SOOCHOW JOURNAL OF MATHEMATICS
Volume 31, No. 1, pp. 31-40, January 2005

L_p-SATURATION THEOREM FOR A LINEAR COMBINATION OF A NEW SEQUENCE OF LINEAR POSITIVE OPERATORS

BY

P. N. AGRAWAL AND ALI J. MOHAMMAD

Abstract. The present paper is a continuation of our work in [2] and [3]. In [2], we studied some direct results in L_p-approximation by a linear combination of a new sequence of linear positive operators introduced by us in [1]. In [3], we established an inverse theorem in L_p-norm for the same sequence. Here our object is to discuss a saturation theorem in L_p-norm for these operators.

1. Introduction

For $f \in L_p[0,\infty)$, $1 \le p < \infty$, we [1] introduced a new sequence of linear positive operators in the following way:

$$M_n(f(t);x) = \int_0^\infty W_n(t,x)f(t)dt, \tag{1.1}$$

where $W_n(t,x) = n\sum_{v=1}^{\infty} p_{n,v}(x)q_{n,v-1}(t) + (1+x)^{-n}\delta(t)$, $\delta(t)$ being the Dirac-delta function, $p_{n,v}(x) = \binom{n+v-1}{v}x^v(1+x)^{-n-v}$ and $q_{n,v}(t) = \frac{e^{-nt}(nt)^v}{v!}$, where x, $t \in [0,\infty)$.

May [4] and Rathore [5] have described a method for forming linear combinations of a sequence of linear positive operators so as to improve the order of approximation. Following their method, in [1] we established some direct theorems for a linear combination of the operators (1.1). The approximation process is described as follows:

Received December 26, 2002; revised July 19, 2004.
AMS Subject Classification. 41A40, 41A36.
Key words. linear positive operators, saturation theorem in L_p-space, linear combination, Steklov mean.

For $k \in N^0$ (the set of nonnegative integers), the linear combination $M_n(f,k,x)$ of the operators $M_{d_jn}(f;x)$, $j = 0,1,\dots,k$ is defined as:

$$M_n(f,k,x) = \sum_{j=0}^{k} C(j,k) M_{d_jn}(f;x), \tag{1.2}$$

where $d_0,\ d_1,\dots,d_k \in N$ (the set of positive integers) and are arbitrary and distinct but fixed and

$$C(j,k) = \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{d_j}{d_j - d_i}, \quad k \neq 0 \quad \text{and} \quad C(0,0) = 1. \tag{1.3}$$

Throughout this paper, let $0 < a < b < \infty$, $C[a,b]$ the set all continuous functions on the interval $[a,b]$, $C^m[a,b]$ the subset of $C[a,b]$ having m-times continuously differentiable functions, C_0 the subset of $C(0,\infty)$ having a compact support, C_0^k the subset of C_0 having k-times continuously differentiable functions, $AC[a,b]$ the class of absolutely continuous functions on $[a,b]$, $BV[a,b]$ the class of functions of bounded variation over $[a,b]$ and $\|.\|_{C[a,b]}$, the sup norm on the space $C[a,b]$.

For $m \in N$ and $f \in L_p[a,b]$, $1 \le p < \infty$, the m-th order integral modulus of smoothness of f is defined as:

$$\omega_m(f,\tau,p,[a,b]) = \sup_{0<\delta\le\tau} \|\Delta_\delta^m f(t)\|_{L_p[a,b-m\delta]},$$

where $\Delta_\delta^m f(t)$ is the m-th order forward difference of the function f with step length δ and $0 < \tau \le (b-a)/m$.

Further we assume that $0 < a_1 < a_3 < a_2 < b_2 < b_3 < b_1 < \infty$, $I_i = [a_i, b_i]$ $(i = 1,2,3)$, $\langle h,g\rangle := \int_0^\infty h(x)g(x)dx$ the inner product on the space $L_p[0,\infty)$ and C denotes a constant not necessarily the same in different cases.

The object of the present paper is to prove the following (**saturation theorem**):

Theorem 1. *Let $f \in L_p[0,\infty)$, $1 \le p < \infty$. Then, in the following statements, the implications* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *and* (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) *hold:*

(i) $\|M_n(f,k,.) - f\|_{L_p(I_1)} = O(n^{-(k+1)})$;

(ii) *f coincides almost everywhere (a.e.) with a function F on I_3 having $2k+2$ derivatives such that:*

(a) *when $p > 1$, $F^{(2k+1)} \in AC(I_3)$ and $F^{(2k+2)} \in L_p(I_3)$,*

(b) *when $p = 1$, $F^{(2k)} \in AC(I_3)$ and $F^{(2k+1)} \in BV(I_3)$;*

(iii) $\|M_n(f,k,.) - f\|_{L_p(I_2)} = O(n^{-(k+1)})$;

(iv) $\|M_n(f,k,.) - f\|_{L_p(I_1)} = o(n^{-(k+1)})$;

(v) *f coincides a.e. with a function F on I_3, where F is $2k+2$ times continuously differentiable on I_3 and satisfies $\sum_{j=k+2}^{2k+2} \frac{Q(j,k,x)}{j!} F^{(j)}(x) = 0$, where $Q(j,k,x)$ are the polynomials occurring in Theorem 4;*

(vi) $\|M_n(f,k,.) - f\|_{L_p(I_2)} = o(n^{-(k+1)})$,

where $O(n^{-(k+1)})$ and $o(n^{-(k+1)})$ terms are with respect to n, where $n \to \infty$.

2. Preliminary Results

In order to prove the saturation theorem, we shall require the following results:

Let $f \in L_p[0,\infty)$, $1 \le p < \infty$. Then, for sufficiently small $\eta > 0$, the Steklov mean $f_{\eta,m}$ of m-th order corresponding to f is defined as follows:

$$f_{\eta,m}(t) = \eta^{-m} \left(\int_{-\eta/2}^{\eta/2} \right)^m \left\{ f(t) + (-1)^{m-1} \Delta^m_{\sum_{i=1}^m t_i} f(t) \right\} \prod_{i=1}^m dt_i, \quad t \in I_1.$$

Lemma 1.([7]) *For the function $f_{\eta,m}(t)$ defined above, we have*

(a) *$f_{\eta,m}(t)$ has derivatives upto order m over I_1, $f_{\eta,m}^{(m-1)} \in AC(I_1)$ and $f_{\eta,m}^{(m)}$ exists a.e. and belongs to $L_p(I_1)$;*

(b) $\|f_{\eta,m}^{(r)}\|_{L_p(I_2)} \le M_r \eta^{-r} \omega_r(f,\eta,p,I_1)$, $r = 1,2,\ldots,m$;

(c) $\|f - f_{\eta,m}\|_{L_p(I_2)} \le M_{m+1} \omega_m(f,\eta,p,I_1)$;

(d) $\|f_{\eta,m}\|_{L_p(I_2)} \le M_{m+2} \|f\|_{L_p(I_1)}$;

(e) $\|f_{\eta,m}^{(m)}\|_{L_p(I_2)} \le M_{m+3} \eta^{-m} \|f\|_{L_p(I_1)}$,

where M_i's are certain constants that depend on i but are independent of f and η.

Lemma 2.([1]) *Let $m \in N^0$, the m-th order moment for the operators (1.1)* *be defined by:*

$$T_{n,m}(x)=M_n((t-x)^m;x)=n\sum_{v=1}^{\infty}p_{n,v}(x)\int_0^{\infty}q_{n,v-1}(t)(t-x)^m dt+(-x)^m(1+x)^{-n}.$$

Then $T_{n,0}(x)=1$, $T_{n,1}(x)=0$ *and*

$$nT_{n,m+1}(x)=x(1+x)T'_{n,m}(x)+mT_{n,m}(x)+mx(x+2)T_{n,m-1}(x), \quad m\geq 1.$$

Further, we have the following consequences of $T_{n,m}(x)$*:*

(i) $T_{n,m}(x)$ *is a polynomial in* x *of degree* m, $m\neq 1$*;*

(ii) *for every* $x\in[0,\infty)$, $T_{n,m}(x)=O(n^{-[(m+1)/2]})$ *where* $[\beta]$ *denotes the integral part of* β.

Lemma 3.([1]) *For* $m\in N^0$*, we define the function* $\mu_{n,m}(t)$ *as:*

$$\mu_{n,m}(t)=n\sum_{v=1}^{\infty}q_{n,v-1}(t)\int_0^{\infty}p_{n,v}(x)(x-t)^m dx.$$

Then $\mu_{n,0}(t)=\frac{n}{n-1}$, $\mu_{n,1}(t)=\frac{2n(1+t)}{(n-1)(n-2)}$ *and there holds the recurrence relation*

$$(n-m-2)\mu_{n,m+1}(t)=t\mu'_{n,m}(t)+(m+2mt+2t+2)\mu_{n,m}(t)+mt(t+2)\mu_{n,m-1}(t),$$

where $n>m+2$*. Consequently:*

(i) $\mu_{n,m}(t)$ *is a polynomial in* t *of degree* m*;*

(ii) *for every* $t\in[0,\infty)$, $\mu_{n,m}(t)=O(n^{-[(m+1)/2]})$.

Lemma 4.([4]) *If* $C(j,k)$, $j=0,1,\dots,k$ *are defined as in* (1.3)*, then*

$$\sum_{j=0}^{k}C(j,k)d_j^{-m}=\begin{cases}1, & m=0,\\ 0, & m=1,2,\dots,k.\end{cases}$$

Lemma 5. *Let* $h\in L_p(I_1)$, $1\leq p<\infty$*, with supp* $h\subset I_1$*. If* h *has* $2k+1$ *derivatives with* $h^{(2k)}\in AC(I_1)$ *and* $h^{(2k+1)}\in L_p(I_1)$*, then for each* $g\in C_0^{2k+2}$ *with supp* $g\subset(0,\infty)$*, the following inequality holds:*

$$|\langle M_n(h,k,x)-h(x),g(x)\rangle|\leq\frac{C}{n^{k+1}}\sum_{r=0}^{2k}\|h^{(r)}\|_{C(I_1)}.$$

Proof. Clearly,

$$\langle M_n(h,k,x)-h(x),g(x)\rangle=\sum_{j=0}^{k}C(j,k)\langle M_{d_jn}(h(t);x),g(x)\rangle-\langle h,g\rangle. \tag{2.1}$$

By using Fubini's theorem and Taylor's expansion of g at $x=t$, we have

$$\begin{aligned}
&\langle M_{d_jn}(h(t);x),g(x)\rangle\\
&=\int_0^\infty\int_0^\infty W_{d_jn}(t,x)h(t)g(x)dtdx\\
&=\int_0^\infty\int_0^\infty W_{d_jn}(t,x)h(t)\left[\sum_{i=0}^{2k+1}\frac{(x-t)^i}{i!}g^{(i)}(t)+\frac{(x-t)^{2k+2}}{(2k+2)!}g^{(2k+2)}(\xi_1)\right]dxdt\\
&\qquad\qquad(\text{ where } \xi_1 \text{ lies between } x \text{ and } t\ )\\
&=\int_0^\infty\left(\int_0^\infty W_{d_jn}(t,x)dx\right)h(t)g(t)dt\\
&\quad+\int_0^\infty\left(\int_0^\infty W_{d_jn}(t,x)(x-t)dx\right)h(t)g'(t)dt\\
&\quad+\sum_{i=2}^{2k+1}\frac{1}{i!}\int_0^\infty\int_0^\infty W_{d_jn}(t,x)(x-t)^ih(t)g^{(i)}(t)dxdt\\
&\quad+\frac{1}{(2k+2)!}\int_0^\infty\int_0^\infty W_{d_jn}(t,x)(x-t)^{2k+2}h(t)g^{(2k+2)}(\xi_1)dxdt\\
&=\sigma_0+\sigma_1+\sum_{i=2}^{2k+1}\sigma_i+\sigma_{2k+2}.
\end{aligned}$$

Applying Lemmas 3 and 4 we obtain $\sum_{j=0}^{k}C(j,k)\sigma_0=\langle h,g\rangle+O(n^{-(k+1)})$ and by Lemma 3 again, we have

$$\begin{aligned}
\sum_{j=0}^{k}C(j,k)\sigma_1&=\sum_{j=0}^{k}C(j,k)\int_0^\infty\left(\int_0^\infty W_{d_jn}(t,x)(x-t)dx\right)h(t)g'(t)dt\\
&=\sum_{j=0}^{k}C(j,k)\int_0^\infty\frac{2d_jn(1+t)}{(d_jn-1)(d_jn-2)}h(t)g'(t)dt.
\end{aligned}$$

Thus, by using the compactness of g' and Lemma 4, we get:

$$\left|\sum_{j=0}^{k}C(j,k)\sigma_1\right|\le\frac{C}{n^{k+1}}\|h\|_{C(I_1)}\quad\text{and}\quad\left|\sum_{j=0}^{k}C(j,k)\sigma_{2k+2}\right|\le\frac{C}{n^{k+1}}\|h\|_{C(I_1)}.$$

Now, let $h_i(t) = h(t)g^{(i)}(t)$, $2 \leq i \leq 2k+1$ then by using Taylor's expansion of h_i at $t = x$, we have $h_i(t) = \sum_{r=0}^{2k-1} \frac{(t-x)^r}{r!} h_i^{(r)}(x) + \frac{(t-x)^{2k}}{(2k)!} h_i^{(2k)}(\xi_2)$ where ξ_2 lies between t and x. Applying Fubini's theorem and Lemma 4, we have for each $i = 2, 3, \ldots, 2k+1$

$$\left| \sum_{j=0}^{k} C(j,k)\sigma_i \right| \leq \frac{C}{n^{k+1}} \left\{ \sum_{r=0}^{2k} \|h_i^{(r)}\|_{C(I_1)} \right\} \leq \frac{C}{n^{k+1}} \left\{ \sum_{r=0}^{2k} \|h^{(r)}\|_{C(I_1)} \right\}.$$

Thus, $\sum_{j=0}^{k} C(j,k) \left(\sum_{i=2}^{2k+1} \sigma_i \right) \leq \frac{C}{n^{k+1}} \sum_{r=0}^{2k} \|h^{(r)}\|_{C(I_1)}$.
Finally, combining the estimates of $\sum_{j=0}^{k} C(j,k)\sigma_i$, $i = 0, 1, \ldots, 2k+2$ with (2.1), the required result follows.

Lemma 6.([2]) *Let $1 < p < \infty$ and $f \in L_p^{(2k+2)}(I_1)$, then for all n sufficiently large, the following inequality holds:*

$$\|M_n(f,k,.) - f\|_{L_p(I_2)} \leq C_1 n^{-(k+1)} \left\{ \|f^{(2k+2)}\|_{L_p(I_1)} + \|f\|_{L_p[0,\infty)} \right\},$$

where $C_1 = C_1(k,p)$.

Let $f \in L_1[0,\infty)$. If f has $2k+1$ derivatives in I_1 with $f^{(2k)} \in AC(I_1)$ and $f^{(2k+1)} \in BV(I_1)$, then for all n sufficiently large, the following inequality holds:

$$\|M_n(f,k,.) - f\|_{L_1(I_2)} \leq C_2 n^{-(k+1)} \left\{ \|f^{(2k+1)}\|_{BV(I_1)} + \|f^{(2k+1)}\|_{L_1(I_2)} + \|f\|_{L_1[0,\infty)} \right\},$$

where $C_2 = C_2(k)$.

Our next result is an inverse theorem in L_p-approximation for $M_n(.,k,x)$.

Theorem 2.([3]) *Let $0 < \alpha < 2k+2$, $f \in L_p[0,\infty)$, $1 \leq p < \infty$, and*

$$\|M_n(f,k,.) - f\|_{L_p(I_1)} = O(n^{-\alpha/2}) \text{ as } n \to \infty,$$

then $\omega_{2k+2}(f,\tau,p,I_3) = O(\tau^\alpha)$ as $\tau \to 0$.

Theorem 3.([6]) *Let $1 \leq p < \infty$, $f \in L_p[a,b]$ and there hold*

$$\omega_m(f,\tau,p,[a,b]) = O(\tau^{r+\alpha}), \quad (\tau \to 0),$$

where m, $r \in N$ and $0 < \alpha < 1$. Then $f(x)$ coincides a.e. on $[c,d] \subset (a,b)$ with a function $F(x)$ possessing an absolutely continuous derivative $F^{(r-1)}(x)$, *the r^{th} derivative $F^{(r)}(x) \in L_p[c,d]$, and there holds $\omega(F^{(r)}, \tau, p, [c,d]) = O(\tau^\alpha)$, $(\tau \to 0)$.*

Let $\alpha > 0$ and $f \in C_\alpha[0,\infty) \equiv \{f \in C[0,\infty) : |f(t)| \le Me^{\alpha t}$ for some $M > 0\}$. We state Voronoskaja type asymptotic result for the operator $M_n(f,k,x)$.

Theorem 4.([1]) *Let $f \in C_\alpha[0,\infty)$ and $f^{(2k+2)}$ exist at a point $x \in [0,\infty)$. Then $\lim_{n\to\infty} n^{k+1}[M_n(f,k,x) - f(x)] = \sum_{m=k+2}^{2k+2}(f^{(m)}(x)/m!)Q(m,k,x)$ and $\lim_{n\to\infty} n^{k+1}[M_n(f,k+1,x) - f(x)] = 0$, where $Q(m,k,x)$ are certain polynomials in x of degree m. Moreover,*

$$Q(2k+1,k,x) = \frac{(-1)^k}{\prod_{j=0}^{k} d_j} Cx^k(x+2)^{k-1}(x^2+3x+3) \quad \textit{and}$$
$$Q(2k+2,k,x) = \frac{(-1)^k}{\prod_{j=0}^{k} d_j}(2k+1)!!\{x(x+2)\}^{k+1},$$

where !! denotes the semi-factorial function.

3. Proof of Theorem 1

We assume that $a_1 < x_1 < x_2 < a_3 < b_3 < y_2 < y_1 < b_1$, and $J_i = [x_i, y_i]$ $(i = 1,2)$. We get from Theorem 2 and 3 that f coincides a.e. on (x_1, y_1) with a function called F such that $F^{(2k)} \in AC(J_1)$ and $F^{(2k+1)} \in L_p(J_1)$. Moreover, for $0 < \beta < 1$,

$$\omega(F^{(2r+1)}, \tau, p, J_1) = O(\tau^\beta), \quad \tau \to 0. \tag{3.1}$$

Let $q \in C_0^{2k+2}$ with $\operatorname{supp} q \subset (a_1, b_1)$ and $q(x) = 1$ if $x \in J_1$. Put $\hat{f}(x) = F(x)q(x)$, $x \in [0,\infty)$ then

$$\|M_n(\hat{f},k,.) - \hat{f}\|_{L_p(J_2)} \le \|M_n(f,k,.) - f\|_{L_p(J_2)} + \|M_n(\hat{f} - f,k,.)\|_{L_p(J_2)}.$$

Because of $\hat{f} = f$ on J_1, the contribution of the second term of the right hand side can be made arbitrarily small as $n \to \infty$. Hence, assuming (i), it follows that

$$\|M_n(\hat{f},k,\cdot) - \hat{f}\|_{L_p(J_2)} = O(n^{-(k+1)}), \quad n \to \infty.$$

Now, if $p > 1$, using Alaoglu's theorem there exists a function $h(x) \in L_p(J_2)$ such that for some subsequence $\{n_j\}$ and for every $g \in C_0^{2k+2}$ with $\operatorname{supp} g \subset (a_1, b_1)$,

$$\lim_{n_j\to\infty} n_j^{k+1}\langle M_{n_j}(\hat{f},k,.) - \hat{f}, g\rangle = \langle h, g\rangle. \tag{3.2}$$

When $p = 1$, the function $\phi_n(x)$ can be defined by:

$$\phi_n(u) = \int_{x_2}^{u} n^{k+1}\{M_n(\hat{f}, k, x) - \hat{f}(x)\}dx$$

are uniformly bounded and are of uniformly bounded variation. Making use of Alaoglu's theorem, it follows that there exists a function $\phi_0(x) \in AV(J_2)$ such that

$$\lim_{n_j\to\infty} n_j^{k+1}\langle M_{n_j}(\hat{f}, k, .) - \hat{f}, g\rangle = -\langle \phi_0, g'\rangle. \tag{3.3}$$

Now, suppose $\hat{f}_{\eta,2k+2}$ is the Steklov mean of $(2k+2)^{th}$ order corresponding to $\hat{f}$, we have

$$\begin{aligned}\langle M_{n_j}(\hat{f}, k, x) - \hat{f}(x), g(x)\rangle &= \langle M_{n_j}(\hat{f} - \hat{f}_{\eta,2k+2}, k, x) - (\hat{f} - \hat{f}_{\eta,2k+2})(x), g(x)\rangle \\ &\quad + \langle M_{n_j}(\hat{f}_{\eta,2k+2}, k, x) - \hat{f}_{\eta,2k+2}(x), g(x)\rangle \\ &= \langle M_{n_j}(\hat{f} - \hat{f}_{\eta,2k+2}, k, x) - (\hat{f} - \hat{f}_{\eta,2k+2})(x), g(x)\rangle \\ &\quad + \frac{1}{n_j^{k+1}}\langle P_{2k+2}(D)\hat{f}_{\eta,2k+2}(x), g(x)\rangle + o\left(\frac{1}{n_j^{k+1}}\right),\end{aligned}$$

in view of Theorem 4, where $P_{2k+2}(D) = \sum_{i=k+2}^{2k+2} \frac{Q(i,k,x)}{i!} D^i$ and $D \equiv \frac{\partial}{\partial x}$.
Let $P^*_{2k+2}(D) = \sum_{i=k+2}^{2k+2} \frac{Q^*(i,k,x)}{i!} D^i$ denote the differential operator adjoint to $P_{2k+2}(D)$, thus,

$$\begin{aligned}& n_j^{k+1}\langle M_{n_j}(\hat{f}, k, x) - \hat{f}(x), g(x)\rangle - \langle \hat{f}_{\eta,2k+2}(x), P^*_{2k+2}(D)g(x)\rangle \\ &= n_j^{k+1}\langle M_{n_j}(\hat{f} - \hat{f}_{\eta,2k+2}, k, x) - (\hat{f} - \hat{f}_{\eta,2k+2})(x), g(x)\rangle + o(1) \\ &\leq C\left\{\sum_{r=0}^{2k} \|\hat{f}^{(r)} - \hat{f}^{(r)}_{\eta,2k+2}\|_{C(I_1)}\right\} + o(1) \\ &\qquad \text{(in view of property (a) of Lemma 1 and Lemma 7).}\end{aligned}$$

Therefore,

$$\begin{aligned}& \lim_{n_j\to\infty} n_j^{k+1}\langle M_{n_j}(\hat{f}, k, x) - \hat{f}(x), g(x)\rangle - \langle \hat{f}_{\eta,2k+2}(x), P^*_{2k+2}(D)g(x)\rangle \\ &\leq C\left\{\sum_{r=0}^{2k} \|\hat{f}^{(r)} - \hat{f}^{(r)}_{\eta,2k+2}\|_{C(I_1)}\right\}.\end{aligned}$$

Taking limit as $\eta \to 0$ and using (3.1), we get

$$\lim_{n_j\to\infty} n_j^{k+1}\langle M_{n_j}(\hat{f}, k, x) - \hat{f}(x), g(x)\rangle - \langle \hat{f}(x), P^*_{2k+2}(D)g(x)\rangle = 0. \tag{3.4}$$

Combining (3.2) and (3.4), we have $\langle \hat{f}(x), P^*_{2k+2}(D)g(x)\rangle = \langle h(x), g(x)\rangle$ and hence

$$h = P_{2k+2}(D)\hat{f} \tag{3.5}$$

as generalized functions.

Now, in view of Theorem 4, we have $Q(2k+2,k,x) \neq 0$. Hence, regarding (3.5) as a genralized first order linear differential equation for $\hat{f}^{(2k+1)}$ with the non-homogeneous terms linearly depending on $\hat{f}^{(i)}$, $0 \leq i \leq 2k$ and h with polynomial coefficients, as $\hat{f}^{(i)} \in C(J_2)$ $(0 \leq i \leq 2k)$ and $h \in L_p(J_2)$ we conclude that $\hat{f}^{(2k+1)} \in AC(J_2)$ and therefore that $\hat{f}^{(2k+2)} \in L_p(J_2)$. Since $\hat{f}$ coincides with F on J_2 it follows that $F^{(2k+1)} \in AC(I_2)$ and that $F^{(2k+2)} \in L_p(I_2)$.

When $p = 1$, proceeding as in the case of $p > 1$ with (3.2) replaced by (3.3) we find that $F^{(2k)} \in AC(I_2)$ and $F^{(2k+1)} \in BV(I_2)$. This completes the proof of implication (i) $\Rightarrow$ (ii).

The implication (ii) $\Rightarrow$ (iii) follows from Lemma 6.

Assuming (iv), since $n^{k+1}\|M_n(f,k,.) - f\|_{L_p(I_1)} \to 0$ as $n \to \infty$, proceeding as in the proof of (i) $\Rightarrow$ (ii) it follows that $n^{k+1}\|M_n(\hat{f},k,.) - \hat{f}\|_{L_p(J_2)} \to 0$ as $n \to \infty$ and hence we find that $h(x)$ and $\phi_0(x)$ are zero functions.
Thus, $P^*_{2k+2}(D)\hat{f}(x) = 0$. This implies that $\hat{f}$ is $2k+2$ times continuously differentiable function. Now, applying Theorem 4 for the function $\hat{f}$

$$\lim_{n_j \to \infty} n_j^{k+1}\langle M_{n_j}(\hat{f},k,.) - \hat{f}, g\rangle = \langle P_{2k+2}(D)\hat{f}, g\rangle. \tag{3.6}$$

Comparing (3.4) and (3.6), we have $P_{2k+2}(D)\hat{f}(x) = 0$. Hence, over I_2, F is $2k+2$ times continuously differentiable function and $P_{2k+2}(D)F(x) = 0$.

Finally, (v) $\Rightarrow$ (vi) follows from Theorem 4.

References

[1] P. N. Agrawal and Ali J. Mohammad, *Linear combination of a new sequence of linear positive operators*, Revista de la U.M.A., 44:1(2003), 33-41.

[2] P. N. Agrawal and Ali J. Mohammad, *On L_p-approximation by a linear combination of a new sequence of linear positive operators*, Turk. J. Math., 27(2003), 389-405.

[3] P. N. Agrawal and Ali J. Mohammad, *On L_p-inverse theorem for a linear combination of a new sequence of linear positive operators*, Boletin AMV, Submitted for publication.

[4] C. P. May, *Saturation and inverse theorems for combinations of a class of exponential type operators*, Canad. J. Math., 28 (1976), 1224-1250.

[5] R. K. S. Rathore, Linear combinations of linear positive operators and generating relations in special functions, Ph.D. Thesis, I.I.T. Delhi, 1973.
[6] T. A. K. Sinha, Restructured sequences of linear positive operators for higher order L_p-approximation, Ph.D. Thesis, I.I.T. Kanpur (India), 1981.
[7] A. F. Timan, Theory of Approximation of Functions of a Real Variable (English Translation), Dover Publications, Inc., N.Y., 1994.

Department of Mathematics, I.I.T. Roorkee, Roorkee-247 667, India.
E-mail: pnappfma@iitr.ernet.in

Department of Mathematics, College of Education, Basrah, Iraq.
E-mail: alijasmoh@rediffmail.com