Approximation by Iterative Combination of a New Sequence of Linear Positive Operators

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Abstract. In[6] Micchelli had introduced a technique of iterative combination to improve the order of approximation by Bernstein polynomials. In the present paper, we have used his technique to improve the order of approximation by a new sequence of linear positive operators introduced by us in [2]. Here, we study the degree of approximation in ordinary and simultaneous approximation by the above combination of these operators.

§1. Introduction

Motivated by the integral modification of Szasz-Mirakian operators by Mazhar and Totik [5] to approximate Lebesgue integrable functions, we [2] introduced a new sequence of linear positive operators M_n as given below:

Let $\alpha > 0$ and $f \in C_\alpha[0,\infty) := \{f \in C[0,\infty) : |f(t)| \leq Me^{\alpha t}$ for some $M > 0\}$. Then,

$$M_n(f(t);x) = n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty} q_{n,\nu-1}(t)f(t)dt + (1+x)^{-n}f(0)\ , \quad (1.1)$$

where $p_{n,\nu}(x) = \binom{n+\nu-1}{\nu} x^\nu(1+x)^{-n-\nu}$ and $q_{n,\nu}(t) = \dfrac{e^{-nt}(nt)^\nu}{\nu!}$, x, $t \in [0,\infty)$.

We may also write the operators (1.1) as $M_n(f(t);x) = \int_0^{\infty} W_n(t,x) f(t)dt$ where the kernel $W_n(t,x) = n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)q_{n,\nu-1}(t) + (1+x)^{-n}\delta(t)$,

Advances in Constructive Approximation
M. Neamtu and E. B. Saff (eds.), pp. 13–24.

ISBN 0-9728482-2-3

$\delta(t)$ being the Dirac-delta function. The space $C_\alpha[0,\infty)$ is normed by $\|f\|_{C_\alpha} = \sup_{0\le t<\infty} |f(t)|e^{-\alpha t}, f \in C_\alpha[0,\infty)$.

The order of approximation by the operators (1.1) is, at best $O(n^{-1})$ however smooth the function may be. Therefore, in order to improve the rate of convergence $O(n^{-1})$ by these operators, the technique of linear combination as described in [4] has been used [2]. There is yet another approach for improving the order of approximation, which was offered by Micchelli [6] by considering the iterative combinations $U_{n,k} = I - (I - B_n)^k$ of the Bernstein polynomials B_n, where $k \in N$ (the set of positive integers). He proved some direct and saturation results for these operators $U_{n,k}$ using semi-group method. Agrawal [1] studied an inverse theorem in simultaneous approximation for the operators $U_{n,k}$.

In the present paper, we have investigated certain problems concerning the degree of approximation in ordinary and simultaneous approximation by the above iterative combination for the operators (1.1). The iterates of the operator M_n are defined by $M_n^0 = I$ and $M_n^r = M_n(M_n^{r-1}), r \in N$. For $k \in N$, the iterative combination $L_{n,k} : C_\alpha[0,\infty) \to C^\infty[0,\infty)$ (the class of infinitely differentiable functions on $[0,\infty)$) of the operators M_n is defined as:

$$L_{n,k}(f(t);x) = (I - (I - M_n)^k)(f(t);x) = \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r} M_n^r(f(t);x). \tag{1.2}$$

Let $m \in N$ and $0 < a < b < \infty$, for sufficient small values of $\eta > 0$, the m^{th} order modulus of continuity $\omega_m(f,\eta;[a,b])$ for a continuous function f on the interval $[a,b]$ is defined as $\omega_m(f,\eta;[a,b]) = \sup\{|\Delta_h^m f(x)| : |h| \le \eta, x, x+mh \in [a,b]\}$, where $\Delta_h^m f(x)$ is the m^{th} order forward difference with step length h. For $m = 1$, $\omega_m(f,\eta;[a,b])$ is written simply as $\omega_f(\eta;[a,b])$ or $\omega(f,\eta;[a,b])$.

Throughout this paper, we denote by $0 < a < b < \infty$, $C[a,b]$ the space of all continuous functions on $[a,b]$, $\|\cdot\|_{C[a,b]}$ the sup-norm on the space $C[a,b]$, $0 < a_1 < a_2 < b_2 < b_1 < \infty$, $I_i = [a_i,b_i], i = 1,2$ and C a constant not necessarily the same in different cases.

§2. Auxiliary Results

In the sequel, we shall require the following results:

For $f \in C_\alpha[0,\infty), \eta > 0$ and $m \in N$, the Steklov mean $f_{\eta,m}$ of m^{th} order corresponding to f is defined by:

$$f_{\eta,m}(x) = \eta^{-m}\left(\int_{-\eta/2}^{\eta/2}\right)^m \left\{f(x) + (-1)^{m-1}\Delta^m_{\sum_{i=1}^m x_i} f(x)\right\}\prod_{i=1}^m dx_i, x \in I_1.$$

Lemma 1 [8]. *For the function $f_{\eta,m}(x)$ defined above, we have*

$(a) f_{\eta,m}(x)$ has derivatives up to order m over I_1;
$(b) \|f_{\eta,m}^{(r)}\|_{C(I_2)} \leq M_r \eta^{-r} \omega_r(f,\eta;I_1), r = 1,2,...,m$;
$(c) \|f - f_{\eta,m}\|_{C(I_2)} \leq M_{m+1}\omega_m(f,\eta;I_1)$;
$(d) \|f_{\eta,m}\|_{C(I_2)} \leq M_{m+2}\|f\|_{C_\alpha}$;
$(e) \|f_{\eta,m}^{(m)}\|_{C(I_2)} \leq M_{m+3}\|f\|_{C(I_1)}$,
where M_i's are certain constants that depend on i but are independent of f and η.

Lemma 2 [7]. *For $m \in N^0$ (the set of non-negative integers), the m^{th} order moment of the Lupas operators is defined by the function $\mu_{n,m}(x) = \sum_{\nu=0}^{\infty} p_{n,\nu}\left(\frac{\nu}{n} - x\right)^m$. Hence $\mu_{n,0}(x) = 1, \mu_{n,1}(x) = 0$ and there holds the recurrence relation $n\mu_{n,m+1}(x) = x(1+x)[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)]$, for $m \geq 1$. Consequently, we have*

(i) $\mu_{n,m}(x)$ is a polynomial in x of degree at most m;
(ii) for every $x \in [0,\infty), \mu_{n,m}(x) = O(n^{-[(m+1)/2]})$ where $[\beta]$ denotes the integer part of β.

Lemma 3 [2]. *Let $m \in N^0$, the m^{th} order moment for the operators (1.1) be defined by:*

$$T_{n,m}(x) = M_n((t-x)^m;x)$$

$$= n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty} q_{n,\nu-1}(t)(t-x)^m dt + (-x)^m(1+x)^{-n}.$$

Then $T_{n,0}(x) = 1, T_{n,1}(x) = 0$ and

$$nT_{n,m+1}(x) = x(1+x)T'_{n,m}(x) + mT_{n,m}(x) + mx(x+2)T_{n,m-1}(x), m \geq 1.$$

Further, we have the following consequences of $T_{n,m}(x)$:

(i) $T_{n,m}(x)$ is a polynomial in x of degree m and in n^{-1} of degree $m-1$, $m \neq 1$;
(ii) for every $x \in [0,\infty), T_{n,m}(x) = O(n^{-[(m+1)/2]})$.

Lemma 4 [2]. *Let δ and γ be any positive real numbers. Then for any $m > 0$ we have*

$$\left\| \int_{|t-x|\geq\delta} W_n(t,x)e^{\gamma t}dt \right\|_{C[a,b]} = O(n^{-m}).$$

Lemma 5 [7]. *There exist the polynomials $q_{i,j,p}(x))$ independent of n and ν such that*

$$\frac{d^p}{dx^p}\left[x^{\nu}(1+x)^{-n-\nu}\right] = \sum_{2i+j\leq p, i,j\geq 0} n^i(\nu-nx)^j q_{i,j,p}(x)x^{\nu-p}(1+x)^{-n-\nu-p}.$$

For every $m \in N^0$, the m^{th} order moment $T^{\{p\}}_{n,m}$ for the operator M^p_n where $p \in N$, is defined by $T^{\{p\}}_{n,m}(x) = M^p_n((t-x)^m; x)$. We denote $T^{\{1\}}_{n,m}(x)$ by $T_{n,m}(x)$.

Lemma 6. *There holds the recurrence relation*

$$T^{\{p+1\}}_{n,m}(x) = \sum_{j=0}^{m}\binom{m}{j}\sum_{i=0}^{m-j} T_{n,i+j}(x)\frac{D^i}{i!}\left(T^{\{p\}}_{n,m-j}(x)\right), (D \equiv \frac{d}{dx}) \quad (2.1)$$

Proof: By the definition, we have

$$T^{\{p+1\}}_{n,m}(x) = M_n(M^p_n((t-x)^m; u); x) = M_n(M^p_n((t-u+u-x)^m; u); x)$$

$$= \sum_{j=0}^{m}\binom{m}{j} M_n((u-x)^j M^p_n((t-u)^{m-j}; u); x).$$

Since $M^p_n((t-u)^{m-j}; u)$ is a polynomial in u of degree $\leq m-j$, by Taylor's expansion, we can write $M^p_n((t-u)^{m-j}; u) = \sum\limits_{i=0}^{m-j}\frac{(u-x)^i}{i!}D^i$ $\left(T^{\{p\}}_{n,m-j}(x)\right)$.
Hence, the equation (2.1) is immediate. □

Lemma 7. *For every $x \in [0,\infty)$ we have*

$$T^{\{p\}}_{n,m}(x) = O(n^{-[(m+1)/2]}). \quad (2.2).$$

Proof: We prove (2.2) by induction on p. For $p = 1$, the result holds from Lemma 3. Assuming the result to be true for p and making use of Lemma 6, we obtain 2.2. □

Lemma 8. *For m^{th} order moment ($m \in N$) of the operators $L_{n,k}$ defined in (1.2) we find that $L_{n,k}((t-x)^m; x) = O(n^{-k})$.*

Proof: We prove this lemma by induction on k. First, for $k = 1$, the result follows from Lemma 3. Assuming the result to be true for k and applying Lemma 6, we prove it for $k+1$.

§3. Main Results

Ordinary Approximation. First, we establish a Voronoskaja type asymptotic formula for the operators $L_{n,k}$.

Theorem 1. *Let $f \in C_\alpha[0,\infty)$. If $f^{(2k)}$ exists at a point $x \in [0,\infty)$ then*

$$\lim_{n\to\infty} n^k \{L_{n,k}(f;x) - f(x)\} = \sum_{i=2}^{2k} \frac{f^{(i)}(x)}{i!} Q(i,k,x), \text{ and} \tag{3.1}$$

$$\lim_{n\to\infty} n^k \{L_{n,k+1}(f;x) - f(x)\} = 0, \tag{3.2}$$

where $Q(i,k,x)$ are certain polynomials in x of degree at most i.

Further, if $f^{(2k-1)}$ exists and is absolutely continuous over the interval $[0,b]$ and $f^{(2k)} \in L_\infty[0,b]$, then for any $[c,d] \subset (0,b)$ there holds

$$\|L_{n,k}(f;x) - f(x)\|_{C[c,d]} \le Cn^{-k} \left\{ \|f\|_{C_\alpha} + \|f^{(2k)}\|_{L_\infty[0,b]} \right\}. \tag{3.3}$$

Proof: By Taylor's expansion of f, we have

$$f(t) = \sum_{j=0}^{2k} \frac{f^{(j)}(x)}{j!}(t-x)^j + \epsilon(t,x)(t-x)^{2k},$$

where $\epsilon(t,x) \to 0$ as $t \to x$ and $|\epsilon(t,x)| \le Ce^{\alpha t}$ for some $C > 0$. Therefore,

$$n^k\{L_{n,k}(f(t);x) - f(x)\} = n^k \sum_{j=1}^{2k} \frac{f^{(j)}(x)}{j!} L_{n,k}((t-x)^j;x)$$

$$+n^k \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} M_n^r(\epsilon(t,x)(t-x)^{2k};x) := \sum\nolimits_1 + \sum\nolimits_2.$$

Using Lemma 8, we have

$$\sum\nolimits_1 = n^k \sum_{j=2}^{2k} \frac{f^{(j)}(x)}{j!} L_{n,k}((t-x)^j;x) = \sum_{j=2}^{2k} \frac{f^{(j)}(x)}{j!} Q(j,k,x) + o(1).$$

Since $\epsilon(t,x) \to 0$ as $t \to x$, thus for a given $\epsilon > 0$, there exists a $\delta > 0$ such that $|\epsilon(t,x)| < \epsilon$ whenever $|t-x| < \delta$. Suppose that $\phi_\delta(t)$ denotes the characteristic function of the interval $(x-\delta, x+\delta)$, then

$$|\sum\nolimits_2| \le n^k \sum_{r=1}^{k} \binom{k}{r} M_n^r(|\epsilon(t,x)|(t-x)^{2k}\phi_\delta(t) : x)$$

$$+n^k \sum_{r=1}^{k} \binom{k}{r} M_n^r(|\epsilon(t,x)|(t-x)^{2k}(1-\phi_\delta(t));x) := J_1 + J_2.$$

To estimate J_1, applying Lemma 7 we have $J_1 \leq \epsilon n^k \sum_{r=1}^{k} \binom{k}{r} M_n^r((t-x)^{2k};x) < \epsilon C$. For an arbitrary $s > 0$, applying Lemma 4 we have

$$|J_2| \leq n^k \sum_{r=1}^{k} \binom{k}{r} M_n^r(Ce^{\alpha t}(t-x)^{2k}(1-\phi_\delta(t));x) < \frac{C}{n^s} = o(1).$$

Since, $\epsilon > 0$ is arbitrary, thus $\sum_2 \to 0$ as $n \to \infty$. Combining the estimates of $\sum_1$ and $\sum_2$, we obtain (3.1).

The equation (3.2) can be proved along similar lines by noting the fact that

$$L_{n,k+1}((t-x)^j;x) = O(n^{-(k+1)}), \forall\ j \in N.$$

Now, we shall prove (3.3). For this purpose let ψ be the characteristic function of $[0,b]$. Thus,

$$L_{n,k}(f(t);x) - f(x) = L_{n,k}(\psi(t)(f(t)-f(x));x)$$
$$+L_{n,k}((1-\psi(t))(f(t)-f(x));x) := \sum\nolimits_3 + \sum\nolimits_4.$$

The estimate of $\sum_4$ can be found in a manner similar to the estimate of J_2. Thus, we have for all $x \in [c,d], \sum_4 \leq Cn^{-k}\|f\|_{C_\alpha}$.
For $t \in [0,b]$ and $x \in [c,d]$, by our hypothesis of f, we can write

$$f(t) - f(x) = \sum_{i=1}^{2k-1} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{1}{(2k-1)!}\int_x^t (t-w)^{2k-1} f^{(2k)}(w)dw.$$

Thus,

$$\sum\nolimits_3 = \sum_{i=1}^{2k-1} \frac{f^{(i)}(x)}{i!}\{L_{n,k}((t-x)^i;x) + L_{n,k}((\psi(t)-1)(t-x)^i;x)\}$$
$$+\frac{1}{(2k-1)!}L_{n,k}\left(\psi(t)\int_x^t (t-w)^{2k-1} f^{(2k)}(w)dw;x\right)$$
$$:= \sum_{i=1}^{2k-1} \frac{f^{(i)}(x)}{i!}\{J_3 + J_4\} + J_5. \tag{3.4}$$

By Lemma 8, we get $J_3 = O(n^{-k})$ uniformly for $x \in [c,d]$.

Proceeding as in the estimate of J_2 and taking into account the fact that $x \in [c,d]$

$$J_4 = O(n^{-k}).$$

By (1.2) and (2.2), we have $J_5 = \|f^{(2k)}\|_{L_\infty[0,b]} O(n^{-k})$.
Combining the estimates $J_3 - J_5$ in (3.4), we obtain

$$\|\sum\nolimits_3\|_{C[c,d]} \leq Cn^{-k}\{\sum_{i=1}^{2k-1} \|f^{(i)}\|_{C[c,d]} + \|f^{(2k)}\|_{L_\infty[0,b]}\}.$$

Finally, following [Thm.1,3], we obtain the required result. □

Theorem 2. *Let $f \in C_\alpha[0,\infty)$, then for sufficiently large n, we have*

$$\|L_{n,k}(f(t);x) - f(x)\|_{C(I_2)} \leq C\{\omega_{2k}(f, n^{-1/2}, I_1) + n^{-k}\|f\|_{C_\alpha}\},$$

where C is independent of f and n.

Proof: Let $f_{\eta,2k}$ be the Steklov mean of $(2k)^{th}$ order corresponding to f. Then,

$$\|L_{n,k}(f(t);x) - f(x)\|_{C(I_2)} \leq \|L_{n,k}(f(t) - f_{\eta,2k}(t);x)\|_{C(I_2)}$$
$$+\|L_{n,k}(f_{\eta,2k}(t);x) - f_{\eta,2k}(x)\|_{C(I_2)} + \|f_{\eta,2k}(x) - f(x)\|_{C(I_2)}$$
$$:= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3.$$

By using the property (c) of Lemma 1, we have $\sum_3 \leq C\omega_{2k}(f,\eta,I_1)$. Using Theorem 1, we get $\sum_2 \leq Cn^{-k} \sum_{j=2}^{2k} \|f^{(j)}_{\eta,2k}\|_{C(I_2)}$. Now, following [Thm.2,3] and making use of the properties (d),(b) of Lemma 1, we have

$$\|f^{(j)}_{\eta,2k}\|_{C(I_2)} \leq C(\|f\|_{C_\alpha} + \eta^{-2k}\omega_{2k}(f,\eta;I_1))(j = 2,3,...,2k).$$

Let a' and b' be such that $a_1 < a' < a_2 < b_2 < b' < b_1$ and ψ be the characteristic function of $[a',b']$. Then, by using Lemma 4 and property (c) of Lemma 1 we get

$$\sum\nolimits_1 \leq C\{\omega_{2k}(f,\eta,I_1) + n^{-m}\|f\|_{C_\alpha}\}.$$

Now, choosing $m \geq k$ and $\eta = n^{-1/2}$ in the estimates of $\sum_1, \sum_2$ and $\sum_3$, the result follows. □

Simultaneous Approximation. Here we extend the results in ordinary approximation to the case of simultaneous approximation. First, we prove that the derivatives of the operators $L_{n,k}$ are approximation processes for corresponding order derivatives of the function, i.e., we prove that $L^{(p)}_{n,k}(f(t);x) \to f^{(p)}(x)$ as $n \to \infty$, $p \in N$.

Theorem 3. *Suppose that $p \in N$, $f \in C_\alpha[0,\infty)$ and $f^{(p)}$ exists at a point $x \in (0,\infty)$, then*

$$\lim_{n\to\infty} L_{n,k}^{(p)}(f(t);x) = f^{(p)}(x). \tag{3.5}$$

Further, if $f^{(p)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, then (3.5) holds uniformly in $x \in [a,b]$.

Proof: To prove the theorem, it is sufficient to show that for every $r \in N$, $\lim_{n\to\infty} \frac{d^p}{dx^p}\{M_n^r(f(t);x)\} = f^{(p)}(x)$ and that it holds uniformly in the uniformity case. By Taylor's expansion of f at x, we have $f(t) = \sum_{i=0}^{p} \frac{f^{(i)}(x)}{i!}(t-x)^i + \epsilon(t,x)(t-x)^p$, where $\epsilon(t,x) \to 0$ as $t \to x$ and $|\epsilon(t,x)| \le Ce^{\alpha t}$ for some $C > 0$. Hence,

$$\frac{d^p}{dx^p}\{M_n^r(f(t);x)\} = \sum_{i=0}^{p} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_n^{(p)}(u,x) M_n^{r-1}((t-x)^i;u)du$$

$$+ \int_0^\infty W_n^{(p)}(u,x) M_n^{r-1}(\epsilon(t,x)(t-x)^p;u)du := \sum\nolimits_1 + \sum\nolimits_2.$$

From Lemma 3 it follows that, $\sum_1 = \left(\frac{(n+p-1)!}{n^p(n-1)!}\right)^r f^{(p)}(x) \to f^{(p)}(x)$, as $n \to \infty$. By using Lemma 5, we have

$$\left|\sum\nolimits_2\right| \le \sum_{2i+j\le p, i,j\ge 0} n^i \frac{|q_{i,j,p}(x)|}{x^p(1+x)^p} n \sum_{\nu=1}^{\infty} p_{n,\nu}(x)|\nu - nx|^j$$

$$\times \int_0^\infty q_{n,\nu-1}(u) M_n^{r-1}(|\epsilon(t,x)||t-x|^p;u)du$$

$$+ \frac{(n+p-1)!}{(n-1)!}(1+x)^{-n-p}|\epsilon(0,x)|x^p := J_1 + J_2.$$

Clearly, $J_2 \to 0$ as $n \to \infty$. To estimate J_1, suppose that $\phi_\delta(t)$ denotes the characteristic function of the interval $(x-\delta, x+\delta)$. Since $\epsilon(t,x) \to 0$ as $t \to x$, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that $|\epsilon(t,x)| < \epsilon$, whenever $0 < |t-x| < \delta$. For $|t-x| \ge \delta$, we have $|\epsilon(t,x)(t-x)^p| \le Ce^{\alpha t}$ for some $C > 0$. Hence,

$$J_1 \le \sup_{2i+j\le p, i,j\ge 0} \frac{|q_{i,j,p}(x)|}{x^p(1+x)^p} \sum_{2i+j\le p, i,j\ge 0} n^{i+1} \sum_{\nu=1}^{\infty} p_{n,\nu}(x)|\nu - nx|^j$$

$$\times\left\{\epsilon\int_0^\infty q_{n,\nu-1}(u)M_n^{r-1}(|t-x|^p\phi_\delta(t);u)du\right.$$

$$\left.+\int_0^\infty q_{n,\nu-1}(u)M_n^{r-1}(Ce^{\alpha t}(1-\phi_\delta(t));u)du\right\}:=J_3+J_4.$$

Now, applying Schwarz inequality, Lemma 2 and Lemma 7 it follows that $J_3=\epsilon O(1)$. Proceeding similarly, applying Schwarz inequality and Lemma 4 we obtain $J_4=o(1)$. Since $\epsilon>0$ is arbitrary, it follows that $J_1=o(1)$ and hence $\sum_2=o(1)$. This completes the proof of (3.5).

The uniformity assertion follows easily from the fact that $\delta(\epsilon)$ in the above proof can be chosen to be independent of $x\in[a,b]$ and all the other estimates hold uniformly in $x\in[a,b]$. In the next theorem, we establish an asymptotic formula for the operator $L_{n,k}^{(p)}$. □

Theorem 4. *Let $f\in C_\alpha[0,\infty)$ and $k,p\in N$. If $f^{(2k+p)}$ exists at some point $x\in(0,\infty)$, then*

$$\lim_{n\to\infty}n^k(L_{n,k}^{(p)}(f;x)-f^{(p)}(x))=\sum_{j=p}^{2k+p}Q(j,k,p,x)f^{(j)}(x), \tag{3.6}$$

where $Q(j,k,p,x)$ are certain polynomials in x. Further, if $f^{(2k+p)}$ exists and is continuous on the interval $(a-\eta,b+\eta)\subset(0,\infty)$, $\eta>0$, then (3.6) holds uniformly on [a,b].

Proof: Since $f^{(2k+p)}$ exists at $x\in(0,\infty)$, it follows that

$$f(t)=\sum_{m=0}^{2k+p}\frac{f^{(m)}(x)}{m!}(t-x)^m+\epsilon(t,x)(t-x)^{2k+p}$$

where $\epsilon(t,x)\to0$ as $t\to x$ and $|\epsilon(t,x)|\le Ce^{\alpha t}$ for some $C>0$. Therefore, we can write

$$L_{n,k}^{(p)}(f(t);x)=\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\int_0^\infty W_n^{(p)}(u,x)\left\{\sum_{m=0}^{2k+p}\frac{f^{(m)}(x)}{m!}\right.$$

$$\left.M_n^{r-1}((t-x)^m;u)+M_n^{r-1}(\epsilon(t,x)(t-x)^{2k+p};u)\right\}du:=\sum\nolimits_1+\sum\nolimits_2.$$

By the consequence of Lemma 3 and Theorem 1, we get

$$\sum\nolimits_1=\sum_{m=p}^{2k+p}\frac{f^{(m)}(x)}{m!}\sum_{i=0}^{m}\binom{m}{i}(-x)^{m-i}L_{n,k}^{(p)}(t^i;x)$$

$$= \sum_{m=p}^{2k+p} \frac{f^{(m)}(x)}{m!} \sum_{i=0}^{m} \binom{m}{i} (-x)^{m-i} \Big\{ D^p x^i$$

$$+n^{-k} \sum_{j=2}^{2k} D^p \Big(\frac{Q(j,k,x)}{j!} D^j x^i\Big) + o(n^{-k}) \Big\}$$

$$= \sum_{m=p}^{2k+p} \frac{f^{(m)}(x)}{m!} p! \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \binom{i}{p} x^{m-p}$$

$$+n^{-k} \sum_{j=p}^{2k+p} Q(j,k,p,x) f^{(m)}(x) + o(n^{-k})$$

$$= f^{(p)}(x) + n^{-k} \sum_{j=p}^{2k+p} Q(j,k,p,x) f^{(m)}(x) + o(n^{-k})$$

(in view of the identities)

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} \binom{i}{p} = \begin{cases} 0, & m > p, \\ (-1)^p, & m = p. \end{cases} \tag{3.7}$$

To estimate $\sum_2$, proceeding along the lines of the proof of $\sum_2 = o(1)$ in Theorem 3, we obtain $n^k \sum_2 = o(1)$. By combining the estimates of $\sum_1$ and $\sum_2$, we get the required result (3.6).

The uniformity assertion follows as in the proof of Theorem 3. □

Theorem 5. *Let $f \in C_\alpha[0,\infty)$ and $p \leq q \leq 2k+p$. If $f^{(q)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, then for sufficiently large n,*

$$\|L_{n,k}^{(p)}(f(t);x) - f^{(p)}(x)\|_{C[a,b]} \leq \max\Big\{ C_1 n^{-(q-p)/2} \omega(f^{(q)}, n^{-1/2}), C_2 n^{-k} \Big\}$$

where $C_1 = C_1(k,p), C_2 = C_2(k,p,f)$ and $\omega(f,\delta)$ denotes the modulus of continuity of f on the interval $(a-\eta, b+\eta)$.

Proof: By our hypothesis

$$f(t) = \sum_{i=0}^{q} \frac{f^{(i)}(x)}{i!} (t-x)^i + \frac{f^{(q)}(\xi) - f^{(q)}(x)}{q!} (t-x)^q \phi_\eta(t)$$

$$+h(t,x)(1-\phi_\eta(t)),$$

where ξ lies between t and x and $\phi_\eta(t)$ is the characteristic function of the interval $(a-\eta, b+\eta)$. For $t \in (a-\eta, b+\eta)$ and $x \in [a,b]$, we get

$$f(t) = \sum_{i=0}^{q} \frac{f^{(i)}(x)}{i!} (t-x)^i + \frac{f^{(q)}(\xi) - f^{(q)}(x)}{q!} (t-x)^q.$$

For $t \in [0,\infty) \setminus (a-\eta, b+\eta)$ and $x \in [a,b]$, we define $h(t,x) = f(t) - \sum_{i=0}^{q} \frac{f^{(i)}(x)}{i!}(t-x)^i$. Clearly, the function $h(t,x)$ is bounded by $Ce^{\alpha t}$ for some $C > 0$. Now,

$$L_{n,k}^{(p)}(f(t);x) = \left[\sum_{i=0}^{q} \frac{f^{(i)}(x)}{i!} L_{n,k}^{(p)}((t-x)^i;x)\right] + \frac{1}{q!} L_{n,k}^{(p)}\Big((f^{(q)}(\xi) - f^{(q)}(x))$$

$$\times (t-x)^q \phi_\eta(t);x\Big) + L_{n,k}^{(p)}(h(t,x)(1-\phi_\eta(t));x) := \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3.$$

Using Theorem 4 and the identities (3.7), we obtain $\sum_1 = f^{(p)}(x) + O(n^{-k})$ uniformly in $x \in [a,b]$. To estimate $\sum_2$ we proceed as follows:

$$\left|\sum\nolimits_2\right| \le \frac{\omega(f^{(q)},\delta)}{q!} \sum_{r=1}^{k} \binom{k}{r} \Bigg[n \sum_{\nu=1}^{\infty} |p_{n,\nu}^{(p)}(x)| \int_0^\infty q_{n,\nu-1}(t) M_n^{r-1}\big(|t-x|^q$$

$$+\delta^{-1}|t-x|^{q+1};u\big)du + \frac{(n+r-1)!}{(n-1)!}(1+x)^{-n-r}\big(|x|^q + \delta^{-1}|x|^{q+1}\big)\Bigg],$$

Now, using Lemma 5 and the Schwarz inequality three times (as in the estimate of J_3 in Theorem 3) in view of Lemmas 2 and 7, it follows that for $s = 0, 1, 2, \ldots$

$$n \sum_{\nu=1}^{\infty} |p_{n,\nu}^{(p)}(x)| \int_0^\infty q_{n,\nu-1}(u) M_n^{r-1}\big(|t-x|^s;u\big)du = O(n^{(p-s)/2}), \qquad (3.9)$$

uniformly in $x \in [a,b]$. Choosing $\delta = n^{-1/2}$ and applying (3.9), we are led to

$$\left\|\sum\nolimits_2\right\| \le C n^{-(p-q)/2} \omega(f^{(q)}, n^{-1/2}).$$

Since $t \in [0,\infty) \setminus (a-\eta, b+\eta)$, we can choose $\delta > 0$ in such a way that $|t-x| \ge \delta$ for all $x \in [a,b]$. Thus, by Lemmas 5 and 4, Schwarz inequality, and Lemma 2, it is easy to see that $\sum_3 = O(n^{-s})$ for any $s > 0$, uniformly on $[a,b]$. Finally, combining the estimates of $\sum_1 - \sum_3$, the required result is immediate. □

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