+x)^r}p_{n,k}(x),$$

and $(x)_k=\begin{cases} x(x+1)(x+2)(x+3)\dots(x+k-1), & k=1,2,\dots \\ 1, & k=0 \end{cases}$.

Note that $\beta_{n,k,0}(x)=p_{n,k}(x)$ and $\beta_{n,k,1}(x)=\beta_{n,k}(x)$.

We define the operators $M_{n,v,r_1,r_2}(f;x)$ as:

$$M_{n,v,r_1,r_2}(f;x)=\frac{(n+r_2-1)}{(n)_{r_1}(n)_{r_2}}\sum_{k=v}^{\infty}\beta_{n,k,r_1}(x)\int_0^{\infty}\beta_{n,k-v,r_2}(t)f(t)dt$$
$$+\frac{1}{(n)_{r_1}}\sum_{k=0}^{v-1}\beta_{n,k,r_1}(x)f(0), x\in[0,\infty), \tag{1.2}$$

where $r_1,r_2\in N^0$ and $f\in C_\propto[0,\infty)$.

Wecan write the operators $M_{n,v,r_1,r_2}(f;x)$ as:

$$M_{n,v,r_1,r_2}(f;x)=\int_0^{\infty} W_{n,v}(t,x)f(t)\,dt,$$

where

$$W_{n,v}(t,x)=\frac{(n+r_2-1)}{(n)_{r_1}(n)_{r_2}}\sum_{k=v}^{\infty}\beta_{n,k,r_1}(x)\,\beta_{n,k-v,r_2}(t)+\sum_{k=0}^{v-1}\delta(t)\beta_{n,k,r_1}(x),$$

where $\delta(t)$ being the Dirac-delta function.

It is easily verify that the operators M_{n,v,r_1,r_2}defined above are linear positive operators. In this paper, we study some direct results include theconvergence, Voronovaskaja-type asymptotic formula and error estimate in terms of modulus of continuity of the function beingapproximated.

2. Preliminary Results

In this section, we establish some lemmas which help us in proving the main results of this paper. For $m \in N^0$, the m-thorder moment of the operator (1.1)is defined as

$$T_{n,m}(x) = \frac{1}{(n)_r}\sum_{k=0}^{\infty} \beta_{n,k,r}\left(\frac{k}{n} - x\right)^m .$$

2.1Lemma. For the function $T_{n,m}(x)$, we have $T_{n,0}(x) = 1$, $T_{n,1}(x) = \frac{rx}{n}$.

Furthermore, we have the following recurrence relation for $T_{n,m}(x)$:

$$nT_{n,m+1}(x) = x(1+x)\left[T'_{n,m}(x) + mT_{n,m-1}(x)\right] + rxT_{n,m}(x), m \geq 1 \qquad (2.1)$$

Consequently, we have $T_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$.

Proof: By direct computation, we have $T_{n,0}(x)$=1, $T_{n,1}(x) = \frac{rx}{n}$.

To prove (2.1). It is clear that the relation is true at $x = 0$.

Now, for $x \in (0, \infty)$, we have:

$$T_{n,m}(x) = \frac{1}{(n)_r}\sum_{k=0}^{\infty} \beta_{n,k,r}(x)\left(\frac{k}{n} - x\right)^m$$

$$T'_{n,m}(x) = \frac{-m}{(n)_r}\sum_{k=0}^{\infty} \beta_{n,k,r}(x)\left(\frac{k}{n} - x\right)^{m-1} + \frac{1}{(n)_r}\sum_{k=0}^{\infty} \beta'_{n,k,r}(x)\left(\frac{k}{n} - x\right)^m$$

$$x(1+x)T'_{n,m}(x) = -mx(1+x)T_{n,m-1}(x) + \frac{1}{(n)_r}\sum_{k=0}^{\infty}[k-(n+r)x]\beta_{n,k,r}(x)\left(\frac{k}{n} - x\right)^m$$

$$x(1+x)\left[T_{n,m}(x) + mT_{n,m-1}(x)\right] = \frac{n}{(n)_r}\sum_{k=0}^{\infty} \beta_{n,k,r}(x)\left(\frac{k}{n} - x\right)^{m+1} - \frac{rx}{(n)_r}\sum_{k=0}^{\infty} \beta_{n,k,r}(x)\left(\frac{k}{n} - x\right)^m .$$

From which (2 .1) is immediate.

From the above lemma, we get:

$$\sum_{k=v}^{\infty} \beta_{n,k,r}(x)(k-nx)^{2j} = n^{2j}\left(T_{n,2j}(x) - \sum_{k=0}^{v-1} \beta_{n,k,r}(x)\left(\frac{k}{n} - x\right)^{2j}\right)$$

$$= n^{2j}\{O\left(n^{-j}\right) + O\left(n^{-u}\right)\} \quad (\text{for } u > 0)$$

$$= O\left(n^{-j}\right), (\text{for } u \geq j\ ). \qquad (2.2)$$

2.2Lemma. Let the function $Y_{n,m}(x), n > m$ and $m \in N^0$, be defined as:

$$Y_{n,m}(x) = \frac{(n+r_2-1)}{(n)_{r_1}(n)_{r_2}}\sum_{k=v}^{\infty} \beta_{n,k,r_1}(x)\int_0^{\infty} \beta_{n,k-v,r_2}(t)(t-x)^m dt + \frac{f(0)}{(n)_{r_1}}\sum_{k=0}^{v-1} \beta_{n,k,r_1}(x).$$

Then$Y_{n,0}(x) = 1$,

$$Y_{n,1}(x) = \frac{\left((r_1 - r_2) + 2\right)x + (1-v)}{n + r_2 - 2} + \frac{\sum_{k=0}^{v-1}(v-1-k)\beta_{n,k,r_1}(x)}{(n)_{r_1}(n+r_2-2)}.$$

And there holds the recurrence relation

$$[(n+r_2) - (m+2)]Y_{n,m+1}(x) = x(1+x)Y'_{n,m}(x) + 2mx(1+x)Y_{n,m-1}(x) + [(1+2x)(m+1) - v + (r_1 - r_2)x]Y_{n,m}(x)$$

$$-\frac{1}{(n)_{r_1}}\sum_{k=0}^{v-1}(k-v+1)(-x)^m\beta_{n,k,r_1}(x). \qquad (2.3)$$

Consequently for each $x\in[0,\infty)$, we have from this recurrence relation that $Y_{n,m}(x)=O\left(n^{-[(m+1)/2]}\right)$.

Proof. The values of $Y_{n,0}(x)$and $Y_{n,1}(x)$are easily follow. We prove the recurrence relationas follows:

$$x(1+x)Y'_{n,m}(x)$$
$$=x(1+x)\left[\frac{(n+r_2-1)}{(n)_{r_1}(n)_{r_2}}\sum_{k=v}^{\infty}\beta'_{n,k,r_1}(x)\int_0^{\infty}\beta_{n,k-v,r_2}(t)(t-x)^m dt\right.$$
$$-m\sum_{k=v}^{\infty}\beta_{n,k,r_1}(x)\int_0^{\infty}\beta_{n,k-v,r_2}(t)(t-x)^{m-1}dt+\frac{1}{(n)_{r_1}}\sum_{k=0}^{v-1}\beta_{n,k,r_1}(x)(-x)^m$$
$$\left.-\frac{m}{(n)_{r_1}}\sum_{k=0}^{v-1}\beta_{n,k,r_1}(x)(-x)^{m-1}\right].$$

Now using the identities $x(1+x)\beta'_{n,k,r_1}(x)=\beta_{n,k,r_1}(x)[k-(n+r_1)x]$, we obtain

$$x(1+x)\left[Y'_{n,m}(x)+mY_{n,m-1}(x)\right]=\frac{(n+r_2-1)}{(n)_{r_1}(n)_{r_2}}\sum_{k=v}^{\infty}\beta_{n,k,r_1}(x)$$
$$\times\left[\int_0^{\infty}[(k-v)-(n+r_2)t]\beta_{n,k-v,r_2}(t)(t-x)^m dt+v\int_0^{\infty}\beta_{n,k-v,r_2}(t)(t-x)^m dt\right.$$
$$\left.+(n+r_2)\int_0^{\infty}\beta_{n,k-v,r_2}(t)(t-x)^m dt+(r_2-r_1)\int_0^{\infty}\beta_{n,k-v,r_2}(t)(t-x)^m dt\right]$$
$$+\frac{1}{(n)_{r_1}}\sum_{k=0}^{v-1}k(-x)^m\beta_{n,k,r_1}(x)+\frac{1}{(n)_{r_1}}\sum_{k=0}^{v-1}(n+r_1)(-x)^m\beta_{n,k,r_1}(x).$$
$$=\frac{(n+r_2-1)}{(n)_{r_1}(n)_{r_2}}\sum_{k=v}^{\infty}\beta_{n,k,r_1}(x)\int_0^{\infty}t(t+1)\beta'_{n,k-v,r_2}(t)(t-x)^m dt+vY_{n,m}(x)$$
$$+(n+r_2)Y_{n,m+1}(x)+(r_2-r_1)Y_{n,m}(x)+\frac{1}{(n)_{r_1}}\sum_{k=0}^{v-1}(k-v)(-x)^m\beta_{n,k,r_1}(x).$$

Integrating by parts, we get:

$$[(n-r_2)-(m+2)]Y_{n,m+1}(x)=x(1+x)Y'_{n,m}(x)$$
$$+2mx(1+x)Y_{n,m-1}(x)[(1+2x)(m+1)-v+(r_1-r_2)x]Y_{n,m}(x)$$
$$-\frac{1}{(n)_{r_1}}\sum_{k=0}^{v-1}(k-v+1)(-x)^m\beta_{n,k,r_1}(x).$$

From the values of $Y_{n,0}(x),Y_{n,1}(x)$,using the indication on m, the recurrence relation above and the fact that $\sum_{k=0}^{v-1}(k-v+1)\beta_{n,k,r_1}(x)=o(1)\ as\ n\to\infty$ we can easily prove that $Y_{n,m}(x)=O\left(n^{-[(m+1)/2]}\right)$.

From the above lemma , we have:

$$\frac{(n+r_2-1)}{(n)_{r_1}(n)_{r_2}}\sum_{k=v}^{\infty}\beta_{n,k,r_1}(x)\int_0^{\infty}\beta_{n,k-v,r_2}(t)(t-x)^{2\gamma}dt$$
$$=Y_{n,2\gamma}(x)-\frac{1}{(n)_{r_1}}\sum_{k=0}^{v-1}\beta_{n,k,r_1}(x)(-x)^{2u}=O(n^{-\gamma})+O(n^{-u})$$
$$=O(n^{-\gamma}),u\geq\gamma \qquad (2.4)$$

Now usinglemma 2.2 we get that $M_{n,v,r_1,r_2}(t^m,x)$is a polynomial in x of degree exactly m for all $m\in N^0$,Further we can write it as:

$$M_{n,v,r_1,r_2}(t^m,x)=\frac{(n+r_1+m-1)!\,(n+r_2-m-2)!}{(n+r_1-1)!\,(n+r_2-2)!}x^m$$
$$+m(m-v)\frac{(n+r_1+m-2)!\,(n+r_2-m-2)!}{(n+r_1-1)!\,(n+r_2-2)!}x^{m-1}$$
$$+o(1) \qquad (2.5)$$

2.3 Lemma Let δ,γ be any two positive real numbers and$[a,b]\subset(0,\infty)$ we have:

$$\left\|\int_{|t-x|\geq\delta}W_{n,v}(x,t)t^{\gamma}dt\right\|_{C[a,b]}=O(n^{-u}),\qquad u=1,2,3,\dots$$

Making use of Schwarz inequality for integration, summation and (2.4), the proof of the lemma easily follows.

2.4 Lemma There exist a polynomials $Q_{i,j,s}(x)$ independent of n and k such that

$$\{x(1+x)^s\}\frac{d^s}{d^sx}\left[\beta_{n,k,r_1}(x)\right]=\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^i\,(k-(n+r_1)x)^jQ_{i,j,s}(x)\beta_{n,k,r_1}(x).$$

3. Main Results

In this section we study the rate of pointwise convergence of an asymptotic formula and an error estimation in terms of modulus of continuity in simultaneous approximation for our operators .

3.1 Theorem Let $f\in C_{\propto}[0,\infty),\propto>0$ and $f^{(s)}$ exists at a point $x\in(0,\infty)$, then
$M_{n,v,r_1,r_2}^{(s)}(f,x)=f^{(s)}(x)+o(1)$as $n\to\infty$. (3.1)

Further, if $f^{(s)}$ exist and is continuous on $(a-\Gamma,b+\Gamma)\subset(0,\infty)$, $\Gamma>0$,then (3.1) hold uniformly in$[a,b]$.

Proof By Taylor's expansion of f, we have

$$f(t)=\sum_{i=0}^{s}\frac{f^{(i)}(x)}{i!}(t-x)^i+\varepsilon(t,x)(t-x)^s,$$

where $\varepsilon(t,x)\to 0$ as $t\to x$.

$$M_{n,v,r_1,r_2}^{(s)}(f;x)=\int_0^{\infty}W_{n,v}^{(s)}(t,x)\,f(t)dt$$
$$=\sum_{i=0}^{s}\frac{f^{(i)}(x)}{i!}\int_0^{\infty}W_{n,v}^{(s)}(t,x)\,(t-x)^idt+\int_0^{\infty}W_{n,v}^{(s)}(t,x)\,\varepsilon(t,x)(t-x)^sdt$$
$$:=\mathcal{H}_1+\mathcal{H}_2.$$

First to estimate $\mathcal{H}_1$ using binomial expansion of $(t-x)^s$, and lemma 2.2, we have:

$$\mathcal{H}_1=\sum_{i=0}^{s}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\int_0^{\infty}W_{n,v}^{(s)}(t,x)\,t^j dt$$
$$=\frac{f^{(s)}(x)}{s!}\Bigg(\frac{(n+r_1+s-1)!\,(n+r_2-s-2)!}{(n+r_1-1)!\,(n+r_2-2)!}s! + terms\ containing\ lower\ powers\ of\ x\Bigg)$$

Using lemma 2.4, we obtain:

$$|\mathcal{H}_2|\le\sum_{\substack{2i+j\le r\\ i,j\ge 0}}n^i\frac{|Q_{i,j,s}(x)|}{x^s(1+x)^s}\frac{(n+r_2-1)}{(n)_{r_1}(n)_{r_2}}\sum_{k=v}^{\infty}\beta_{n,k,r_1}(x)\,|(k-(n+r_1)x)|^j\int_0^{\infty}\beta_{n,k-v,r_2}(x)\,|\varepsilon(t,x)||t-x|^s dt$$
$$+\frac{1}{(n)_{r_1}}\sum_{\substack{2i+j\le s\\ i,j\ge 0}}n^i\frac{|Q_{i,j,s}(x)|}{x^s(1+x)^s}\sum_{k=0}^{v-1}\beta_{n,k,r_1}(x)\,\big|(k-(n+r_1)x)^j\big||\varepsilon(0,x)|x^s$$
$$\coloneqq\mathcal{H}_3+\mathcal{H}_4.$$

Since $\varepsilon(t,x)\to 0$ as $t\to x$, then for a given $\varepsilon>0$ there exists $\delta>0$ such that $|\varepsilon(t,x)|<\varepsilon$ whenever $0<|t-x|<\delta$.For $|t-x|\ge\delta$,there exists a constant $C>0$ such that $|\varepsilon(t,x)(t-x)^s|\le C|t-x|^\gamma$.

$$\mathcal{H}_3\le\sum_{\substack{2i+j\le s\\ i,j\ge 0}}n\sum_{k=v}^{\infty}\beta_{n,k,r_1}(x)\,|k-(n+r_1)x|^j\left\{\varepsilon\int_{|t-x|<\delta}\beta_{n,k-v,r_2}(x)|t-x|^s dt+\int_{|t-x|\ge\delta}\beta_{n,k-v,r_2}(x)|t-x|^\gamma dt\right\}:=\mathcal{H}_5+\mathcal{H}_6.$$

Where

$$\sup_{\substack{2i+j\le r\\ i,j\ge 0}}\frac{|Q_{i,j,s}(x)|}{x^s(1+x)^s}=:M(x)=C\,,x\in(0,\infty)$$

Now, by applying the Cauchy-Schwarz inequality for integration and summation respectively, we have :

$$\mathcal{H}_5\le\varepsilon C\sum_{\substack{2i+j\le s\\ i,j\ge 0}}n^i\sum_{k=v}^{\infty}\beta_{n,k,r_1}(x)\left(\frac{1}{(n)_{r_1}}\sum_{k=v}^{\infty}\beta_{n,k,r_1}(x)\,|k-(n+r_1)+x|^{2j}\right)^{1/2}$$
$$\times\left(\frac{(n+r_2+1)}{(n)_{r_1}(n)_{r_2}}\sum_{k=v}^{\infty}\beta_{n,k,r_1}(x)\int_0^{\infty}\beta_{n,k-v,r_2}(x)(t-x)^{2s}dt\right)^{1/2}.$$

Using (2.4), we have:

$$\mathcal{H}_5\le\varepsilon C\,O\big(n^{-S/2}\big)\sum_{\substack{2i+j\le s\\ i,j\ge 0}}n^iO\big(n^{j/2}\big)=\varepsilon O(1).$$

Again using the Schwarz inequality for integration and then for summation, in view of (2.2) and (2.4) we have :

$$\mathcal{H}_6 \leq C \sum_{\substack{2i+j\leq s\\ i,j\geq 0}} n^i \frac{(n+r_2+1)}{(n)_{r_1}(n)_{r_2}} \sum_{k=v}^{\infty} \beta_{n,k,r_1}(x)\,|k-(n+r_1)x|^j$$

$$\times \left(\int_{|t-x|\geq\delta} \beta_{n,k-v,r_2}(x)dt \right)^{1/2} \left(\int_{|t-x|\geq\delta} \beta_{n,k-v,r_2}(x)t^{2\gamma}dt \right)^{1/2}$$

$$\leq C \sum_{\substack{2i+j\leq s\\ i,j\geq 0}} n^i \left(\frac{1}{(n)_{r_1}} \sum_{k=v}^{\infty} \beta_{n,k,r_1}(x)\,|k-(n+r_1)x|^{2j} \right)^{1/2}$$

$$\times n^i \left(\frac{(n+r_2+1)}{(n)_{r_1}(n)_{r_2}} \sum_{k=v}^{\infty} \beta_{n,k,r_1}(x) \int_0^{\infty} \beta_{n,k-v,r_2}(x)t^{2\gamma}dt \right)^{1/2}$$

$$\mathcal{H}_6 = \sum_{\substack{2i+j\leq s\\ i,j\geq 0}} n^i\, O(n^{j/2})O(n^{-\gamma/2}) = O(n^{(s-\gamma)/2}) = o(1)\,.$$

Thus, due to arbitrariness of $\varepsilon > 0$, it follows that $\mathcal{H}_3 = o(1)$. Also $\mathcal{H}_4 \to 0$ as $n \to \infty$ and hence $\mathcal{H}_2 = o(1)$. Collecting the estimates of $\mathcal{H}_1$ and $\mathcal{H}_2$,we have $M^{(s)}_{n,v,r_1,r_2}(f,x) = f^{(s)}(x) + o(1)$as $n \to \infty$.

To prove the uniformly assertion, it is sufficient to remark that $\delta(\varepsilon)$ in above prove can be chosen to be independent of $x \in [a,b]$and also that the order estimates holds uniformly in $[a,b]$.

3. 2. Theorem

Let $f \in C_\alpha[0,\infty)$, $\gamma > 0$. if $f^{(s+2)}$ exists at a point $x \in (0,\infty)$, then

$$\lim_{n\to\infty} n\left[M^{(s)}{}_{n,v,r_1,r_2}(f,x) - f^{(s)}(x)\right] = [s(s+1) + (s-1)(r_1-r_2)]f^{(s)}(x)$$
$$+[x(2s+2+r_1-r_2) + (s-v+1)]f^{(s+1)}(x) + x(1+x)f^{(s+2)}(x). \qquad (3.2)$$

Further, if $f^{(s+2)}$ exist and is continuous on $(a-\Gamma, b+\Gamma) \subset (0,\infty)$, $\Gamma > 0$,then (3.2) hold uniformly in $[a,b]$.

Proof . Using Taylor's expansion of f, we have

$$f(t) = \sum_{i=0}^{s+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{s+2},$$

where $\varepsilon(t,x) \to 0$ as $t \to x$ and $\varepsilon(t,x) = O(t-x)^\gamma, t \to \infty\ for\ \gamma > 0$.

$$n\left[M^{(s)}{}_{n,v,r_1,r_2}(f,x) - f^{(s)}(x)\right]$$
$$= n\left[\sum_{i=0}^{s+2} \frac{f^{(i)}(x)}{i!} \int_0^{\infty} W^{(s)}_{n,v}(t,x)(t-x)^i dt - f^{(s)}(x)\right.$$
$$\left. + \int_0^{\infty} W^{(s)}_{n,v}(t,x)\,\varepsilon(t,x)(t-x)^{(s+2)}dt\right] \qquad =: \Psi_1 + \Psi_2$$

$$\Psi_1 = n\sum_{i=0}^{s+2} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \int_0^{\infty} W^{(s)}_{n,v}(t,x)t^i dt - nf^{(s)}(x)$$
$$= \frac{f^{(s)}(x)}{s!} n\left[M^{(s)}{}_{n,v,r_1,r_2}(t^s,x) - (s!)\right]$$
$$+ \frac{f^{(s+1)}(x)}{(s+1)!} n\left[(s+1)(-x)M^{(s)}{}_{n,v,r_1,r_2}(t^s,x) + M^{(s)}{}_{n,v,r_1,r_2}(t^{s+1},x)\right]$$

$$+\frac{f^{(s+2)}(x)}{(s+1)!}n\left[\frac{(s+2)(s+1)}{2}x^2 M^{(s)}{}_{n,v,r_1,r_2}\left(t^{(s)},x\right)\right.$$
$$\left.+(s+2)(-x)M^{(s)}{}_{n,v,r_1,r_2}\left(t^{(s+1)},x\right)+M^{(s)}{}_{n,v,r_1,r_2}\left(t^{(s+2)},x\right)\right]$$

By (2.5), we have

$$\Psi_1 = nf^{(s)}(x)\left[\frac{(n+r_1+s-1)!\,(n+r_2+s-1)!}{(n+r_1-1)!\,(n+r_2-2)!}-1\right]$$
$$+n\frac{f^{(s+1)}(x)}{(s+1)!}\left[(s+1)(-x)\frac{(n+r_1+s-1)!\,(n+r_2+s-2)!}{(n+r_1-1)!\,(n+r_2-2)!}s!\right.$$
$$+\frac{(n+r_1+s-1)!\,(n+r_2+s-2)!}{(n+r_1-1)!\,(n+r_2-2)!}(s+1)!\,x$$
$$\left.+(s+1)(s+1-v)s!\frac{(n+r_1+s-1)!\,(n+r_2+s-3)!}{(n+r_1-1)!\,(n+r_2-2)!}\right]$$

$$+\frac{f^{(s+2)}(x)}{(s+2)!}\left[\frac{(s+1)(s+2)}{2}x^2\frac{(n+r_1+s-1)!\,(n+r_2-s-2)!}{(n+r_1-1)!\,(n+r_2-2)!}s!\right.$$
$$+(s+2)(-x)\frac{(n+r_1+s)!\,(n+r_2-s-3)!}{(n+r_1-1)!\,(n+r_2-2)!}(s+1)!\,x$$
$$+(s+1)(s+1-v)s!\frac{(n+r_1+s-1)!\,(n+r_2-s-3)!}{(n+r_1-1)!\,(n+r_2-2)!}$$
$$+\frac{(s+2)!\,x^2}{2}\frac{(n+r_1+s+1)!\,(n+r_2-s-4)!}{(n+r_1-1)!\,(n+r_2-2)!}$$
$$\left.+(s+2)(s+2-v)(s+1)!\,x\frac{(n+r_1+s)!\,(n+r_2-s-4)!}{(n+r_1-1)!\,(n+r_2-2)!}\right]+o(1).$$

In order to complete the proof of the theorem it is sufficient to show that $\Psi_2 \to 0$ as $n \to \infty$ by follows proceeding along the line of the proof of Theorem 3.1.

To prove the uniformly assertion, it is sufficient to remark that $\delta(\varepsilon)$ in above prove can be chosen to be independent of $x \in [a,b]$ and also that the order estimatesholds uniformly in$[a,b]$.

3.3Theorem.

Let $f \in C_{\propto}[0,\infty)$ for some $\propto > 0$ and $s \le q \le s+2$. if $f^{(q)}$ exists and is continuous on $(a-\tau, b+\tau) \subset (0,\infty), \tau > 0$, then for sufficiently large n,

$$\left\|M^{(s)}{}_{n,v,r_1,r_2}(f,x)-f^{(s)}(x)\right\|_{C[a,b]}$$
$$\le Z_1 n^{-1}\sum_{i=s}^{q}\left\|f^{(i)}\right\|_{C[a,b]}+Z_2 n^{-1/2}\omega_{f^{(q)}}\left(n^{-1/2};(a-\tau,b+\tau)\right)+O(n^{-2}),$$

where Z_1, Z_2 are constants independent of f and $n, \omega_f(\delta)$is the modulus of continuity of f on $(a-\tau, b+\tau)$and$\|.\|_{C[a,b]}$ denotes the sup-norm on $[a,b]$.

Proof. By a finite Taylor's expansion of f, we have :

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(\Gamma)-f^{(q)}(x)}{q!}(t-x)^q\varphi(t)+g(t,x)\big(1-\varphi(t)\big),$$

where Γ lies between t and x, and $\varphi(t)$ is the characteristic function of the interval $(a-\tau, b+\tau)$.

For $t \in (a-\tau, b+\tau)$and $x \in [a,b]$, we get

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(\Gamma)-f^{(q)}(x)}{q!}(t-\Gamma)^q.$$

For $t\in[0,\infty)\setminus(a-\tau,b+\tau)$and $x\in[a,b]$, we define

$g(t,x)=f(t)-\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i$, now

$$M^{(s)}{}_{n,v,r_1,r_2}(f,x)-f^{(s)}(x)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\int_0^\infty W^{(s)}{}_{n,v}(t,x)\,(t-x)^i dt-f^{(s)}(x)$$

$$+\int_0^\infty W^{(s)}{}_{n,v}(t,x)\left(\frac{f^{(q)}(\Gamma)-f^{(q)}(x)}{q!}(t-x)^q\varphi(t)\right)dt$$

$$+\int_0^\infty W^{(s)}{}_{n,v}(t,x)\,g(t,x)\big(1-\varphi(t)\big)dt.$$

$:=\Upsilon_1+\Upsilon_2+\Upsilon_3$.

By using (2.5), we get

$$\Upsilon_1=\sum_{i=s}^{q}\frac{f^{(i)}(x)}{i!}\sum_{j=s}^{i}\binom{i}{j}(-x)^{i-j}\frac{d^r}{dx^r}\left(\frac{(n+r_1+j-1)!\,(n+r_2-j-2)!}{(n+r_1-1)!\,(n+r_2-2)!}x^j\right.$$

$$\left.+j(j-v)\frac{(n+r_1+j-2)!\,(n+r_2-j-2)!}{(n+r_1-1)!\,(n+r_2-2)!}x^{j-1}\right)+O(n^{-2})-f^{(s)}(x).$$

$\|\Upsilon_1\|_{C[a,b]}=Z_1n^{-1}\sum_{i=s}^{q}\left\|f^{(i)}\right\|_{C[a,b]}+O(n^{-2})$uniformly on$[a,b]$.

Next, we estimate Υ_2 as

$$|\Upsilon_2|\le\int_0^\infty\left|W^{(s)}{}_{n,v}(t,x)\right|\left(\frac{\left|f^{(q)}(\Gamma)-f^{(q)}(x)\right|}{q!}|t-x|^q\varphi(t)\right)dt$$

$$\le\frac{\omega_{f^{(q)}}(\delta)}{q!}\int_0^\infty\left|W^{(s)}{}_{n,v}(t,x)\right|\left(1+\frac{|t-x|}{\delta}\right)|t-x|^q dt$$

$$\le\frac{\omega_{f^{(q)}}(\delta)}{q!}\left[\frac{(n+r_2-1)}{(n)_{r_1}(n)_{r_2}}\sum_{k=v}^{\infty}\beta_{n,k,r_1}(x)\int_0^\infty\beta_{n,k-v,r_2}(t)(|t-x|^q+\delta^{-1}|t-x|^{q+1})dt\right.$$

$$\left.+\frac{1}{(n)_{r_1}}\sum_{k=0}^{v-1}\beta_{n,k,r_1}(x)(|x|^q+\delta^{-1}|x|^{q+1})\right],$$

Now, for $u=0,1,2,\ldots$using Schwartz inequalityfor integration and then summation, (2.2) and lemma 2.3 we have

$$\frac{(n+r_2-1)}{(n)_{r_1}(n)_{r_2}}\sum_{k=v}^{\infty}\beta_{n,k,r1}(x)|k-(n+r_1)x|^j\int_0^\infty\beta_{n,k-v,r_2}(t)|t-x|^u dt$$

$$\le\frac{(n+r_2-1)}{(n)_{r_1}(n)_{r_2}}\sum_{k=v}^{\infty}\beta_{n,k,r_1}(x)|k-(n+r_1)x|^j$$

$$\times\left\{\left(\int_0^\infty\beta_{n,k-v,r_2}(t)dt\right)^{1/2}\left(\int_0^\infty\beta_{n,k-v,r_2}(t)(t-x)^{2u}dt\right)^{1/2}\right\}$$

$\le\left(\sum_{k=v}^{\infty}\beta_{n,k,r_1}(x)|k-(n+r_1)x|^{2j}\right)^{1/2}$

$$\times\left(\frac{(n+r_2-1)}{(n)_{r_1}(n)_{r_2}}\sum_{k=v}^{\infty}\beta_{n,k,r1}(x)\int_0^{\infty}\beta_{n,k-v,r_2}(t)(t-x)^{2s}dt\right)^{1/2}$$

$= O\left(n^{j/2}\right)O\left(n^{-u/2}\right) = O\left(n^{(j-u)/2}\right)$, uniformly on$[a,b]$. (3.3)

Therefore, by (2.2),(2.4) andlemma 2.4,we get:

$$\frac{(n+r_2-1)}{(n)_{r_1}(n)_{r_2}}\sum_{k=v}^{\infty}\left|\beta_{n,k,r_1}^{(s)}(x)\right|\int_0^{\infty}\beta_{n,k-v,r_2}(t)|t-x|^u dt$$

$$\leq \frac{(n+r_2-1)}{(n)_{r_1}(n)_{r_2}}\sum_{k=v}^{\infty}\sum_{\substack{2i+j\leq s\\ i,j\geq 0}}|k-(n+r_1)x|\frac{\left|Q_{i,j,s}(x)\right|}{x^s(1+x)^s}\beta_{n,k,r_1}(t)\int_0^{\infty}\beta_{n,k-v,r_2}(t)|t-x|^u dt$$

$$\leq\left(\sup_{\substack{2i+j\leq r\\ i,j\leq 0}}\sup_{x\in[a,b]}\frac{\left|Q_{i,j,s}(x)\right|}{x^s(1+x)^s}\right)\sum_{\substack{2i+j\leq r\\ i,j\leq 0}}n^i\left(\sum_{k=v}^{\infty}\beta_{n,k,r_1}(x)|k\right.$$

$$\left.-(n+r_1)x|^j\int_0^{\infty}\beta_{n,k-v,r_2}(t)|t-x|^u dt\right).$$

$\leq C\sum_{\substack{2i+j\leq r\\ i,j\leq 0}} n^i O\left(n^{(j-u)/2}\right) = O\left(n^{(s-u)/2}\right)$, uniformly on $[a,b]$, (3.4)

where

$$M=\sup_{\substack{2i+j\leq r\\ i,j\leq 0}}\sup_{x\in[a,b]}\frac{\left|Q_{i,j,s}(x)\right|}{x^s(1+x)^s}.$$

By choosing $\delta = n^{-1/2}$ and making use of (3.4), we get

$$\|\Upsilon_2\|_{C[a,b]}\leq\frac{\omega_{f^{(q)}}\left(n^{-1/2}\right)}{q!}\left[O\left(n^{(s-q)/2}\right)+n^{1/2}O\left(n^{(s-q-1)/2}\right)+O(n^{-m})\right].$$

$$\leq Z_2 n^{-(q-s)/2}\omega_{f^{(q)}}\left(n^{-1/2}\right).\ (m>0)\ .$$

Since $t\in[0,\infty)\setminus(a-\tau,b+\tau)$, we can choose $\delta>0$ in such a way that $|t-x|\geq\delta$ for all $x\in[a,b]$. Thus, by lemma2.4, we get:

$$|\Upsilon_3|\leq\frac{(n+r_2-1)}{(n)_{r_1}(n)_{r_2}}\sum_{k=v}^{\infty}\sum_{\substack{2i+j\leq s\\ i,j\geq 0}}n^i|k$$

$$-(n+r_1)x|\frac{\left|Q_{i,j,s}(x)\right|}{x^s(1+x)^s}\beta_{n,k,r_1}(x)\int_{|t-x|\geq\delta}\beta_{n,k-v,r_2}(t)|g(t,x)|dt$$

$$+\sum_{k=0}^{v-1}\sum_{\substack{2i+j\leq s\\ i,j\geq 0}}n^i|k-(n+r_1)x|\frac{\left|Q_{i,j,s}(x)\right|}{x^s(1+x)^s}\beta_{n,k,r_1}(x)\,|g(0,x)|.$$

For$|t-x|\geq\delta$, we can find a constant K such that $|g(t,x)|\leq K|t-x|^{\sigma}$, where σ is any integer greater than or equal to $\{\propto,q\}$.

Now using Schwarz inequality for integration and then for summation, (2.2) and(2.4) it easyfollows that $\Upsilon_3 = O(n^{-u})$for any $u>0$,uniformlyon$[a,b]$.

Combining the estimates of Υ_1,Υ_2 and Υ_3, the required result is immediate.

References

[1] P. N. Agrawal and Kareem J. Thamer, Approximation of unbounded functions by a new sequence of linear positive operators, J. of Math. Anal. Appl. 225, 660-672, (1998)

[2] G.Doğru. Direct results for certain family of integral type operators,Journal of Inequal. in Pure and Appl. Math., Volume 6,Issue 1, Article21(2005).

[3]M. A.Gupta,M.S.Noor, Beniwal, and M.L.Gupta, On simultaneous approximation for certain BaskakovDurrmeyer type operators, J.Inequal. in Pure and Appl. Math.,Volume 7,Issue 4, Article 125,(2006).

[4]A. J. Mohammad and A. K. Hassan. Ageneralization of Baskakov operators of summation-integral- Phillips type form. Basrah J. of Sci. Vol.31(1), 28-4(2013).

[5] P.Maheshwair , Saturation theorem for combination of modified Beta operators in L_p-spaces, Appl.Math. E-Notes, 7(2007), 32-37.

[6] A. Sahai and G. Prasad. On simultaneous approximation by modified Lupas operators, J. of Approx.Theory,45(1985), 122-128.

[7] H.M. Srivastava and V. Gupta, A certain family of summation integral type operators, Math.Comput.Modelling, 37(2003),1307-1315.

تعميم عائلة المجموع– تكامل من نوع مؤثر باسكسكوف

علي جاسم محمد و صفاء عبد الشهيد و امل خليل حسين

قسم الرياضيات/ كلية التربية للعلوم الصرفة

قسم الرياضيات/ كلية العلوم / جامعة البصرة

المستخلص

في هذا البحث , قدمنا تعميم لعائلة المجموع– تكامل من نوع مؤثر باسكسكوف, أولا برهنا بأن هذه العائلة متقاربة بواسطة تطبيق مبرهنة كورفكن للمتسلسلة الجبرية لمؤثرات خطية موجبة. ثانيا ناقشنا صيغة ڤورونوفسكي للتقارب لهذه العائلة. أخيرا أوجدنا تخمين الخطأ لهذه العائلة باستخدام مقياس الاستمرارية.

الكلمات الرئيسية: المؤثر الخطي الموجب ,صيغة فورونوفسكي للتقارب ,التقارب النقطي ,مقياس الاستمرارية.