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Approximation by Linear Combination of Generalization of Phillips Baskakov Operators of Summation-Integral Form

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Abstract

The aim of the present paper is to study the phenomena of simultaneous approximation (approximation of derivatives of functions by the corresponding order derivatives of operators) by the linear combination $S_{n,v}(.,k,x)$ of $S_{n,v}$. We establish a Voronovskaja-type asymptotic formula and obtain an error estimate in terms of the modulus of continuity for these operators.

1.Introduction

In [1] Agrawal and Thamer proposed a new sequence of linear positive operators M_n called integral Baskakov – type operators to approximate unbounded continuous functions on $[0,\infty)$ and it is defined as follows

Let

$$\alpha > 0, f \in C_\alpha[0,\infty) = \{f \in C[0,\infty): |f(t)| \leq M(1+t)^\alpha for\ some\ M > 0\}$$

Then,

$$M_n(f(t);x) = (n-1)\sum_{v=1}^{\infty} p_{n,v}(x)\int_0^\infty p_{n,v-1}(t)f(t)dt + (1+x)^{-n}f(0), \quad (1.1)$$

Where $p_{n,v}(x) = \binom{n+v-1}{v}x^v(1+x)^{-(n+v)}$

$, x \in [0,\infty)$.

After that [5] Mohammad and Hassan defined and studied the following generalization form of summation-integral-Phillips Baskakov type operators:

$$S_{n,v}(f(t),x) = (n-1)\sum_{k=v}^{\infty} p_{n,k}(x)\int_0^\infty p_{n,k-v}(t)\, f(t)dt + f(0)\sum_{k=0}^{v-1} p_{n,k}\,(x) \quad (1.2)$$

Where $f(0)\sum_{k=0}^{v-1} p_{n,k}\,(x) = 0$ whenever $v = 0$.

In [7] Thamer and Ibrahem introduced a simultaneous approximation with linear combination of operators (1.1). The linear combination is defined as follows:

Let $d_0, d_1, \ldots, d_k$ be $(k+1)$ arbitrary but fixed distinct positive integers. Then, following [2] Agrawal and Sinha, the linear combination $M_n(f,k,x)$ of $M_{d_j n}(f,x), j = 0,1,2,\ldots,k$ is given by

$$M_n(f,k,x) = \frac{1}{\Delta}\begin{vmatrix} M_{d_0 n}(f;x) & d_0^{-1} & d_0^{-2} & \cdots & d_0^{-k} \\ M_{d_1 n}(f;x) & d_1^{-1} & d_1^{-2} & \cdots & d_1^{-k} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ M_{d_k n}(f;x) & d_k^{-1} & d_k^{-2} & \cdots & d_k^{-k} \end{vmatrix} \quad (1.3)$$

where Δ is the Vandermonde determinant obtained by replacing the operator column of the above determinant by the entries 1. We have

$$M_n(f,k,x) = \sum_{j=0}^{k} C(j,k)\, M_{d_j n}(f;x), \quad (1.4)$$

where

$$C(j,k) = \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{d_j}{d_j - d_i},$$

$$k \neq o\ and\ C(0,0) = 1 \quad (1.5)$$

Through this paper, we denote by $C[a,b]$ the apace of all continuous function on the interval $[a,b]$, $\|\cdot\|_{C[a,b]}$ denotes the norm of the space $C[a,b]$ which is defined as: $\|f\|_{C[a,b]} = \sup_{t\in[a,b]} |f(t)|$ and C denotes a constant not necessarily the same in different cases.

In this paper we study a Voronovaskaja-type asymptotic formula and an error estimate of the modulus of continuity of the function approximated by operators $S_{n,v}^{(r)} = (\cdot,k,x)$, where $r \in N$.

Definition 1.1[6]: A function f is at most the order of a function g as $x \to \infty$, if there is a positive integer A for which $\frac{f(x)}{g(x)} \leq A, g(x) \neq 0$, for x sufficiently large. We indicate this by writing $f = O(g)$, i.e. (f is big – oh of g).

Definition 1.2[6]: A function f is at smaller the order of a function g as $x \to \infty$, if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0, g(x) \neq 0$, we indicate this by writing $f = o(g)$, i.e. (f is little – oh of g).

2.Auxiliary Results

Lemma 2.1[3]: For $m \in N^0$, the m^{th} order moment of Baskakov operators is define by:

$$\mu_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x)\left(\frac{k}{n} - x\right)^m.$$

Hence, $\mu_{n,0}(x) = 1, \mu_{n,1}(x) = 0$ and

$$n\mu_{n,m+1}(x) = x(1+x)\left[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)\right], \qquad m \geq 1$$

Consequently

(i) $\mu_{n,m}(x)$ is a polynomial in x of degree at most m.

(ii) For every $x \in [0,\infty), \mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$.

Lemma 2.2[5]: Let the function $T_{n,m,v}(x), m \in N^0$ be defined as

$$T_{n,m,v}(x) = S_{n,v}((t-x)^m; x) = (n-1)\sum_{k=v}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k-v}(t)(t-x)^m dt + \sum_{k=0}^{v-1} p_{n,k}(x)(-x)^m.$$

Then

$$T_{n,0,v}(x) = 1, T_{n,1,v}(x) = \frac{2x+1-v}{n-2} - \frac{1}{n-2}\left[\sum_{k=0}^{v-1}(k+1-v)\, p_{n,k}(x)\right],$$

$$T_{n,2,v}(x) = \frac{2nx^2 + 2nx + v^2 - 3v + 2 + 6x + 6x^2 - 6vx}{(n-2)(n-3)} + \frac{1}{(n-2)(n-3)}(-v^2 + 3v - 2 + 2nx - 2nvx - 6x + 6vx)\sum_{k=0}^{v-1} p_{n,k}(x) + \frac{1}{(n-2)(n-3)}(2v - 3 + 2nx - 6x)\sum_{k=0}^{v-1} k\, p_{n,k}(x) - \frac{1}{(n-2)(n-3)}\sum_{k=0}^{v-1} k^2\, p_{n,k}(x)$$

Also, we have the following recurrence relation for $T_{n,m,v}(x)$ whenever

$n > m + 2$,

$$(n-m-2)T_{n,m+1,v}(x) = x(1+x)T'_{n,m,v}(x) + \big((2x+1)m+2x+1 - v\big)T_{n,m,v}(x) + 2mx(1+x)T_{n,m-1,v}(x) + (-x)^m \sum_{k=0}^{v-1}(v-1-k)P_{n,k}(x).$$

And for every $x \in [0,x), T_{n,m,v}(x) = O\left(n^{-\left[\frac{(m+1)}{2}\right]}\right)$.

Lemma 2.3[4]: There exist polynomials $q_{i,j,r}(t)$ independent of n and v such that

$$t^r(1+t)^r \frac{d^r}{dt^r} p_{n,v}(t) = \sum_{\substack{2i+m\le r \\ i,m\ge 0}} n^i (v-nt)^m q_{i,m,r}(t) p_{n,v}(t).$$

Lemma 2.4: For $f \in N$ and n sufficiently large, there holds

$$S_{n,v}((t-x)^m,k,x) = n^{-(k+1)}\{Q(r,k.x) + o(1)\},$$

where $Q(r,k.x)$ is a certain polynomial in x of degree r.

Proof: By using the Lemma 2.2, we can write

$$T_{d_j n,m}(x) = \frac{P_1(x)}{(d_j n)^{[(m+1)/2]}} + \frac{P_2(x)}{(d_j n)^{[(m+1)/2]+1}} + \cdots + \frac{P_{[m/2]}(x)}{(d_j n)^{m-1}}$$

for certain polynomial $P_i, i = 1,2,3,\dots,[m/2]$ in x of degree at most m, it follows that

$$S_{n,v}((t-x)^m,k,x) = \sum_{j=0}^{k} C(j,k)\, T_{d_j n,m}(x)$$

$$= \frac{1}{\Delta}\begin{vmatrix} T_{d_0 n,m}(x) & d_0^{-1} & d_0^{-2} \dots & d_0^{-k} \\ T_{d_1 n,m}(x) & d_1^{-1} & d_1^{-2} \dots & d_1^{-k} \\ \vdots & \vdots & \vdots \ \cdots & \vdots \\ T_{d_k n,m}(x) & d_k^{-1} & d_k^{-2} \dots & d_k^{-k} \end{vmatrix}$$

$$= n^{-(k+1)}\{Q(m,k,x) + o(1)\}, \quad m = 1,2,3,\dots$$

3. Main results

Theorem 3.1: Let $f \in C_\alpha[0,\infty)$ admitting a derivative of order $(2k+2)$ at a point $x \in (0,\infty]$. Then

$$\lim_{n\to\infty} n^{k+1}\left[S_{n,v}(f,k,x) - f(x)\right] = \sum_{m=1}^{2k+2} \frac{f^{(m)}}{m!} Q(m,k,x) \qquad (3.1)$$

and

$$\lim_{n\to\infty} n^{k+1}\left[S_{n,v}(f,k+1,x) - f(x)\right] = 0 \qquad (3.2)$$

where $Q(m,k,x)$ are certain polynomials in x of degree m.

Proof: By the hypothesis,

$$f(t) = \sum_{m=0}^{2k+2} \frac{f^{(m)}(x)}{m!}(t-x)^m + \varepsilon(t,x)(t-x)^{2k+2},$$

where $\varepsilon(t,x) \to 0$ as $t \to x$.

In view of $S_{n,v}(1,k,x) = 1$, we have

$$n^{k+1}[S_{n,v}(f,k,x)-f(x)]$$
$$=n^{k+1}\sum_{m=1}^{2k+2}\frac{f^{(m)}(x)}{m!}S_{n,v}((t-x)^m,k,x)$$
$$+n^{k+1}\sum_{j=0}^{k}C(j,k)S_{d_jn}(\varepsilon(t,x)(t-x)^{2k+2};x)$$
$$:=I_1+I_2.$$

Clearly,

$$I_1=\sum_{m=1}^{2k+2}\frac{f^{(m)}(x)}{m!}\sum_{k=0}^{k}C(j,k)\,T_{d_jn,m}(x).$$

Using Lemma 2.4, we have

$$I_1=\sum_{m=1}^{2k+2}\frac{f^{(m)}(x)}{m!}Q(m,k,x)+o(1).$$

The expressions for $Q(2k+1,k,x)$ and $Q(2k+2,k,x)$ can be easily obtained on an application of Lemmas 2.2 and 2.4.

Next, we show that $I_2\to 0$ as $n\to\infty$. For a given $\varepsilon>0$, there exist a $\delta>0$ such that $|\varepsilon(t,x)|<\varepsilon$, whenever $0<|t-x|<\delta$, and for $|t-x|\geq\delta$, $|\varepsilon(t,x)(t-x)^{2k+2}|\leq A|t-x|^{\gamma}$ where $\gamma$is any integer $>2k+2$ and A is a constant.

Let $\chi(t)$ be a characteristic function of the interval $(x-\delta,x+\delta)$, then

$$|I_2|\leq n^{k+1}\sum_{j=0}^{k}|C(j,k)|\,S_{d_jn}(|\varepsilon(t,x)(t-x)^{2k+2}\chi(t);x|)$$
$$+n^{k+1}\sum_{j=0}^{k}|C(j,k)|\,S_{d_jn}(|\varepsilon(t,x)(t-x)^{2k+2}(1-\chi(t));x|)$$
$$:=I_3+I_4$$

In view of Lemma 2.2, we have

$$I_3\leq\varepsilon n^{k+1}\left(\sum_{j=0}^{k}|C(j,k)|\right)\max_{0\leq j\leq k}\left[S_{d_jn}((t-x)^{2k+2},x)\right]$$
$$\leq\varepsilon O(n^{k+1})O\left(n^{-(k+1)}\right)=\varepsilon O(1)$$

Using Schwarz inequality for integration and then for summation and Lemma 2.2, we have

$$I_4\leq An^{k+1}\sum_{j=0}^{k}|C(j,k)|\,S_{d_jn}(|t-x|^{\gamma}(1-\chi(t));x)$$
$$\leq An^{k+1}\sum_{j=0}^{k}|C(j,k)|\left\{S_{d_jn}((t-x)^{2\gamma};x)\right\}^{1/2}$$
$$=O\left(n^{(2k+2-\gamma)/2}\right)=o(1)$$

Since $\varepsilon>0$ is arbitrary, combining the estimates of I_3 and I_4 we conclude that $I_2\to 0$ as $n\to\infty$.

The assertion (3.2) can be proved in a similar manner as

$$S_n((t-x)^m,k+1,x)=O\left(n^{-(k+2)}\right).$$

For all $m=k+3,k+4,\ldots,2k+2$.

■

Theorem 3.2: Let $f\in C_\alpha[0,\infty)$ and be bounded on every finite subinterval of $[0,\infty)$ admitting a derivative of order $2k+r+2$ at a fixed $x\in(0,\infty)$. Let $f(t)=O(t^\alpha)$ as $t\to\infty$ for some $\alpha>0$ then we have

$$\lim_{n\to\infty} n^{k+1}\left[S_n^{(r)}(f,k,x) - f^{(r)}(x)\right] = \sum_{i=r}^{2k+r+2} f^{(i)}(x)\, Q(i,k,r,x) \qquad (3.3)$$

and

$$\lim_{n\to\infty} n^{k+1}\left[S_n^{(r)}(f,k+1,x) - f^{(r)}(x)\right] = 0, \qquad (3.4)$$

where $Q(i,k,r,x)$ are certain polynomials in x.

Further, the Limits (3.3) and (3.4) hold uniformly in [a, b], if $f^{(2k+r+2)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$.

Proof: By Tayior's expansion, we have

$$f(t) = \sum_{i=0}^{2k+r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{2k+r+2}$$

where $\varepsilon(t,x) \to 0$ as $t \to x$.

Then

$$n^{k+1}\left[S_n^{(r)}(f,k,x) - f^{(r)}(x)\right] = n^{k+1}\left[\sum_{i=0}^{2k+r+2} \frac{f^{(i)}(x)}{i!} S_n^{(r)}\left((t-x)^i,k,x\right) + \sum_{j=0}^{k} c(j,k)\, S_{d_jn}^{(r)}\left(\varepsilon(t,x)(t-x)^{2k+r+2};x\right) - f^{(r)}(x)\right]$$

$$:= I_1 + I_2.$$

By using Lemma 2.2 and Theorem 3.1, we have

$$I_1 = n^{k+1}\left[\sum_{i=r}^{2k+r+2} \frac{f^{(i)}(x)}{i!} S_n^{(r)}\left((t-x)^i,k,x\right) - f^{(r)}(x)\right]$$

$$= n^{k+1}\left[\sum_{i=r}^{2k+r+2} \frac{f^{(i)}(x)}{i!} \sum_{m=0}^{i} \binom{i}{m}(-x)^{i-m} S_n^{(r)}(t^m,k,x) - f^{(r)}(x)\right]$$

$$= n^{k+1}\left[\sum_{i=r}^{2k+r+2} \frac{f^{(i)}(x)}{i!} \sum_{m=0}^{i} \binom{i}{m}(-1)^{i-m}x^{i-m} \times \left[D^r x^i + n^{-(k+1)}\left\{\sum_{j=k+2}^{2k+2} D^r\left(\frac{D^j x^i}{j!} Q(j,k,x)\right) + o(1)\right\}\right]\right] - f^{(r)}(x)$$

$$= n^{k+1}\left[\sum_{i=r}^{2k+r+2} \frac{f^{(i)}(x)}{i!} r! \sum_{m=0}^{i} \binom{i}{m}\binom{m}{r}(-1)^{i-m}x^{i-r} - f^{(r)}(x)\right] + \sum_{i=r}^{2k+2+r} Q(i,k,r,x)\, f^{(i)}(x) + o\left(n^{-(k+1)}\right)$$

$$= \sum_{i=r}^{2k+2+r} Q(i,k,r,x)\, f^{(i)}(x) + o\left(n^{-(k+1)}\right).$$

Where

$$\sum_{m=0}^{i} (-1)^m \binom{i}{m}\binom{m}{r} = \begin{cases} 0 & ,i > r \\ (-1)^r & ,i = r \end{cases}.$$

Now, using Lemma 2.3, we get

$$|I_2|$$

$$\leq n^{k+1}\left[\sum_{j=0}^{k} c(j,k) \sum_{\substack{2i+m\leq r\\ i,m\geq o}} (d_j n)^i \frac{|q_{m,j,r}(x)|}{x^r(1+x)^r}(d_j n\right.$$

$$-1)\sum_{k=p}^{\infty} P_{d_j n,k}(x)\,|k$$

$$-d_j nx|^j \int_0^{\infty} P_{d_j n,k-p}(t)\,|\varepsilon(t,x)||t$$

$$-x|^{2k+r+2}dt$$

$$+\sum_{\substack{2i+m\leq r\\ i,m\geq o}} (d_j n)^i \frac{|q_{i,m,r}(x)|}{x^r(1+x)^r}\sum_{k=0}^{p-1} P_{d_j n,k}(x)\big(k$$

$$\left.-d_j nx\big)^j|\varepsilon(0,x)|x^{2k+r+2}\right]$$

$$:= I_3 + I_4$$

Since $\varepsilon(t,x) \to 0$ as $t \to x$, then for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\varepsilon(t,x)| < \varepsilon$, whenever $0 < |t-x| < \delta$. For $|t-x| \geq \delta$, there exist a constant C such that $|\varepsilon(t,x)(t-x)^r| \leq C|t-x|^{\gamma}$.

$$I_3$$

$$\leq Cn^{k+1}\sum_{j=0}^{k} c(j,k) \sum_{\substack{2i+j\leq r\\ i,j\geq o}} (d_j n)^i (d_j n$$

$$-1)\sum_{k=p}^{\infty} P_{d_j n,k}(x)\,|k$$

$$-d_j nx|^j \left\{\varepsilon \int_{|t-x|<\delta} P_{n,k-p}(t)|t\right.$$

$$-x|^{2k+r+2}dt$$

$$\left.+\int_{|t-x|\geq\delta} P_{n,k-p}(t)|t-x|^{2k+\gamma+2}dt\right\}$$

$$:= I_5 + I_6$$

Where

$$\sup_{\substack{2i+m\leq r\\ i,m\geq o}} \frac{|q_{i,m,r}(x)|}{x^r(1+x)^r} := M(x) = C$$

$$\forall x \in (0,\infty)$$

Now, applying Schwartz inequality for integration and summation, we get

$$I_5$$

$$\leq \varepsilon\, Cn^{k+1}\sum_{j=0}^{k} c(j,k) \sum_{\substack{2i+j\leq r\\ i,j\geq o}} (d_j n)^i (d_j n$$

$$-1)\sum_{k=p}^{\infty} P_{d_j n,k}(x)\,|k$$

$$-d_j nx|^j \left(\int_0^{\infty} P_{n,k-p}(t)dt\right)^{\frac{1}{2}} \left(\int_0^{\infty} P_{n,k-p}(t)|t\right.$$

$$\left.-x|^{2(2k+r+2)}dt\right)^{\frac{1}{2}}$$

as $\int_0^{\infty} P_{n,k-p}(t)dt = \frac{1}{n-1}$,we have

$$I_5$$

$$\leq \varepsilon\, Cn^{k+1}\sum_{j=0}^{k} c(j,k) \sum_{\substack{2i+j\leq r\\ i,j\geq o}} (d_j n)^i \left(\sum_{k=p}^{\infty} P_{d_j n,k}(x)\,(k\right.$$

$$\left.-d_j nx)^{2j}\right)^{\frac{1}{2}} \Bigg((d_j n$$

$$-1)\sum_{k=p}^{\infty} P_{d_j n,k}(x)\int_0^{\infty} P_{n,k-p}(t)(t$$

$$\left.-x)^{2(2k+r+2)}dt\right)^{\frac{1}{2}}$$

$$\leq \varepsilon\, Cn^{k+1} O\left(n^{-(2k+r+2)/2}\right) \sum_{\substack{2i+j\leq r\\ i,j\geq o}} n^i\, O\left(n^{\frac{j}{2}}\right)$$

$$= \varepsilon O(1).$$

Now, by using Schwarz inequality for integration and then for summation

$$I_6 \leq Cn^{k+1} \sum_{j=0}^{k} c(j,k) \sum_{\substack{2i+j\leq r\\ i,j\geq o}} (d_j n)^i (d_j n - 1) \times \sum_{k=p}^{\infty} P_{d_j n,k}(x)\, |k - d_j nx|^j \int_{|t-x|\geq\delta} P_{n,k-p}(t) |t - x|^{2k+\gamma+2} dt$$

$$\leq Cn^{k+1} \sum_{j=0}^{k} c(j,k) \sum_{\substack{2i+j\leq r\\ i,j\geq o}} (d_j n)^i (d_j n - 1) \times \sum_{k=p}^{\infty} P_{d_j n,k}(x)\, |k - d_j nx|^j \left(\int_{|t-x|\geq\delta} P_{n,k-p}(t) dt\right)^{\frac{1}{2}} \times \left(\int_{|t-x|\geq\delta} P_{n,k-p}(t)(t - x)^{2(2k+\gamma+2)}\right)^{\frac{1}{2}}$$

$$\leq Cn^{k+1} \sum_{j=0}^{k} c(j,k) \sum_{\substack{2i+j\leq r\\ i,j\geq o}} (d_j n)^i \times \left(\sum_{k=p}^{\infty} P_{d_j n,k}(x)\left(k - d_j nx\right)^{2j}\right)^{\frac{1}{2}} \times \left((d_j n - 1) \sum_{k=p}^{\infty} P_{d_j n,k}(x) \int_0^{\infty} P_{n,k-p}(t)(t - x)^{2(2k+\gamma+k)}\right)^{\frac{1}{2}}$$

$$\leq Cn^{k+1} O\left(n^{\frac{-(2k+r+2)}{2}}\right) \sum_{\substack{2i+j\leq r\\ i,j\geq o}} n^i\, O\left(n^{\frac{j}{2}}\right) = o(1)$$

Now, since $\varepsilon > 0$ is arbitrary, it follows that $I_2, I_4 \to 0$ as $n \to \infty$. The assertion (3.4) can be proved along similar lines by noting that

$$S_n((t-x)^m, k+1, x) = O\left(n^{-(k+2)}\right),$$

for $m = k+3, k+4, \ldots$.

The uniformity assertion follows easily from the fact that $\delta(\varepsilon)$ in the above proof can be chosen to be independent of $x \in [a,b]$ and all the other estimates hold uniformly on $[a,b]$. ■

Theorem 3.3: Let $1 \leq p \leq 2k+2$ and $f \in C_\alpha[0,\infty)$ for some $\alpha > 0$. If $f^{(p+r)}$ exists and continuous on $(a-\eta, b+\eta), \eta > 0$, then for sufficiently large n,

$$\left\|S_n^{(r)}(f,k,x)-f^{(r)}\right\|_{c[a,b]} \leq Max\left(C_1 n^{-p/2}\omega_{f^{(p+r)}}\left(n^{-1/2}\right),\left(C_2 n^{-(k+1)}\right)\right),$$

where $\omega_{f^{(p+r)}}(\delta)$ denoted the modulus continuity of $f^{(p+r)}$ on $(a-\eta,b+\eta)$, $C_1=C_1(k,p,r)$, $C_2=C_2(k,p,r,f)$ are constant and $\|.\|$ denoted the sup-norm on $[a,b]$.

Proof: By the hypothesis

$$f(t) = \sum_{i=0}^{p+r}\frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{\left(f^{(p+r)}(\xi)-f^{(p+r)}(x)\right)}{(p+r)!}(t-x)^{(p+r)}\chi(t) + h(t,x)\left(1-\chi(t)\right), \qquad (3.5)$$

where ξ lies between t and x,and $\chi(t)$ is the characteristic function of the interval $(a-\eta,b+\eta)$. Operating on this equality by $S_n^{(r)}(.,k,x)$ and breaking the right hand side into three parts I_1, I_2 and I_3 say, corresponding to the three terms on the right hand side of (3.5) as in the proof of Theorem 3.2, we have

$I_1=f^{(r)}(x)+O\left(n^{-(k+1)}\right)$, uniformly for all $x\in[a,b]$.

For every $\delta>0$, we have

$$\left|f^{(p+r)}(\xi)-f^{(p+r)}(x)\right| \leq \omega_{f^{(p+r)}}(|\xi-x|) \leq \omega_{f^{(p+r)}}(|t-x|) \leq \left(1+\frac{|t-x|}{\delta}\right)\omega_{f^{(p+r)}}(\delta). \qquad (3.6)$$

Since

$$I_2=\frac{1}{(p+r)!}S_n^{(r)}\left(\left(f^{(p+r)}(\xi)-f^{(p+r)}(x)\right)(t-x)^{p+r}\chi(t),k,x\right)$$

Using (3.6), we have

$$|I_2| \leq \frac{\omega_{f^{(p+r)}}(\delta)}{(p+r)!}\left[\sum_{j=0}^{k}|C(j,k)|\left(d_j n-1\right)\sum_{l=v}^{\infty}\left|p_{n,l}^{(r)}(x)\right|\int_0^\infty p_{n,l-v}(t)(|t-x|^{p+r}+\delta^{-1}|t-x|^{p+r+1})dt + \sum_{l=0}^{r-1}p_{n,l}^{(r)}(x)(|x|^{p+r}+\delta^{-1}|x|^{p+r+1})\right],$$

$\delta>0.$

Now, for $s=0,1,2,\ldots$, using Schwartz inequality for integration ,summation, Lemmas 2.1 and 2.2, we have

$$(n-1)\sum_{l=v}^{\infty}p_{n,l}(x)|l-nx|^j\int_0^\infty p_{n,l-v}(t)|t-x|^s\,dt$$

$$\leq O\left(n^{j/2}\right)O\left(n^{-s/2}\right)=O\left(n^{(j-s)/2}\right),$$
uniformly on $[a,b]$. (3.7)

Therefore, by Lemma 2.3, we get

$$\sum_{j=0}^{k}|C(j,k)|\left(d_j n-1\right)\sum_{l=v}^{\infty}\left|p_{d_j n,v}^{(r)}(x)\right|\int_o^\infty p_{d_j n,l-v}(t)\,|t-x|^s dt$$

$$\leq \sum_{j=0}^{k} |C(j,k)| \left(d_j n - 1\right) \sum_{l=v}^{\infty} \sum_{\substack{2i+m\leq r\\ i,m\geq 0}} (d_j n)^i \, |l - nx|^j \frac{|q_{i,m,r}(t)|}{x^r(1+x)^r} p_{d_j n,l}(x)$$

$$\times \int_o^{\infty} p_{d_j n, l-v}(t)\, |t-x|^s dt$$

$$\leq \left(\sup_{\substack{2i+m\leq r\\ i,m\geq 0}} \sup_{x\in[a,b]} \frac{|q_{i,m,r}(t)|}{x^r(1+x)^r} \right) \sum_{j=0}^{k} |C(j,k)|$$

$$\times \sum_{\substack{2i+m\leq r\\ i,m\geq 0}} (d_j n)^i \left[\sum_{l=v}^{\infty} p_{d_j n,l}(x)\, |l - d_j nx|^m \int_o^{\infty} p_{d_j n,l-v}(t)\, |t-x|^s dt \right]$$

$$= C \sum_{\substack{2i+m\leq r\\ i,m\geq 0}} n^i \, O\left(n^{(m-s)/2}\right)$$

$= O\left(n^{(r-s)/2}\right)$, uniformly on [a,b].(3.8)

$$C = \sup_{\substack{2i+m\leq r\\ i,m\geq 0}} \sup_{x\in[a,b]} \frac{|q_{i,m,r}(t)|}{x^r(1+x)^r}$$

choosing $\delta = n^{-1/2}$ and applying (3.8), it follows that

$$I_2 = \omega_{f^{(p+r)}}\left(n^{-1/2}\right) O\left(n^{-p/2}\right).$$

For $x \in [a,b]$and $t \in [0,\infty)\backslash(a-\eta, b+\eta)$, we can choose a $\delta > 0$ in such a way that $|t-x| \geq \delta$. Hence

$|I_3|$

$$\leq \sum_{j=0}^{k} |C(j,k)| \left[\left(d_j n - 1\right) \sum_{l=v}^{\infty} \sum_{\substack{2i+m\leq r\\ i,m\geq 0}} (d_j n)^i \, |l - d_j n|^m \frac{|q_{i,m,r}(t)|}{x^r(1+x)^r} p_{d_j n,l}(x) \times \int_{|t-x|\geq\delta} p_{d_j n,l-v}(t)\, |h(t,x)| dt + \sum_{l=0}^{v-1} \sum_{\substack{2i+m\leq r\\ i,m\geq 0}} n^i \, |l - d_j n|^m \frac{|q_{i,m,r}(t)|}{x^r(1+x)^r} p_{d_j n,l}(x) |h(0,x)| \right]$$

For $|t-x| \geq \delta$, we can find a constant $C > 0$ such that $|h(t-x)| \leq C|t-x|^\gamma$.

Hence, using Schwarz inequality for integration and then for summation, Lemmas 2.1 and 2.2it easily follows that

$|I_3| = O(n^{-s})$ for any $s > 0$, uniformly on $[a,b]$.

Choosing $s > k+1$ and then combining the estimates of I_1, I_2 and I_3, the required result is immediate. ■

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التقريب باستخدام التركيب الخطي لتعميم مؤثر باسكوكوف فيلبس بصيغة مجموع- تكامل

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الخلاصة

إن الهدف من هذا البحث هو دراسة ظواهر التقريب المتعدد(تقريب مشتقات الدوال بواسطة مشتقات المؤثرات المقابلة لها) باستخدام التركيب الخطي $S_{n,v}(.,k,x)$ للمؤثر $S_{n,v}$. وإثبات صيغة فرونفسكي المشابهة وإيجاد الخطأ المخمن بواسطة معيار الاستمرارية لهذه المؤثرات.