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The Voronovskaya Theorem for q-Analogue of Szasz-Mirakjan Operators

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Abstract.

In this present paper, we define q-Analogue sequence of Szasz-Mirakjan Operators $R_{n,q}(f,x)$ and introduce some direct results of these operators. First, we show that this sequence of operators $R_{n,q}(f,x)$ converges to $f(x)$ as n tends to ∞. Also, we defined the m^{th} order moment $T_{n,m}(x)$ for $R_{n,q}(f,x)$ and we then find a recurrence relation for $T_{n,m}(x)$. Finally ,we find and prove a Voronovskaya –type asymptotic formula of this operator.

Keywords: q –Szasz-Mirakjan Operators, linear positive operators, m^{th} order moment and Voronovskaya – type asymptotic formula.

1. Introduction

In 1987, Lupas [2] introduced and studied the first q-analogue of Bernstein operators which is defined as:

$$B_n(f,q;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1-x)_q^{n-k} f\left(\frac{[k]_q}{[n]_q}\right).$$

$0 \le x \le 1$ *and* $q \in (0,1)$.

In 1997, Rempulsa and Skorupka [5] introduced and studied new four sequences of linear positive operators which are

$$L_n^{(1)}(f;x) = \sum_{k=0}^{\infty} a_{n,k}(x) f\left(\frac{2k}{n}\right),$$

$$L_n^{(2)}(f;x) = \frac{n}{2}\sum_{k=0}^{\infty} a_{n,k}(x) \int_{I_{n,k}} f(t)dt\,,$$

$$L_n^{(3)}(f;x) = \frac{f(0)}{(1+\sinh(nx))} + \sum_{k=0}^{\infty} b_{n,k}(x) f\left(\frac{2k+1}{n}\right),$$

$$L_n^{(4)}(f;x) = \frac{f(0)}{(1+\sinh(nx))} + \frac{n}{2}\sum_{k=0}^{\infty} b_{n,k}(x) \int_{I_{n,k}^*} f(t)dt.$$

where $a_{n,k}(x) = \frac{1}{\cosh(nx)}\frac{(nx)^{2k}}{(2k)!}$, $b_{n,k}(x) = \frac{1}{(1+\sinh(nx))}\frac{(nx)^{2k+1}}{(2k+1)!}$, $x \in [0,\infty)$, $I_{n,k} = \left[\frac{2k}{n}, \frac{2k+2}{n}\right]$, $I_{n,k}^* = \left[\frac{2k+1}{n}, \frac{2k+3}{n}\right]$, $n \in N$ and $k \in N^0$.

In 2006, Aral and Gupta [1] introduced and studied q-generalization of Szasz-Mirakjan operators defined as:

$$S_n^q(f)(x) = E_q\left(-[n]\frac{x}{b_n}\right)\sum_{k=0}^{\infty}\frac{([n]x)^k}{[k]!\,(b_n)^k} f\left(\frac{[k]!\,b_n}{[n]}\right)$$

$0 \le x < \frac{b_n}{1-q^n}$, b_n is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = 0$.

In 2010, Salman [3] introduced modifications for the operators, $L_n^{(\lambda)}, \lambda = 1,2,3,4$ and studied Voronovskaya theorem for these modifications.

We mention some basic definitions and notations used in q-calculus, details can be found in [4] and [7].

$$[n]_q := 1 + q + \cdots q^{n-1} = \frac{1-q^n}{1-q}, \qquad (n = 1,2,\ldots), \quad [0]_q = 0$$

$[n]_q! := [1]_q[2]_q \ldots [n]_q$ for $n \in N$, and $[0]! = 1$.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!},$$

$$(t-x)_q^n = (t-x)_q(t-qx)_q(t-q^2x)_q \ldots (t-q^{n-1}x)_q.$$

The q-derivative of a function $f(x)$, denoted by $D_q f$, is defined by $\left(D_q f\right)(x) := \frac{f(qx)-f(x)}{(q-1)x}$, $x \neq 0$, and the higher q-derivatives as $D_q^0 f := f$,

$D_q^n f := D_q\left(D_q^{n-1} f\right)$, $\quad (n = 1,2,\ldots)$.

The formula for the q-derivative of a product and quotient are

$$D_q\big(f(x)g(x)\big) = f(qx)\,D_q(\,g(x)) + g(x)(D_q\,f(x)) \tag{1.1}$$

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)\,D_q\,f(x) - f(qx)\,D_q\,g(x)}{g(x)g(qx)}. \tag{1.2}$$

The q-analogue of the classical exponential function e^x is

$$e_q^x = \sum_{k=0}^{\infty}\frac{x^k}{[k]_q!} = \frac{1}{(1+(1-q)x)_q^{\infty}}, \quad E_q^x = \sum_{k=0}^{\infty} q^{k(k-1)/2}\frac{x^k}{[k]_q!} = (1-(1-q)x)_q^{\infty}$$

$$where\ (1-a)_q^{\infty} = \prod_{j=0}^{\infty}\left(1-q^j a\right).$$

The q-derivative of exponential function is defined by $D_q e_q^{nx} = [n]e_q^{nx}$, $D_q E_q^{nx} = [n]E_q^{qnx}$.

The q-analogues of the hyperbolic functions $(see[7])$.

$$cosh_q([n]\,x) = \frac{e_q^{nx}+e_q^{-nx}}{2} = \sum_{k=0}^{\infty}\frac{[n]^{2k}x^{2k}}{[2k]_q!};$$

$$sinh_q([n]\,x) = \frac{e_q^{nx}-e_q^{-nx}}{2} = \sum_{k=0}^{\infty}\frac{[n]^{2k+1}x^{2k+1}}{[2k+1]_q!};$$

$$tanh_q([n]\,x) = \frac{sinh_q([n]x)}{cosh_q([n]x)} = \frac{e_q^{nx}-e_q^{-nx}}{e_q^{nx}+e_q^{-nx}}.$$

In this paper, we introduce a new sequence of q- Szasz-Mirakjan type operators to approximation a function $f(x)$ which belongs to the space
$C_\alpha[0,\infty)=\{f\in C[0,\infty):|f(t)|\le C\,e_q(\alpha x)$ for some $C,\alpha>0$ and $q_n\in(0,1)\}$ as:

$$R_{n,q_n}(f,x)\equiv\sum_{k=0}^{\infty}a_{n,k}(q_n,x)\,f\left(\frac{[2k]_{q_n}}{[n]_{q_n}}\right) \qquad (1.3)$$

$$where\;\; a_{n,k}(q_n,x)=\frac{1}{cosh_{q_n}([n]x)}\frac{[n]_{q_n}^{2k}x^{2k}}{[2k]_{q_n!}}$$

We show that the sequence $R_{n,q}(f,x)$ converges to the function $f(x)$ as n tends to ∞ by applying the Korovkin's conditions [8]. Also, we define the m[th] order moment $T_{n,m}(x)$ for the operators $R_{n,q}(f,x)$ and then find a recurrence relation for $T_{n,m}(x)$. Finally ,we find and prove a Voronovskaya –type asymptotic formula of this operator.

Now, we begin to state some properties of the q- hyperbolic functions.

Lemma 1.1.

For $n\in N$ and $x\in(0,\infty)$ *we have*

$i)\; D_q\,cosh_q([n]\,x)=[n]\,sinh_q([n]\,x);$ (1.4)

$ii)\; D_q\,sinh_q([n]\,x)=[n]\,cosh_q([n]\,x);$ (1.5)

$iii)\; D_q\,tanh_q([n]\,x)=[n]\big(1-tanh_q([n]\,x)\,tanh_q([n]q\,x)\big).$ (1.6)

Proof:

Using (1.1), (1.2) and the direct computation, the above results follow immediately.

The next lemma gives us some properties of the weight functions $a_{n,k}(q,x)$ of q –Szasz-Mirakjan Operators.

Lemma 1.2:

For $x\in(0,\infty)$, we have:

1) $\displaystyle\sum_{k=0}^{\infty}a_{n,k}(q_n,x)=1;$

2) $\displaystyle\sum_{k=0}^{\infty}a_{n,k}(q_n,x)[2k]_{q_n}=[n]_{q_n}x\,tanh_{q_n}([n]x);$

3) $\displaystyle\sum_{k=0}^{\infty}a_{n,k}(q_n,x)[2k]_{q_n}^2=[n]_{q_n}\,x\,tanh_q([n]\,x)+q_n[n]_{q_n}^2x^2.$

Proof:

By direct computations ,we get :

1) $\displaystyle\sum_{k=0}^{\infty}a_{n,k}(q_n,x)=1$

2) $\displaystyle\sum_{k=0}^{\infty}a_{n,k}(q_n,x)\,[2k]_{q_n}$

$$=\frac{1}{cosh_{q_n}([n]x)}\sum_{k=1}^{\infty}\frac{[n]_{q_n}^{2k}x^{2k}}{[2k]_{q_n}!}[2k]_{q_n}$$

$$=\frac{[n]_{q_n}x}{cosh_{q_n}([n]x)}\sum_{k=1}^{\infty}\frac{[n]_{q_n}^{2k-1}x^{2k-1}}{[2k-1]_{q_n}!}$$

$$=\frac{[n]_{q_n}x}{cosh_{q_n}([n]x)}\sum_{k=0}^{\infty}\frac{[n]_{q_n}^{2k+1}x^{2k+1}}{[2k+1]_{q_n}!}$$

$$= \frac{[n]_{q_n} x}{cosh_{q_n}([n]x)} sinh_{q_n}([n]\, x)$$
$$= [n]_{q_n}\, x\, tanh_{q_n}([n]x).$$

3) $\sum_{k=0}^{\infty} a_{n,k}(q_n, x)\, [2k]_{q_n}^2$

$$= \frac{1}{cosh_{q_n}([n]x)} \sum_{k=1}^{\infty} \frac{[n]_{q_n}^{2k} x^{2k}}{[2k]_{q_n}!} [2k]_{q_n}^2$$
$$= \frac{[n]_{q_n} x}{cosh_{q_n}([n]x)} \sum_{k=1}^{\infty} \frac{[n]_{q_n}^{2k-1} x^{2k-1}}{[2k-1]_{q_n}!} [2k]_{q_n}$$

Using the fact $[2k]_q = 1 + q[2k-1]_q$, we get

$$\sum_{k=0}^{\infty} a_{n,k}(q_n, x)\, [2k]_{q_n}^2 = \frac{[n]_{q_n} x}{cosh_{q_n}([n]x)} \sum_{k=1}^{\infty} \frac{[n]_{q_n}^{2k-1} x^{2k-1}}{[2k-1]_{q_n}!}$$
$$+ q_n \frac{[n]_{q_n}^2 x^2}{cosh_{q_n}([n]x)} \sum_{k=1}^{\infty} \frac{[n]_{q_n}^{2k-2} x^{2k-2}}{[2k-1]_{q_n}!} [2k-1]_{q_n}$$
$$= \frac{[n]_{q_n} x}{cosh_{q_n}([n]x)} \sum_{k=0}^{\infty} \frac{[n]_{q_n}^{2k+1} x^{2k+1}}{[2k+1]_{q_n}!} + q_n \frac{[n]_{q_n}^2 x^2}{cosh_{q_n}([n]x)} \sum_{k=0}^{\infty} \frac{[n]_{q_n}^{2k} x^{2k}}{[2k]_{q_n}!}$$
$$= [n]_{q_n}\, x\, tanh_q([n]\, x) + q_n [n]_{q_n}^2 x^2. \qquad \square$$

The next theorem shows that, the convergence for the operators $R_{n,q_n}(t, x)$ to the function being approximated.

Theorem 1.1.

For any $f \in C_\alpha[0, \infty)$, we have $R_{n,q_n}(f, x) \to f(x) \quad as\ n \to \infty$.

Proof : By using Lemma 1.2 and direct computation, we get

$i)\ R_{n,q_n}(1, x) = 1 \qquad (1.7)$

$ii)\ R_{n,q_n}(t, x) = x\, tanh_{q_n}([n]x) \to x \quad as\ n \to \infty \qquad (1.8)$

$iii)\ R_{n,q_n}(t^2, x) = \frac{1}{[n]_{q_n}} x\, tanh_q([n]\, x) + q_n x^2 \to x^2\ as\ n \to \infty. \qquad (1.9)\ \square$

Lemma 2.2:

For the weight functions $a_{n,k}(q_n, x)$, we have the following properties:

$i)\ xD_{q_n} a_{n,k}(q_n, x)$

$$= [2k]_{q_n} a_{n,k}(q_n, x) - [n]_{q_n}\, x\, tanh_{q_n}([n]x)\, a_{n,k}(q_n, q_n x); \qquad (1.10)$$

$ii)$ Suppose that $\emptyset_{n,m}(q_n, x) = \sum_{k=0}^{\infty} a_{n.k}(q_n. x)[2k]_{q_n}^m$ then,

$$\emptyset_{n,m+1}(q_n, x) = xD_{q_n} \emptyset_{n,m}(q_n, x) + [n]_{q_n}\, x\, tanh_{q_n}([n]x)\, \emptyset_{n,m}(q_n, q_n x). \qquad (1.11)$$

Proof: Using (1.2), we have:

$i)\ a_{n,k}(q_n, x) = \frac{1}{cosh_{q_n}([n]x)} \frac{[n]_{q_n}^{2k} x^{2k}}{[2k]_{q_n}!}$

$D_{q_n} a_{n,k}(q_n, x)$

$$= \frac{[n]_{q_n}^{2k}}{[2k]_{q_n}!} \left(\frac{cosh_{q_n}([n]q_n x)\, [2k]_{q_n} x^{2k-1} - (q_n x)^{2k} [n]_{q_n} sinh_{q_n}([n]x)}{cosh_{q_n}([n]x)\, cosh_{q_n}([n]q_n x)} \right)$$

$$D_{q_n} a_{n,k}(q_n, x) = \frac{[n]_{q_n}^{2k} [2k]_{q_n} x^{2k-1}}{cosh_{q_n}([n]x)\, [2k]_{q_n}!} - \frac{[n]_{q_n}^{2k} (q_n x)^{2k} [n]_{q_n}}{cosh_{q_n}([n]q_n x)\, [2k]_{q_n}!} tanh_q([n]\, x)$$

$$D_{q_n} a_{n,k}(q_n, x) = \frac{[2k]_{q_n}}{x} a_{n,k}(q_n, x) - [n]_{q_n} tanh_{q_n}([n]x)\, a_{n,k}(q_n, q_n x).$$

Hence,

$x\, D_{q_n}\, a_{n,k}(q_n,x) = [2k]_{q_n}\, a_{n,k}(q_n,x) - [n]_{q_n} x\, tanh_{q_n}([n]x)\, a_{n,k}(q_n,q_nx).$ □

Now, we are going to prove (ii) by using the previous property (i) as:

$$D_{q_n}\, \emptyset_{n,m}(q_n,x) = \sum_{k=0}^{\infty} D_{q_n}\, a_{n.k}(q_n,x)[2k]_{q_n}^{m}$$

$$xD_{q_n}\, \emptyset_{n,m}(q_n,x) = \sum_{k=0}^{\infty} a_{n.k}(q_n.x)[2k]_{q_n}^{m+1}$$

$$- [n]_{q_n}\, x\, tanh_{q_n}([n]x) \sum_{k=0}^{\infty} a_{n,k}(q_n,q_nx)[2k]_{q_n}^{m}$$

$\emptyset_{n,m+1}(q_n,x) = xD_{q_n}\, \emptyset_{n,m}(q_n,x) + [n]_{q_n}\, x\, tanh_{q_n}([n]x)\, \emptyset_{n,m}(q_n,q_nx).$ □

2- The m^{th} Order Moment for $R_{n,q_n}(f,x)$.

For $m \in N$, the mth order moment of $R_{n,q_n}(f;x)$ is denoted by

$$T_{n,m}(x) = R_{n,q_n}\left((t-x)_{q_n}^{m};x\right) = \sum_{k=0}^{\infty} a_{n,k}(q_n,x)(t-x)_{q_n}^{m}$$

Theorem 2.1:

For the function $T_{n,m}(x)$, we have

1) $T_{n,0}(x) = 1$ (2.1)

2) $T_{n,1}(x) = x\, tanh_{q_n}([n]x) - x$ (2.2)

3) $T_{n,2}(x) = \dfrac{x}{[n]_{q_n}} tanh_{q_n}([n]x) - [2]_{q_n} x^2\, tanh_{q_n}([n]x) + 2q_nx^2.$ (2.3)

Proof:

Using Theorem 1.1 and direct computation we get:

1) $T_{n,0}(x) = 1$

2) $T_{n,1}(x) = R_{n,q_n}(t;x) - xR_{n,q_n}(1;x)$

$= x\, tanh_{q_n}([n]x) - x$

3) $T_{n,2}(x) = R_{n,q_n}(t^2;x) - q_nxR_{n,q_n}(t;x) - xR_{n,q_n}(t;x) + q_nx^2R_{n,q_n}(1;x)$

$$= \frac{x}{[n]_{q_n}} tanh_{q_n}([n]x) + q_nx^2 - q_nx^2\, tanh_{q_n}([n]x) - x^2\, tanh_{q_n}([n]x) + q_nx^2$$

$$= \frac{x}{[n]_{q_n}} tanh_{q_n}([n]x) - [2]_{q_n}x^2\, tanh_{q_n}([n]x) + 2q_nx^2.$$

Furthermore, for $m \geq 1$, we have the following recurrence relation

$$[n]_{q_n}\, T_{n,m+1}(x) = x\, D_{q_n}\, T_{n,m}(x) - [n]_{q_n} q_n^m x\, T_{n,m}(x) + [n]_{q_n} x\, tanh_{q_n}([n]x)\, T_{n,m}(q_nx) + ([m]x - [n]_{q_n} x^2 tanh_{q_n}([n]x)(1-q_n^m))\, T_{n,m-1}(q_nx). \quad (2.4)$$

To prove result (2.4) using the identities (1.1)and (1.10), we have

$$xD_{q_n}\, T_{n,m}(x) = -[m]x \sum_{k=0}^{\infty} a_{n,k}(q_n,q_nx)\,(t-x)_{q_n}^{m-1} + \sum_{k=0}^{\infty} (t-x)_{q_n}^{m} D_{q_n}\, a_{n,k}(q_n,x)$$

$$xD_{q_n}\, T_{n,m}(x) + [m]x\, T_{n,m-1}(q_nx)$$

$$= \sum_{k=0}^{\infty} \left([2k]_{q_n}\, a_{n,k}(q_n,x) - [n]_{q_n}\, x\, tanh_{q_n}([n]x)\, a_{n,k}(q_n,q_nx)\right)(t-x)_{q_n}^{m}.$$

Using the following facts

$$(t-x)_q^m = (t-x)_q(t-qx)_q^{m-1} = (t-qx)_q^m - x(1-q^m)(t-qx)_q^{m-1}$$

we get,

$$xD_{q_n}T_{n,m}(x)+[m]xT_{n,m-1}(q_nx)=[n]_{q_n}\sum_{k=0}^{\infty}a_{n,k}(q_n,x)(t-q_n^mx)\,(t-x)^m$$

$$+[n]_{q_n}q_n^mx\sum_{k=0}^{\infty}a_{n,k}(q_n,x)\,(t-x)^m-[n]_{q_n}x\,tanh_{q_n}([n]x)\sum_{k=0}^{\infty}a_{n,k}(q_n,q_nx)$$

$$(t-q_nx)^m+[n]_{q_n}x^2\,tanh_{q_n}([n]x)(1-q_n^m)\sum_{k=0}^{\infty}a_{n,k}(q_n,q_nx)(t-q_nx)^{m-1}$$

$$xD_{q_n}T_{n,m}(x)+[m]x\,T_{n,m-1}(q_nx)=[n]_{q_n}T_{n,m+1}(x)+[n]_{q_n}q_n^mxT_{n,m}(x)$$
$$-[n]_{q_n}x\,tanh_{q_n}([n]x)\,T_{n,m}(q_nx)+[n]_{q_n}x^2tanh_{q_n}([n]x)(1-q_n^m)\,T_{n,m-1}(q_nx)$$

Hence,

$$[n]_{q_n}T_{n,m+1}(x)=x\,D_{q_n}T_{n,m}(x)-[n]_{q_n}q_n^mx\,T_{n,m}(x)$$
$$+[n]_{q_n}x\,tanh_{q_n}([n]x)\,T_{n,m}(q_nx)+([m]x$$
$$-[n]_{q_n}x^2tanh_{q_n}([n]x)(1-q_n^m))\,T_{n,m-1}(q_nx).$$

This completes the consequence (2.4). □

3. Voronovskaya type-asymptotic formula.

The next theorem gives the error that occurs by the approximation of the function f to the operator $R_{n,q_n}(f,x)$. It turns out that the degree of approximation is $O([n]^{-1})$.

Theorem 3.1. Let $f\in C_\alpha[0,\infty)$, and $D_{q_n}^2f(x)$ exists and be continuous at a point $x\in(0,\infty)$, then

$$\lim_{n\to\infty}[n]_{q_n}\big(R_{n,q_n}(f;x)-f(x)\big)=\frac{x}{2}f''(x). \tag{3.1}$$

Proof. By q −Taylor's formula for f, we have

$$f(t)=f(x)+D_{q_n}f(x)(t-x)+\frac{1}{[2]_{q_n}!}D_{q_n}^2f(x)(t-x)_{q_n}^2$$
$$+\varepsilon_{q_n}(t;x)(t-x)_{q_n}^2 \tag{3.2}$$

where $\varepsilon_{q_n}(t;x)\to 0$ as $t\to x$.

Using (2.2), (2.3) and (3.2), we get

$$\mathrm{R}_{n,q_n}(f(t),x)=f(x)+D_{q_n}f(x)\mathrm{R}_{n,q_n}\big((t-x),x\big)$$
$$+\frac{1}{[2]_{q_n}!}D_{q_n}^2f(x)\mathrm{R}_{n,q_n}\big((t-x)_{q_n}^2,x\big)+\mathrm{R}_{n,q_n}\big(\varepsilon_{q_n}(t;x)(t-x)_{q_n}^2,x\big).$$

Using Theorem 2.1, we get

$$\mathrm{R}_{n,q_n}(f(t),x)=f(x)+D_{q_n}f(x)T_{n,1}(x)+\frac{1}{[2]_{q_n}!}D_{q_n}^2f(x)T_{n,2}(x)$$
$$+\mathrm{R}_{n,q_n}\big(\varepsilon_{q_n}(t;x)(t-x)_{q_n}^2,x\big)$$

$$\lim_{n\to\infty}[n]_{q_n}\big(\mathrm{R}_{n,q_n}(f;x)-f(x)\big)=\lim_{n\to\infty}[n]_{q_n}\Big(D_{q_n}f(x)T_{n,1}(x)\Big)+\lim_{n\to\infty}[n]_{q_n}$$
$$\times\left(\frac{1}{[2]_{q_n}!}D_{q_n}^2f(x)T_{n,2}(x)\right)+\lim_{n\to\infty}[n]_{q_n}\Big(\mathrm{R}_{n,q_n}\big(\varepsilon_{q_n}(t;x)(t-x)_{q_n}^2,x\big)\Big)$$

$$\lim_{n\to\infty}[n]_{q_n}\big(\mathrm{R}_{n,q_n}(f;x)-f(x)\big)=f'(x)\lim_{n\to\infty}[n]_{q_n}x\big(tanh_{q_n}([n]x)-1\big)$$
$$+\frac{1}{2}f''(x)\lim_{n\to\infty}[n]_{q_n}\left(\frac{x}{[n]_{q_n}}tanh_{q_n}([n]x)-\big([2]_{q_n}tanh_{q_n}([n]x)-2q_n\big)x^2\right)$$
$$+\lim_{n\to\infty}[n]_{q_n}E_{n,q_n}(x).$$

Hence,

$$\lim_{n\to\infty}[n]_{q_n}\left(R_{n,q_n}(f;x)-f(x)\right)=\frac{x}{2}f''(x)+\lim_{n\to\infty}[n]_{q_n}E_{n,q_n}(x).$$

To complete the proof of our consequence it is sufficient to prove that

$$\lim_{n\to\infty}[n]_{q_n}E_{n,q_n}(x)=0.$$

Now,

$$E_{n,q_n}(x)=R_{n,q_n}\left(\varepsilon_{q_n}(t;x)(t-x)_{q_n}^2,x\right)=\sum_{k=0}^{\infty}a_{n,k}(q_n,x)\,\varepsilon_{q_n}(t;x)(t-x)_{q_n}^2$$

Now, we show that $\lim_{n\to\infty}[n]_{q_n}E_{n,q_n}(x)=0$

$$[n]_{q_n}\left|E_{n,q_n}(x)\right|\le[n]_{q_n}\sum_{|t-x|<\delta}a_{n,k}(q_n,x)\left|\varepsilon_{q_n}(t;x)(t-x)_{q_n}^2\right|$$
$$+[n]_{q_n}\sum_{|t-x|>\delta}a_{n,k}(q_n,x)\left|\varepsilon_{q_n}(t;x)(t-x)_{q_n}^2\right|.$$

Let $[n]_{q_n}E_{n,q_n}(x)=I_1+I_2$.

Since $\varepsilon_{q_n}(t;x)\to 0$ as $t\to x$, then for given $\varepsilon>0$, there exists a $\delta>0$ such that $\left|\varepsilon_{q_n}(t;x)\right|<\varepsilon$, whenever $0<|t-x|<\delta$

$$I_1\le[n]_{q_n}\sum_{|t-x|<\delta}a_{n,k}(q_n,x)\left|\varepsilon_{q_n}(t;x)(t-x)_{q_n}^2\right|$$
$$\le\varepsilon[n]_{q_n}\sum_{|t-x|<\delta}a_{n,k}(q_n,x)\,(t-x)_{q_n}^2$$
$$\le\varepsilon[n]_{q_n}T_{n,2}(x).$$

Since $\varepsilon>0$ is an arbitrary real the number , it follows that

$I_1=o(1)$ as $n\to\infty$.

If $|t-x|>\delta$, then $\left|\varepsilon_{q_n}(t;x)(t-x)_{q_n}^2\right|\le M|t-x|_{q_n}^{\alpha}$ for some constant $M>0$

$$I_2\le[n]_{q_n}\sum_{|t-x|\ge\delta}a_{n,k}(q_n,x)\left|\varepsilon_{q_n}(t;x)(t-x)_{q_n}^2\right|$$
$$\le[n]_{q_n}M\sum_{|t-x|\ge\delta}a_{n,k}(q_n,x)|t-x|_{q_n}^{\alpha}.$$

By the Holder's inequality, we have

$$I_2\le[n]_{q_n}M\left(\sum_{|t-x|\ge\delta}a_{n,k}(q_n,x)\right)^{1/2}\left(\sum_{|t-x|\ge\delta}a_{n,k}(q_n,x)|t-x|_{q_n}^{2\alpha}\right)^{1/2}$$
$$\le[n]_{q_n}M\left(\sum_{|t-x|\ge\delta}a_{n,k}(q_n,x)|t-x|_{q_n}^{4}\right)^{1/2}$$
$$\le[n]_{q_n}M\left|T_{n,4}(x)\right|^{1/2}.$$

By the direct computation and (2.4) we have

$$T_{n,3}(x)=q_n^3x^3\left(tanh_{q_n}([n]x)-1\right)+q_n[3]_{q_n}x^3\left(tanh_{q_n}([n]x)-1\right)$$
$$+\frac{x^2}{[n]_{q_n}}\left(q_n\left(1+[2]_{q_n}\right)-[3]_{q_n}\right)$$

$$T_{n,4}(x) = q_n^2 x^4 \left(2q_n^4 + [5]_{q_n} + q_n^2 - 2q_n [4]_{q_n} tanh_{q_n}([n]x)\right) + \frac{q_n}{[n]} x^3 \left(q_n^2 \left(1 + [2]_{q_n} + [3]_{q_n}\right) tanh_{q_n}([n]x) + \left([5]_{q_n} + q_n^2\right) tanh_q([n]\, x) - [4]_{q_n}\left(1 + [2]_{q_n}\right)\right) + \frac{1}{[n]_{q_n}^2} x^2 \left(q_n\left([2]_{q_n}\left(1 + [2]_{q_n}\right) + 1\right) - [4]_{q_n} tanh_{q_n}([n]x)\right) + \frac{1}{[n]_{q_n}^3} x \, tanh_q([n]\, x)$$

$I_2 = o(1)$ as $n \to \infty$. Combining the result of I_1 and I_2, our theorem is hold. □

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مبرهنة روسكي من النمط q لمؤثرات زار–ماركجان

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المستخلص:
في بحثنا هذا، سنعرف متتابعة من المؤثرات Szasz-Mirakjan من النمط q هي $R_{n,q}(f,x)$ ونقدم بعض النتائج المباشرة لهذه المؤثرات. أولا، سنبين ان هذه المتتابعة من المؤثرات $R_{n,q}(f,x)$ تتقارب للدالة $f(x)$ عندما n تقترب ∞ . ايضا، عرفنا العزم من الرتبة m ، $T_{n,m}(x)$ للمؤثرات $R_{n,q}(f,x)$ و منها وجدنا الصيغة التكرارية لها. اخيرا، وجدنا وبرهنا الصيغة المشابهه من نمط –Voronovskaya لهذه المؤثرات.

المفاتيح الدلالية: مؤثرات زار-ماركجان من النمط q ، المؤثرات الخطية الموجبة، العزم من الرتبة m و الصيغة المشابهه لصيغة روسكي