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On the Iterative Combination of Integral Baskakov-Type operators

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ABSTRACT

In [6] Micchelli had introduced a technique of iterative combination to improve the order of approximation by Bernstein polynomials. In the present paper, we have used his technique to improve the order of approximation by a new sequence of linear positive operators introduced by Agrawal and Thamer in [2] called the integral Baskakov-type operators.

1. INTRODUCTION

Agrawal and Thamer [2] introduced a new sequence of linear positive operators M_n as given below:

Let $\alpha > 0$ and $f \in C_\alpha[0,\infty) := \left\{ f \in C[0,\infty) : |f(t)| \le M\,(1+t)^\alpha \text{ for some } M > 0 \right\}$.

Then,

$$M_n(f(t);x) = (n-1)\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^\infty p_{n,\nu-1}(t) f(t)\,dt + (1+x)^{-n} f(0), \qquad (1.1)$$

where $p_{n,\nu}(x) = \binom{n+\nu-1}{\nu} x^\nu (1+x)^{-n-\nu}$ and $x \in [0,\infty)$.

The space $C_\alpha[0,\infty)$ is normed by $\|f\|_{C_\alpha} := \sup\limits_{0\le t<\infty} |f(t)|(1+t)^{-\alpha}$, $f \in C_\alpha[0,\infty)$. Alternatively, the operators (1.1) may be written as

$$M_n(f(t);x) = \int_0^\infty W_n(t,x) f(t)\,dt,$$

where the kernel

$$W_n(t,x) = (n-1)\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\, p_{n,\nu-1}(t) + (1+x)^{-n}\,\delta(t),$$

$\delta(t)$ being the Dirac-delta function.

The order of approximation by the operators (1.1) is, at best $O(n^{-1})$ whatsoever smooth the function may be. Therefore, in order to improve the rate of convergence $O(n^{-1})$ by these operators, the technique of linear combination introduced by May [5] and Rathore [7] has been used [3]. There is yet another approach for improving the order of approximation, which was offered by Micchelli [6] by considering the iterative combinations $U_{n,k} = I - (I - B_n)^k$ of the Bernstein polynomials B_n, where $k \in N$ (the set of positive integers). He proved some direct and saturation results for these operators $U_{n,k}$ using semi-group method. Agrawal [1] obtained an inverse theorem in simultaneous approximation for the operators $U_{n,k}$.

In the present paper, we have considered Micchelli combination for the operators (1.1) and proved some direct results concerning the degree of approximation.

The iterates of the operator M_n are defined by

$$M_n^0 = I \text{ and } M_n^k = M_n(M_n^{k-1}),\ k \in N.$$

Now, we define the operators $L_{n,k} : C_\alpha[0,\infty) \to C^\infty[0,\infty)$ (the class of infinitely differentiable functions on $[0,\infty)$) as:

$$L_{n,k}(f(t);x) = \left(I - (I - M_n)^k\right)(f(t);x) = \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} M_n^r (f(t);x). \quad (1.2)$$

Let $m \in N$ and $0 < a < b < \infty$, for sufficient small values of $\eta > 0$, the m-th order modulus of continuity $\omega_m(f,\eta;[a,b])$ for a continuous function f on the interval $[a,b]$ is defined as:

$$\omega_m(f,\eta;[a,b]) = \sup\left\{\left|\Delta_h^m f(x)\right| : |h| \le \eta,\ x, x+mh \in [a,b]\right\},$$

where $\Delta_h^m f(x)$ is the m-th order forward difference with step length h. For $m=1$, $\omega_m(f,\eta;[a,b])$ is written simply as $\omega_f(\eta;[a,b])$ or $\omega(f,\eta;[a,b])$.

Throughout this paper, we denote by $C[a,b]$ the space of all continuous functions on the interval $[a,b]$, $\|.\|_{C[a,b]}$ the sup-norm on the space $C[a,b]$, $0 < a_1 < a_2 < b_2 < b_1 < \infty$, $I_i = [a_i, b_i]$, $i = 1, 2$ and C denotes a constant not necessarily the same in different cases.

2. PRELIMINARIES

In the sequel, we shall require the following results:

For $f \in C_\alpha[0,\infty)$, $\eta > 0$ and $m \in N$, the Steklov mean $f_{\eta,m}$ of m-th order corresponding to f is defined by:

$$f_{\eta,m}(x) = \eta^{-m} \left(\int_{-\eta/2}^{\eta/2}\right)^m \left\{ f(x) + (-1)^{m-1} \Delta^m_{\sum_{i=1}^{m} x_i} f(x) \right\} \prod_{i=1}^{m} dx_i\ ,\ x \in I_1.$$

Lemma 1 [8]**.** For the function $f_{\eta,m}(t)$ defined above, we have

(a) $f_{\eta,m}(t)$ has derivatives upto order m over I_1;

(b) $\left\|f_{\eta,m}^{(r)}\right\|_{C(I_2)} \le C_r\, \eta^{-r}\, \omega_r(f,\eta;I_1)$, $r = 1, 2, \ldots, m$;

(c) $\left\| f - f_{\eta,m} \right\|_{C(I_2)} \le C_{m+1} \ \omega_m(f,\eta;I_1)$;

(d) $\left\| f_{\eta,m} \right\|_{C(I_2)} \le C_{m+2} \left\| f \right\|_{C_\alpha}$;

(e) $\left\| f_{\eta,m}^{(m)} \right\|_{C(I_2)} \le C_{m+3} \ \left\| f \right\|_{C(I_1)}$,

where C_i's are certain constants that depend on i but are independent of f and η.

Let the *m*-th order moment for the operators (1.1) be defined by:

$$T_{n,m}(x) = M_n((t-x)^m;x) = (n-1)\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty} p_{n,\nu-1}(t)(t-x)^m\,dt + (-x)^m(1+x)^{-n} .$$

Lemma 2 [2]. For the function $T_{n,m}(x)$, there follow

$T_{n,0}(x) = 1$, $T_{n,1}(x) = \dfrac{2x}{n-2}$ and

$$(n-m-2)T_{n,m+1}(x) = x(1+x)\,T'_{n,m}(x) + [(2x+1)m + 2x]T_{n,m}(x)$$
$$+ 2mx(1+x)T_{n,m-1}(x),\ n > m-2$$

Further, we have the following consequences of $T_{n,m}(x)$:

(i) $T_{n,m}(x)$ is polynomial in x of degree m;

(ii) For every $x \in [0,\infty)$, $T_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$, where $[(m+1)/2]$ denotes the integer part of $(m+1)/2$;

(iii) The coefficients of n^{-k} in $T_{n,2k}(x)$ and $T_{n,2k-1}(x)$ are $\dfrac{(2k)!\ \{x(x+2)\}^k}{k!}$ and $\dfrac{(2k-1)!}{(k-1)!}\{k(1+2k)-1\}\ \{x(x+2)\}^{k-1}$ respectively.

Lemma 3. [3] Let δ and γ be any positive real numbers. Then for any $m > 0$ we have:

$$\left\| \int_{|t-x|\ge\delta} W_n(t,x)\ t^\gamma\ dt \right\|_{C(I_1)} = O(n^{-m}) .$$

For every $m \in N^0$, the *m*-th order moment $T_{n,m}^{\{p\}}$ for the operator M_n^p where $p \in N$, is defined by $T_{n,m}^{\{p\}}(x) = M_n^p((t-x)^m;x)$. We denote $T_{n,m}^{\{1\}}(x)$ by $T_{n,m}(x)$.

Lemma 4. There holds the recurrence relation

$$T_{n,m}^{\{p+1\}}(x)=\sum_{j=0}^{m}\binom{m}{j}\sum_{i=0}^{m-j}T_{n,i+j}(x)\,\frac{D^i}{i!}\left(T_{n,m-j}^{\{p\}}(x)\right), \tag{2.1}$$

where $D\equiv\dfrac{d}{dx}$.

Proof. By the definition, we have

$$T_{n,m}^{\{p+1\}}(x)=M_n\left(M_n^p((t-x)^m;u);x\right)=M_n\left(M_n^p((t-u+u-x)^m;u);x\right)$$

$$=\sum_{j=0}^{m}\binom{m}{j}M_n\left((u-x)^j M_n^p((t-u)^{m-j};u);x\right)$$

since $M_n^p((t-u)^{m-j};u)$ is a polynomial in u of degree $\le m-j$, by Taylor's expansion, we can write

$$M_n^p\left((t-u)^{m-j};u\right)=\sum_{i=0}^{m-j}\frac{(u-x)^i}{i!}D^i\left(T_{n,m-j}^{\{p\}}(x)\right).$$

Hence, the equation (2.1) is immediate. ■

Lemma 5. For every $x\in[0,\infty)$, we have

$$T_{n,m}^{\{p\}}(x)=O(n^{-[(m+1)/2]}), \tag{2.2}$$

where $[(m+1)/2]$ denotes the integer part of $(m+1)/2$.

Proof. We prove (2.2) by induction on p. For $p=1$, the result holds from Lemma 2. Suppose the result is true for p, we shall prove it for $p+1$. Now, $T_{n,m-j}^{\{p\}}(x)=O(n^{-[(m-j+1)/2]})$ and $T_{n,m-j}^{\{p\}}(x)$ is polynomial in x of degree $\le m-j$, it follows that

$$D^i\left(T_{n,m-j}^{\{p\}}(x)\right)=O(n^{-[(m-j+1)/2]}), \text{ for every } i.$$

Now, by using Lemma 4, we get

$$T_{n,m}^{\{p+1\}}(x)=O\left(\sum_{j=0}^{m}\sum_{i=0}^{m-j}n^{-[(m-j+1)/2]-[(i+j+1)/2]}\right)$$

$$=O\left(\sum_{j=0}^{m}\sum_{i=0}^{m-j}n^{-[(m+i+1)/2]}\right).$$

Therefore, by the induction hypothesis, we obtain the result (2.2). ■

Lemma 6. For *m*-th order moment ($m\in N$) of the operators $L_{n,k}$ defined in (1.2) we find that

$$L_{n,k}((t-x)^m;x)=O(n^{-k}).$$

Proof. We prove by induction on k. First, for $k=1$, the result follows from Lemma 2. Suppose the result is true for k, we shall prove it for $k+1$.

$$L_{n,k+1}((t-x)^m;x)=\sum_{r=1}^{k+1}(-1)^{r+1}\binom{k+1}{r}T_{n,m}^{\{r\}}(x)$$

$$=\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}T_{n,m}^{\{r\}}(x)+\sum_{r=1}^{k+1}(-1)^{r+1}\binom{k}{r-1}T_{n,m}^{\{r\}}(x):=\Sigma_1+\Sigma_2\ .$$

Clearly, $\Sigma_1=L_{n,k}((t-x)^m;x)$. Now, using Lemma 4, we have

$$\Sigma_2=-\sum_{r=0}^{k}(-1)^{r+1}\binom{k}{r}T_{n,m}^{\{r+1\}}(x)$$

$$=-\sum_{j=1}^{m-1}\binom{m}{j}\sum_{i=0}^{m-j}T_{n,i+j}(x)\ \frac{D^i}{i!}\Big(L_{n,k}\big((t-x)^{m-j};x\big)\Big)$$

$$-\sum_{i=1}^{m}T_{n,i}(x)\ \frac{D^i}{i!}\Big(L_{n,k}\big((t-x)^m;x\big)\Big)-L_{n,k}\Big((t-x)^m;x\Big).$$

Therefore,

$$\Sigma_1+\Sigma_2=-\sum_{j=1}^{m-1}\binom{m}{j}\sum_{i=0}^{m-j}T_{n,i+j}(x)\ \frac{D^i}{i!}\Big(L_{n,k}\big((t-x)^{m-j};x\big)\Big)$$

$$-\sum_{i=1}^{m}T_{n,i}(x)\frac{D^i}{i!}\Big(L_{n,k}\big((t-x)^m;x\big)\Big)$$

$$=O(n^{-(k+1)})\ .$$

Hence, the required result follows. ■

3. MAIN RESULTS

First, we establish a Voronoskaja-type asymptotic formula for the operators $L_{n,k}$.

Theorem 1. Let $f\in C_\alpha[0,\infty)$. If $f^{(2k)}$ exists at a point $x\in[0,\infty)$ then

$$\lim_{n\to\infty}n^k\big\{L_{n,k}(f;x)-f(x)\big\}=\sum_{i=2}^{2k}\frac{f^{(i)}(x)}{i!}Q(i,k,x)\ , \qquad (3.1)$$

and

$$\lim_{n\to\infty}n^k\big\{L_{n,k+1}(f;x)-f(x)\big\}=0\ , \qquad (3.2)$$

where $Q(i,k,x)$ are certain polynomials in x of degree at most i.

Further, if $f^{(2k-1)}$ exists and is absolutely continuous over the interval $[0,b]$ and $f^{(2k)} \in L_\infty[0,b]$, then for any $[c,d] \subset (0,b)$ there holds

$$\left\| L_{n,k}(f;x) - f(x) \right\|_{C[c,d]} \le C\, n^{-k} \left\{ \|f\|_{C_\alpha} + \left\| f^{(2k)} \right\|_{L_\infty[0,b]} \right\}. \tag{3.3}$$

Proof. By Taylor's expansion of f, we have

$$f(t) = \sum_{j=0}^{2k} \frac{f^{(j)}(x)}{j!}(t-x)^j + \varepsilon(t,x)(t-x)^{2k},$$

where $\varepsilon(t,x) \to 0$ as $t \to x$ and $|\varepsilon(t,x)| \le C(1+t)^\alpha$ for some $C > 0$. Therefore,

$$n^k \left\{ L_{n,k}(f(t);x) - f(x) \right\} = n^k \sum_{j=1}^{2k} \frac{f^{(j)}(x)}{j!} L_{n,k}\left((t-x)^j;x\right)$$

$$+ n^k \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} M_n^r \left(\varepsilon(t,x)(t-x)^{2k};x \right) := \sum\nolimits_1 + \sum\nolimits_2 .$$

Using Lemma 6, we have

$$\sum\nolimits_1 = n^k \sum_{j=2}^{2k} \frac{f^{(j)}(x)}{j!} L_{n,k}\left((t-x)^j;x\right) = \sum_{j=2}^{2k} \frac{f^{(j)}(x)}{j!} Q(j,k,x) + o(1).$$

Since $\varepsilon(t,x) \to 0$ as $t \to x$, thus for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\varepsilon(t,x)| < \varepsilon$ whenever $|t-x| < \delta$. Suppose that $\phi_\delta(t)$ denotes the characteristic function of the interval $(x-\delta, x+\delta)$, then

$$\left| \sum\nolimits_2 \right| \le n^k \sum_{r=1}^{k} \binom{k}{r} M_n^r \left(|\varepsilon(t,x)| (t-x)^{2k} \phi_\delta(t);x \right)$$

$$+ n^k \sum_{r=1}^{k} \binom{k}{r} M_n^r \left(|\varepsilon(t,x)| (t-x)^{2k} (1-\phi_\delta(t));x \right) := J_1 + J_2 .$$

To estimate J_1, applying Lemma 5 we have

$$J_1 \le \varepsilon\, n^k \sum_{r=1}^{k} \binom{k}{r} M_n^r \left((t-x)^{2k};x \right) < \varepsilon\, C .$$

For an arbitrary $s > 0$, applying Lemma 3 we have

$$|J_2| \le n^k \sum_{r=1}^{k} \binom{k}{r} M_n^r \left(C(1+t)^\alpha\ (t-x)^{2k} (1-\phi_\delta(t));x \right) < \frac{C}{n^s} = o(1).$$

Since, $\varepsilon > 0$ is arbitrary, thus $\sum_2 \to 0$ as $n \to \infty$. Combining the estimates of $\sum_1$ and $\sum_2$, we obtain (3.1).

The equation (3.2) can be proved along similar lines by noting the fact that

$$L_{n,k+1}((t-x)^j; x) = O(n^{-(k+1)}), \ \forall \ j \in N .$$

Now, we shall prove (3.3). For this purpose let ψ be the characteristic function of $[0,b]$. Thus,

$$L_{n,k}(f(t);x) - f(x) = L_{n,k}(\psi(t)(f(t)-f(x));x)$$
$$+ L_{n,k}((1-\psi(t))(f(t)-f(x));x) := \sum_3 + \sum_4 .$$

The estimate of $\sum_4$ can be found in a manner similar to the estimate of J_2. Thus, we have for all $x \in [c,d]$

$$\sum_4 \le C\, n^{-k} \|f\|_{C_\alpha} .$$

For $t \in [0,b]$ and $x \in [c,d]$, by our hypothesis of f, we can write f

$$f(t) - f(x) = \sum_{i=1}^{2k-1} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{1}{(2k-1)!}\int_x^t (t-w)^{2k-1} f^{(2k)}(w)\,dw .$$

Thus,

$$\sum_3 = \sum_{i=1}^{2k-1} \frac{f^{(i)}(x)}{i!} L_{n,k}\left(\psi(t)(t-x)^i; x\right)$$
$$+ \frac{1}{(2k-1)!} L_{n,k}\left(\psi(t)\int_x^t (t-w)^{2k-1} f^{(2k)}(w)\,dw; x\right)$$
$$= \sum_{i=1}^{2k-1} \frac{f^{(i)}(x)}{i!}\left\{L_{n,k}\left((t-x)^i; x\right) + L_{n,k}\left((\psi(t)-1)(t-x)^i; x\right)\right\}$$
$$+ \frac{1}{(2k-1)!} L_{n,k}\left(\psi(t)\int_x^t (t-w)^{2k-1} f^{(2k)}(w)\,dw; x\right)$$
$$:= \sum_{i=1}^{2k-1} \frac{f^{(i)}(x)}{i!}\{J_3 + J_4\} + J_5 . \tag{3.4}$$

By Lemma 6, we get $J_3 = O(n^{-k})$ uniformly for $x \in [c,d]$.

Proceeding as in the estimate of J_2 and taking account of the fact that $x \in [c,d]$

$$J_4 = O(n^{-k}) .$$

By (1.2) and (2.2), we have

$$J_5 = \left\| f^{(2k)} \right\|_{L_\infty[0,b]} O(n^{-k}) .$$

Combining the estimates $J_3 - J_5$ in (3.4), we obtain

$$\| \Sigma_3 \|_{C[c,d]} \le C\, n^{-k} \left\{ \sum_{i=1}^{2k-1} \left\| f^{(i)} \right\|_{C[c,d]} + \left\| f^{(2k)} \right\|_{L_\infty[0,b]} \right\} .$$

Finally, applying Goldberg and Meir's [4, p.5] property, we obtain the required result. ■

Theorem 2. Let $f \in C_\alpha[0,\infty)$, then for sufficient large n, we have

$$\left\| L_{n,k}(f(t);x) - f(x) \right\|_{C(I_2)} \le C \left\{ \omega_{2k}(f, n^{-1/2}, I_1) + n^{-k} \|f\|_{C_\alpha} \right\},$$

where C independent of f and n.

Proof. Let $f_{\eta,2k}$ be the steklov mean of $2k$-th order corresponding to f. Then,

$$\left\| L_{n,k}(f(t);x) - f(x) \right\|_{C(I_2)} \le \left\| L_{n,k}(f(t) - f_{\eta,2k}(t);x) \right\|_{C(I_2)}$$

$$+ \left\| L_{n,k}(f_{\eta,2k}(t);x) - f_{\eta,2k}(x) \right\|_{C(I_2)} + \left\| f_{\eta,2k}(x) - f(x) \right\|_{C(I_2)} := \Sigma_1 + \Sigma_2 + \Sigma_3 .$$

By using the property (c) of Lemma 1, we have $\Sigma_3 \le C\, \omega_{2k}(f, \eta, I_1)$.

Using Theorem 1, we get $\Sigma_2 \le C\, n^{-k} \sum_{j=2}^{2k} \left\| f_{\eta,2k}^{(j)} \right\|_{C(I_2)}$.

Now, by using Goldberg and Meir's [4, p.5] and properties (d), (b) of Lemma 1, we have

$$\left\| f_{\eta,2k}^{(j)} \right\|_{C(I_2)} \le C \left\{ \left\| f_{\eta,2k} \right\|_{C(I_2)} + \left\| f_{\eta,2k}^{(2k)} \right\|_{C(I_2)} \right\} \quad (j = 2,\, 3,\, \ldots ,2k)$$

$$\le C\, n^{-k} \left(\|f\|_{C_\alpha} + \left\| f_{\eta,2k}^{(2k)} \right\|_{C(I_2)} \right)$$

$$\le C\, n^{-k} \left(\|f\|_{C_\alpha} + \eta^{-2k} \omega_{2k}(f, \eta; I_1) \right)$$

Let a' and b' be such that $a_1 < a' < a_2 < b_2 < b' < b_1$ and ψ be the characteristic function of $[a', b']$. Then, by using Lemma 3 and property (c) of Lemma 1 we get

$$\Sigma_1 \le \left\| L_{n,k}(\psi(t)(f(t) - f_{\eta,2k}(t));x) \right\|_{C(I_2)} + \left\| L_{n,k}((1-\psi(t))(f(t) - f_{\eta,2k}(t));x) \right\|_{C(I_2)}$$

$$\le C \left\{ \left\| f - f_{\eta,2k} \right\|_{C[a',b']} + n^{-m} \|f\|_{C_\alpha} \right\}, \forall\ m > 0$$

$$\le C \left\{ \omega_{2k}(f, \eta, I_1) + n^{-m} \|f\|_{C_\alpha} \right\}.$$

Now, choosing $m \geq k$ and $\eta = n^{-1/2}$ in the estimates of $\sum_1$, $\sum_2$ and $\sum_3$, the result follows.

■

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المستخلص

في [6] مايكلي قدم تقنية التركيب التكراري لتحسين رتبة التقريب بواسطة متعددات حدود برنستين0 في البحث الحالي، استخدمنا تقنيته لتحسين رتبة التقريب بواسطة متتابعة جديدة من المؤثرات الخطية الموجبة والمقدمة من قبل أكروال وثامر في [2] والمسماة مؤثر نوع- باسكاكوف التكاملي.