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On Simultaneous Approximation for a Generalization of Baskakov–Beta operators

Ali J Mohammad *Mudhar Abbas Majeed*

Abstract

This paper introduces and studies a generalization of Baskakov-Beta operators, denotes it by $B_{n,p}(f;x)$, where $p \in N^0$ (N^0 the set of nonnegative integers). First, we show that $B_{n,p}^{(r)}(f;x)$ is an approximate process for $f^{(r)}(x)$ as $n \to \infty$ where $r \in N^0$. Next, we discuss a Voronovaskaja-type asymptotic formula for $B_{n,p}^{(r)}(f;x)$. Finally, we present a theorem which gives us an estimate of the degree of approximation by the operator $B_{n,p}(f;x)$.

1. Introduction

In the resent paper, we assume that M is a constant not necessarily the same in different cases.

For $f \in C_\alpha[0,\infty) \equiv \{f \in C[0,\infty): |f(x)| \leq M(1+t)^\alpha \text{ for some } M > 0, \alpha > 0\}$, Gupta [2, 3] defined the Baskakov-Beta operator $B_n(f(t);x)$ as:

$$B_n(f(t);x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} \beta_{n,k}(t) f(t)\, dt,$$

where

$$p_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}, \qquad \beta_{n,k}(t) = \frac{1}{B(k+1,n)} \frac{t^k}{(1+t)^{n+k+1}}$$

and $B(k,n) = \dfrac{\Gamma(k)\Gamma(n)}{\Gamma(n+k)} = \dfrac{(k-1)!\,(n-1)!}{(n+k-1)!}$.

Recently, many researchers defined and studied deferent sequences of linear positive operators. The results were found for these operators (i.e. order of approximation, Voronovaskaja formula and the estimate of the degree of approximation) are similar. For some important contributions in this directions, we refer here to [1, 4, 5].

In this paper, we define and study the following family of of Baskakov–Beta operators:

$$B_{n,p}(f(t);x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} \beta_{n,k+p}(t) f(t)\, dt, \; p \in N^0.$$

Clearly $B_{n,p}(f(t);x) = B_n(f(t);x)$ whenever $p = 0$.

Actually, the operator $B_{n,p}(f(t);x)$ can be written as:

$$B_{n,p}(f(t);x) = \int_0^\infty W_{n,p}(t,x)f(t)dt,$$

where the kernel $W_{n,p}(t,x) = \sum_{k=0}^{\infty} p_{n,k}(x)\beta_{n,k+p}(t)$.

The norm $\|.\|_\alpha$ on the space $C_\alpha[0,\infty)$ is defined as:
$\|f(t)\|_\alpha = \sup_{t\in[0,\infty)} |f(t)|\,(1+t)^{-\alpha}$.

First, we give a theorem shows that $B_{n,p}^{(r)}(f;x)$ is an approximate process for $f^{(r)}(x)$ as $n\to\infty$. Then, we discuss a Voronovaskaja-type asymptotic formula for the operator $B_{n,p}^{(r)}(f;x)$. Finally, we present a theorem which gives us an estimate of the degree of approximation by the operator $B_{n,p}(f;x)$.

2. Basic Results

For $m \in N^0$, the *m*-th order moment of the Baskakov operators is defined by the function

$$\lambda_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x)\left(\frac{k}{n} - x\right)^m.$$

Lemma 1.1 [6].

For the function $\lambda_{n,m}(x)$, we have $\lambda_{n,0}(x) = 1$, $\lambda_{n,1}(x) = 0$ and there holds the recurrence relation
$n\lambda_{n,m+1}(x) = x(1+x)\left(\lambda'_{n,m}(x) + m\lambda_{n,m-1}(x)\right)$, for $m \ge 1$.
Consequently, we have
$\lambda_{n,m}(x)$ is a polynomial in x of degree at most m;
for every $x\in[0,\infty)$, $\lambda_{n,m}(x) = O(n^{-[(m+1)/2]})$ where $[(m+1)/2]$ denotes the integer part of $(m+1)/2$.
For $m \in N^o$, we define the *m*-th order moment of the operator $B_{n,p}(f(t);x)$ as:

$$T_{n,p,m}(x) = B_{n,p}((t-x)^m;x) = \sum_{k=0}^{\infty} p_{n,k}(x)\int_0^\infty \beta_{n,k+p}(t)\,(t-x)^m\,dt.$$

Lemma 1.2:

For the function $T_{n,m}(x)$ define above, we have:

$T_{n,p,0}(x) = 1$. $T_{n,p,1}(x) = \dfrac{x+p+1}{n-1}$ and

$$T_{n,p,2}(x) = \frac{2nx^2 + 2nx + 2x^2 + 4x + 2 + p^2 + (3+4x)p}{(n-1)(n-2)}$$

and there holds the recurrence relation

$$(n-m-1)T_{n,p,m+1}(x) = x(1+x)\left\{T'_{n,p,m}(x) + 2mT_{n,p,m-1}(x)\right\}$$

$+((2m+1)x+m+p+1)\, T_{n,p,m}(x)$.

Consequently, we have

$T_{n,p,m}(x)$ is a polynomial in x of degree m;

for every $x \in [0,\infty)$, $T_{n,p,m}(x) = O(n^{-[(m+1)/2]})$ where $[(m+1)/2]$ denotes the integer part of $(m+1)/2$.

Proof: By direct computation, we get the values of $T_{n,p,0}(x)$, $T_{n,p,1}(x)$ and $T_{n,p,2}(x)$.

To prove the recurrence relation (1.1), we have:

For $x = 0$, the recurrence relation clearly holds.

For $x \in (0,\infty)$, we have

$$T'_{n,p,m}(x) = \sum_{k=0}^{\infty} p'_{n,k}(x) \int_0^{\infty} \beta_{n,k+p}(t)(t-x)^m\, dt - m T_{n,m-1}(x).$$

By using the following two relations:

$$x(1+x)p'_{n,k}(x) = (k-nx)p_{n,k}(x)$$

$$t(1+t)\beta'_{n,k+p}(t) = \beta_{n,k+p}(t)((k+p)-(n+1)t)$$

we get:

$$x(1+x)\{T'_{n,p,m}(x) + m T_{n,p,m-1}(x)\} = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} (k-nx)\, \beta_{n,k+p}(t)\, (t-x)^m\, dt.$$

Now, using the equality $k - nx = (k+p)-(n+1)t + (n+1)(t-x) + (x-p)$, we have

$$x(1+x)[T'_{n,p,m}(x) + m T_{n,p,m-1}(x)] = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} \beta'_{n,k+p}(t) t(1+t)(t-x)^m\, dt$$

$$+(n+1)T_{n,p,m+1}(x) + (x-p)T_{n,p,m}(x).$$

Using the equality $t(1+t) = (t-x)^2 + (1+2x)(t-x) + x(1+x)$, we have

$$x(1+x)[T'_{n,p,m}(x) + m T_{n,p,m-1}(x)] - (n+1)T_{n,p,m+1}(x) - (x-p)T_{n,p,m}(x)$$

$$= \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} \beta'_{n,k+p}(t)\, (t-x)^{m+2}\, dt + (1+2x)\sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} \beta'_{n,k+p}(t)(t-x)^{m+1} dt$$

$$+x(1+x)\sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} \beta'_{n,k+p}(t)(t-x)^m\, dt.$$

Integrating by parts, the recurrence relation (1.1) is immediate.

From the values of $T_{n,p,0}(x)$ and $T_{n,p,1}(x)$, it is clear that the consequences (i) and (ii) hold for $m=0$ and $m=1$. The consequence (i) can be proved easily by using (1.1) and the induction on m.

We sketch below the proof of the consequence (ii).

Suppose that the consequence (ii) be true for m, then by (1.1), we have

$$(n-m-1)T_{n,p,m+1}(x) = O(n^{-[(m+1)/2]}) + O(n^{-[(m+1)/2]}) + O(n^{-[m/2]})$$

$$=\begin{cases} O(n^{-(m-1)/2}), \text{ if } m \text{ is odd} \\ O(n^{-m/2}), \quad \text{if } m \text{ is even} \end{cases}.$$

Then,

$$T_{n,p,m+1}(x)=\begin{cases} O(n^{-(m+1)/2}), \text{ if } m \text{ is odd} \\ O(n^{-(m+2)/2}), \text{ if } m \text{ is even} \end{cases}.$$

Hence, for every $x\in[0,\infty)$, $T_{n,p,m+1}(x)=O(n^{-[(m+2)/2]})$. Thus, consequence (ii) holds for $m+1$. Consequently, by mathematical induction, it holds for all $m\in N^0$.

Lemma 1.3:

For $n\in N^o$ and $x\in[0,\infty)$, we have:

$B_{n,p}(t^m;x)$ is a polynomial in x of degree m. Further, we can write it as:

$$B_{n,p}(t^m;x)=\frac{(n-m-1)!}{(n-1)!\,(n-1)!}[\,(n+m-1)!\,x^m+m\,(m+p)\,(n+m-2)!\,x^{m-1}\,]+O(n^{-2}).$$

Making used Lemmas 1.1, 1.2 and the direct computation, the proof of this lemma easily follows, hence the details are omitted.

Corollary. Let δ and γ be any two positive real numbers and $[a,b]\subset(0,\infty)$. Then, for any $s>0$ we have,

$$\sup_{x\in[a,b]}\left|\sum_{k=0}^{\infty}p_{n,k}(x)\int_{|t-x|\geq\delta}\beta_{n,k+p}(t)t^{\gamma}\,dt\right|=O(n^{-s}).$$

Making use of Schwarz inequality for integration and then for summation and Lemma 1.2, the proof of the corollary easily follows, hence the details are omitted.

Lemma 1.4: [6] For $x\in(0,\infty)$ and $r\in N$, there exist the polynomials $q_{i,j,r}(x)$ independent of n and k such that

$$p_{n,k}^{(r)}(x)=x^{-r}(1+x)^{-r}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^i(k-nx)^j\,q_{i,j,r}(x)\,p_{n,k}(x),$$

3. Main Results

First, we prove that for $r\in N$, $B_{n,p}^{(r)}(f;x)$ is approximation process for $f^{(r)}(x)$ as $n\to\infty$.

Theorem 3.1: Let $f\in C_\alpha[0,\infty)$ and $f^{(r)}, r\in N$ exists at a point $x\in(0,\infty)$. Then, we have

$B_{n,p}^{(r)}(f;x)\to f^{(r)}(x)$ as $n\to\infty$.

Proof: By Taylor expansion of f, we have

$$f(t)=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}(t-x)^i+\varepsilon(t,x)(t-x)^r,$$

where $\varepsilon(t,x)\to 0$ as $t\to x$. Hence,

$$B_{n,p}^{(r)}(f;x)=\int_0^\infty W_{n,p}^{(r)}(t,x)\,f(t)\,dt$$

$$=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_{n,p}^{(r)}(t,x)(t-x)^i\,dt+\int_0^\infty W_{n,p}^{(r)}(t,x)\,\varepsilon(t,x)(t-x)^r\,dt$$

$$=:R_1+R_2.$$

First, we estimate R_1, using the binomial expansion of $(t-x)^i$ and Lemma 1.3, we have

$$R_1=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\sum_{v=0}^{i}\binom{i}{v}(-x)^{i-v}\frac{\partial^r}{\partial x^r}\int_0^\infty W_{n,p}(t,x)\,t^v\,dt$$

$$=\frac{f^{(r)}(x)}{r!}\frac{d^r}{dx^r}\left[\frac{(n-r-1)!(n+r-1)!}{(n-1)!\,(n-1)!}x^r+\text{terms containing lower powers of } x\right]$$

$$=\frac{f^{(r)}(x)}{r!}\left[\frac{(n-r-1)!(n+r-1)!}{(n-1)!\,(n-1)!}r!\right]\to f^{(r)}(x)\quad\text{as } n\to\infty.$$

Next, applying Lemma 1.4, we obtain

$$|R_2|\le\sum_{\substack{2i+j\le r\\ i,j\ge 0}}n^i\frac{q_{i,j,r}(x)}{\{x(1+x)\}^r}\sum_{k=0}^{\infty}|k-nx|^j\,p_{n,k}(x)\int_0^\infty\beta_{n,k+p}(t)\,|\varepsilon(t,x)|\,|t-x|^r\,dt.$$ Since

$\varepsilon(t,x)\to 0$ as $t\to x$, for a given $\varepsilon>0$ there exists a δ such that $|\varepsilon(t,x)|<\varepsilon$ whenever $0<|t-x|<\delta$. If $\gamma\ge\max\{\alpha,r\}$, where γ is any integer, then we can find a constant $M>0$, $|\varepsilon(t,x)(t-x)^r|\le M|t-x|^\gamma$, for $|t-x|\ge\delta$. Therefore

$$|R_2|\le M\sum_{\substack{2i+j\le r\\ i,j\ge 0}}n^i\sum_{k=0}^{\infty}|k-nx|^j\,p_{n,k}(x)$$

$$\times\left\{\varepsilon\int_{|t-x|<\delta}\beta_{n,k+p}(t)\,|t-x|^r\,dt+\int_{|t-x|\ge\delta}\beta_{n,k+p}(t)\,|t-x|^\gamma\,dt\right\}$$

$$=:R_3+R_4.$$

Applying the Cauchy–Schwarz inequality for integration and then for summation, we obtain

$$R_3\le M\varepsilon\sum_{\substack{2i+j\le r\\ i,j\ge 0}}n^i\{n^{2j}\mu_{n,2j}(x)\}^{\frac12}\{T_{n,p,2r}(x)\}^{\frac12}.$$

Using Lemmas 1.1 and 1.2, we get

$R_3=\varepsilon\,O(n^{r/2})\,O(n^{-r/2})=\varepsilon O(1)=o(1)$.

Again using the Cauchy–Schwarz inequality for integration and then for summation, Lemmas 1.1 and Corollary, we get

$$R_4 \le M \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \sum_{k=0}^{\infty} |k-nx|^j \, p_{n,k}(x) \int_{|t-x|\ge\delta} \beta_{n,k+p}(t)\, |t-x|^{\gamma}\, dt$$

$$\le M \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \left\{\sum_{k=0}^{\infty} (k-nx)^{2j} p_{n,k}(x)\right\}^{\frac{1}{2}}$$

$$\times \left\{ \sum_{k=0}^{\infty} p_{n,k}(x) \int_{|t-x|\ge\delta} \beta_{n,k+p}(t)\,(t-x)^{2\gamma}\, dt \right\}^{\frac{1}{2}}$$

$$= \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\, O(n^{j/2})\, O(n^{-s/2}) = O(n^{(r-s)/2}) = o(1)\ \text{,for } s > r.$$

Collecting the estimates of $R_1 - R_2$, we obtain the required result.

Our next result is a Voronovaskaja-type asymptotic formula for the operators $B_{n,p}^{(r)}(f,x)$, $r = 1,2,\ldots.$

Theorem 3.2:

Let $f \in C_{\alpha}[0,\infty)$ for $\alpha > 0$ and $f^{(m+2)}$ exists at a point $x \in (0,\infty)$. Then

$$\lim_{n\to\infty} n\left\{B_{n,p}^{(m)}(f,x) - f^{(m)}(x)\right\} = m^2 f^{(m)}(x)$$

$$+ (2mx + x + m + p + 1) f^{(m+1)}(x) + x(1+x) f^{(m+2)}(x)$$

Proof: Using Taylor's expansion of f , we have

$$f(t) = \sum_{i=0}^{m+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{m+2}$$

where $\varepsilon(t,x) \to 0$ as $t \to x$ and $\varepsilon(t,x) = O((t-x)^{\gamma})$, $t \to \infty$ for $\gamma > 0$.
Then, we get

$$B_{n,p}^{(m)}(f(t),x) - f^{(m)}(x) = \sum_{i=0}^{m+2} \frac{f^{(i)}(x)}{i!} \int_0^{\infty} W_{n,p}^{(m)}(t,x)(t-x)^i\, dt - f^{(m)}(x)$$

$$+ \int_0^{\infty} W_{n,p}^{(m)}(t,x)\, \varepsilon(t,x)\, (t-x)^{m+2}\, dt$$

$$=: E_1 + E_2.$$

Using Lammas 1.2 and 1.3 , we have

$$E_1 = \sum_{i=m}^{m+2} \frac{f^{(i)}(x)}{i!} \sum_{v=m}^{i} \binom{i}{v} (-x)^{i-v} B_{n,p}^{(m)}(t^v;x) - f^{(m)}(x)$$

$$= \frac{f^{(m)}(x)}{m!}\left[B_{n,p}^{(m)}(t^m;x) - m!\right]$$

$$+\frac{f^{(m+1)}(x)}{(m+1)!}\left[(m+1)(-x)B_{n,p}^{(m)}(t^m;x)+B_{n,p}^{(m)}(t^{m+1};x)\right]$$

$$+\frac{f^{(m+2)}(x)}{(m+2)!}$$

$$\times\left[\frac{(m+2)(m+1)}{2}x^2B_{n,p}^{(m)}(t^m;x)+(m+2)(-x)B_{n,p}^{(m)}(t^{m+1};x)+B_{n,p}^{(m)}(t^{m+2};x)\right]$$

$$=\frac{f^{(m)}(x)}{m!}\left[\frac{(n-m-1)!(n+m-1)!}{(n-1)!(n-1)!}m!-m!\right]$$

$$+\frac{f^{(m+1)}(x)}{(m+1)!}\left[(m+1)(-x)\frac{(n-m-1)!(n+m-1)!}{(n-1)!(n-1)!}m!+\frac{(n-m-2)!(n+m)!}{(n-1)!(n-1)!}(m+1)!x\right.$$

$$\left.+(m+1)(m+p+1)\frac{(n-m-2)!(n+m-1)!}{(n-1)!(n-1)!}m!\right]$$

$$+\frac{f^{(m+2)}(x)}{(m+2)!}\left[\frac{(m+2)(m+1)}{2}x^2\frac{(n-m-1)!(n+m-1)!}{(n-1)!(n-1)!}m!\right.$$

$$+(m+2)(-x)$$

$$\times\left\{\frac{(n-m-2)!(n+m)!}{(n-1)!(n-1)!}(m+1)!x+(m+1)(m+p+1)\frac{(n-m-2)!(n+m-1)!}{(n-1)!(n-1)!}m!\right\}$$

$$\left.+\frac{(n-m-3)!(n+m+1)!}{(n-1)!(n-1)!}\frac{(m+2)!}{2}x^2+(m+2)(m+p+2)\frac{(n-m-3)!(n+m)!}{(n-1)!(n-1)!}(m+1)!x\right]$$

$$+O(n^{-2})$$

Hence in order to prove the theorem it suffices to show that $nE_2 \to 0$ as $n \to \infty$, which follows on proceeding along the lines of proof of $R_2 \to 0$ as $n \to \infty$ in Theorem 3.1.

Now, we present a theorem, which gives an estimate of the degree of approximation by $L_n^{(r)}(.;x)$ for smooth functions.

Theorem 3.3.

Let $f \in C_\alpha[0,\infty)$ for some $\alpha > 0$ and $r \le q \le r+2$. If $f^{(q)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty)$, $\eta > 0$, then for sufficiently large n,

$$\left\|B_{n,p}{}^{(r)}(f(t);x)-f^{(r)}(x)\right\| \le C_1 n^{-1}\sum_{i=r}^{q}\left\|f^{(i)}\right\| + C_2 n^{-1/2}\omega_{f^{(r+1)}}(n^{-1/2})+O(n^{-2}),$$

where C_1, C_2 are both independent of f and n, $\omega_f(\delta)$ is the modulus of continuity of f on $(a-\eta, b+\eta)$, and $\|.\|$ means the sup-norm on $[a,b]$.

Proof: By our hypothesis

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q\chi(t)+h(t,x)(1-\chi(t)),$$

where ξ lies between t, x, and $\chi(t)$ is the characteristic function of the interval $(a-\eta, b+\eta)$.

For $t \in (a-\eta, b+\eta)$ and $x \in [a,b]$, we get

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q .$$

For $t \in [0,\infty)\setminus(a-\eta, b+\eta)$ and $x \in [a,b]$, we define

$$h(t,x)=f(t)-\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i .$$

Now,

$$B_{n,p}^{(r)}(f(t);x)-f^{(r)}(x)=\left[\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_{n,p}^{(r)}(t,x)(t-x)^i\,dt-f^{(r)}(x)\right]$$

$$+\int_0^\infty W_{n,p}^{(r)}(t,x)\left\{\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q\chi(t)\right\}dt+\int_0^\infty W_{n,p}^{(r)}(t,x)h(t,x)(1-\chi(t))\,dt$$

$$:=E_1+E_2+E_3 .$$

By using Lemmas 1.2 and 1.3, we get

$$E_1=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\frac{d^r}{dx^r}\left[\int_0^\infty W_{n,p}(t,x)t^j\,dt\right]-f^{(r)}(x)$$

$$=\sum_{i=r}^{q}\frac{f^{(i)}(x)}{i!}\sum_{j=r}^{i}\binom{i}{j}(-x)^{i-j}$$

$$\times\frac{d^r}{dx^r}\left[\frac{(n+j-1)!(n-j-1)!}{((n-1)!)^2}x^j+j(j-1)\frac{(n+j-2)!(n-j-1)!}{((n-1)!)^2}x^{j-1}+O(n^{-2})\right]-f^{(r)}(x).$$

Consequently, $\|E_1\|\le C_1\,n^{-1}\left(\sum_{i=r}^{q}\left\|f^{(i)}\right\|\right)+O(n^{-2})$, uniformly in $x\in[a,b]$.

To estimate E_2 we proceed as follows:

$$|E_2|\le\int_0^\infty\left|W_n^{(r)}(t,x)\right|\left\{\frac{\left|f^{(q)}(\xi)-f^{(q)}(x)\right|}{q!}|t-x|^q\chi(t)\right\}dt$$

$$\le\frac{\omega_{f^{(q)}}(\delta)}{q!}\int_0^\infty\left|W_{n,p}^{(r)}(t,x)\right|\left(1+\frac{|t-x|}{\delta}\right)|t-x|^q\,dt$$

$$\le\frac{\omega_{f^{(q)}}(\delta)}{q!}\sum_{k=0}^{\infty}\left|p_{n,k}^{(r)}\right|\int_0^\infty\beta_{n,k+p}(t)\left(|t-x|^q+\delta^{-1}|t-x|^{q+1}\right)dt,\delta>0.$$

Now, for $s=0,1,2,\ldots$, we have

$$\sum_{k=1}^{\infty}p_{n,k}(x)|k-nx|^j\int_0^\infty\beta_{n,k+p}(t)|t-x|^s\,dt$$

$$\leq \sum_{k=1}^{\infty} p_{n,k}(x)\left|k-nx\right|^{j}\left[\left(\int_{0}^{\infty}\beta_{n,k+p}(t)dt\right)^{1/2}\left(\int_{0}^{\infty}\beta_{n,k+p}(t)(t-x)^{2s}\,dt\right)^{1/2}\right]$$

$$\leq \left[\sum_{k=1}^{\infty} p_{n,k}(x)(k-nx)^{2j}\right]^{1/2}\left[\sum_{k=1}^{\infty} p_{n,k}(x)\int_{0}^{\infty} b_{n,k+p}(t)(t-x)^{2s}\,dt\right]^{1/2}$$

$$= O(n^{j/2})O(n^{-s/2}) = O(n^{(j-s)/2}),$$

uniformly in $x \in [a,b]$, in view of Lammas 1.1 and 1.2.
Therefore by Lemma 1.4, we get

$$\sum_{k=1}^{\infty}\left|p_{n,k}^{(r)}(x)\right|\int_{0}^{\infty}\beta_{n,k+p}(t)\left|t-x\right|^{s}dt$$

$$\leq \sum_{k=1}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i}\left|k-nx\right|^{j}\frac{\left|q_{i,j,r}(x)\right|}{x^{r}(1+x)^{r}}p_{n,k}(x)\int_{0}^{\infty}\beta_{n,k+p}(t)\left|t-x\right|^{s}dt$$

$$\leq C\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i}\left[\sum_{k=1}^{\infty} p_{n,k}(x)\left|k-nx\right|^{j}\int_{0}^{\infty}\beta_{n,k+p}(t)\left|t-x\right|^{s}dt\right] = C\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i}\,O(n^{(j-s)/2})$$

$$= O(n^{(r-s)/2}),$$

uniformly in $x \in [a,b]$, where $C = \sup\limits_{\substack{2i+j\leq r\\ i,j\geq 0}}\ \sup\limits_{x\in[a,b]}\dfrac{\left|q_{i,j,r}(x)\right|}{x^{r}(1+x)^{r}}$.

Choosing $\delta = n^{-1/2}$ and applying the above results, we are led to

$$\| E_2 \| \leq \frac{\omega_{f^{(q)}}(n^{-1/2})}{q!}\left[O(n^{(r-q)/2}) + n^{1/2}O(n^{(r-q-1)/2})\right],$$

$$\leq C_2\, n^{-(r-q)/2}\,\omega_{f^{(q)}}(n^{-1/2}).$$

Since $t \in [0,\infty)\setminus(a-\eta, b+\eta)$, we can choose $\delta > 0$ in such a way that $|t-x| \geq \delta$ for all $x \in [a,b]$.
Thus, by Lemma 1.4, we obtain

$$|E_3| \leq n\sum_{k=1}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i}\left|k-nx\right|^{j}\frac{\left|q_{i,j,r}(x)\right|}{x^{r}(1+x)^{r}}p_{n,k}(x)\int_{|t-x|\geq\delta}\beta_{n,k+p}(t)\left|h(t,x)\right|dt.$$

For $|t-x| \geq \delta$, we can find a constant $M > 0$ such that $|h(t,x)| \leq M\,t^{\alpha}$. Finally using Schwarz inequality for integration and then for summation, Lemma 1.1, and Corollary, it easily follows that $E_3 = O(n^{-s})$ for any $s > 0$, uniformly on $[a,b]$.
Combining the estimates of E_1, E_2, E_3, the required result is immediate.

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حول التقريب المتزامن لتعميم لمؤثر Baskakov–Beta

علي جاسم محمد   مضر عباس مجيد

الخلاصة

هذا البحث يقدم ويدرس تعميم لمؤثر Baskakov-Beta نرمز له $B_{n,p}(f;x)$، حيث $p \in N^{\circ}$ (N° مجموعة الأعداد الصحيحة غير السالبة). أولا، بينا أن $B_{n,p}^{(r)}(f;x)$ متقارب لـ $f^{(r)}(x)$ عندما $n \to \infty$ حيث $r \in N^{\circ}$. ثم ناقشنا الصيغة المقاربة لنمط Voronovaskaja. أخيرا، قدمنا مبرهنة تعطينا تخمين الخطأ لدرجة التقريب بواسطة المؤثر $B_{n,p}(f;x)$.