THE SPHERICITY TEST IN REPEATED MEASURES MODEL

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Abstract

The sphericity test for repeated measures model is studied. The likelihood ratio criterion of this test is obtained.

Key words : Repeated Measures Model, Likelihood Ratio Criterion, Sphericity Test

1- Introduction

The repeated measures model (RMM) is one of the most widely used models in experimental designs, especially in biological, agricultural, educational and psychological research [4]. It occurs in analysis of variance when a particular experimental unit receives several treatments [3]. Many literatures that are considered univariate the assumption is made that a set of random variables are independent and have a common variance. Mauchly (1940) [5], was studied the test of the hypothesis that the sample from n-variate population is in fact from a population from which the variances are equal and the correlations are all zero. A population having this symmetry will be called " spherical ". Anderson (1984) [2], was studied a test based on repeated sets of observations, that is, he used a sample of p-component vectors from a multivariate normal distribution to test that hypothesis. Al-Mouel (2004) [1], was studied a generalization for the sphericity test by letting $Y_1,\ldots,Y_n$ be independent of each other, and identically distributed $N_p(\mu,\Sigma)$ and considering the partition $Y_i=[Y_{i1},Y_{i2},\ldots,Y_{ik}]'$,

$\mu=[\mu_1,\mu_2,\ldots,\mu_k]'$, $\Sigma=\begin{bmatrix}\Sigma_{11} & \Sigma_{12} & \ldots & \Sigma_{1k}\\ \Sigma_{21} & \Sigma_{22} & \ldots & \Sigma_{2k}\\ \vdots & \vdots & & \vdots\\ \Sigma_{k1} & \Sigma_{k2} & \ldots & \Sigma_{kk}\end{bmatrix}$, where Y_{ir} and μ_r are $p_r\times 1$

vectors and Σ_{rr} is $p_r \times p_r$ matrices $(r = 1,2,\ldots,k)$ with $\sum_{r=1}^{k} p_r = p$. He tests the null hypothesis

$$H_0 : \Sigma = \begin{bmatrix} I_{q_1} \otimes \Lambda_{11} & 0 & \cdots & 0 \\ 0 & I_{q_2} \otimes \Lambda_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I_{q_k} \otimes \Lambda_{kk} \end{bmatrix}, \qquad (1)$$

where Λ_{rr} is $m_r \times m_r$ matrices with $q_r \times m_r = p_r, r = 1,2,\ldots,k$, I_s denote the $s \times s$ identity matrix and $\otimes$ be the Kroncker product between two matrices [6].

And he shows the criterion for H_0 is

$$\Lambda = \frac{|A|^{\frac{n}{2}}}{\prod_{r=1}^{k} \left(\frac{|B_r|}{q_r^{m_r}}\right)^{\frac{n q_r}{2}}}, \qquad (2)$$

where $$A = \sum_{i=1}^{n} (Y_i - \overline{Y})(Y_i - \overline{Y}) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix}, \qquad (3)$$

A_{rr} are $p_r \times p_r$ matrices, $B_r = \sum_{i=1}^{q_r} A_{rr,ii}$, and

$$A_{rr} = \sum_{i=1}^{n} (Y_{ir} - \overline{Y_r})(Y_{ir} - \overline{Y_r})' = \begin{bmatrix} A_{rr,11} & A_{rr,12} & \cdots & A_{rr,1q} \\ A_{rr,21} & A_{rr,22} & \cdots & A_{rr,2q} \\ \vdots & \vdots & & \vdots \\ A_{rr,q_r 1} & A_{rr,q_r 2} & \cdots & A_{rr,q_r q_r} \end{bmatrix}, \qquad (4)$$

for $r = 1,2,\ldots,k.$ In this paper we study the sphericity test in repeated measures model of Gabbara (1994) [4] as an application of generalized sphericity test .

2- Repeated Measures Model (RMM) of Gabbara (1994) [4]

In this section, we state the RMM of Gabbara (1994) [4], which is given below. Let Y_{ijk} be the $(j,k)^{th}$ observation on the i^{th} individual (experimental unit), for $i=1,\ldots,m,\ j=1,\ldots,a,$ and $k=1,\ldots,b,$ where a is the number of rows and b is the number of columns. Let $Y_i=(Y_{i11},\ldots,Y_{iab})'$ denote the $ab\times 1$ vector of observations on the i^{th} individual, and $\mu_{ijk}=E(Y_{ijk}),\ \mu_i=E(Y_i).$ It is assumed that Y_i are independently normally distributed with mean vector μ_i and common covariance Σ, which is positive definite matrix. Let all measurements have the same variance σ^2, and that every pair of measurements that comes from :

(i) different columns and different rows treatments,

(ii) the same column but different rows treatments,

(iii) different columns but the same row treatments,

having $\sigma^2\rho_1,\sigma^2\rho_2,\sigma^2\rho_3$ as their respective covariances and every pair of measurements that comes from different individuals have covariance zero. In symbols :

$$COV(Y_{ijk},Y_{i'j'k'})=\begin{cases}\sigma^2 & if\ \ i=i',\ j=j',k=k' \\ \sigma^2\rho_3 & if\ \ i=i',\ j=j',k\neq k' \\ \sigma^2\rho_2 & if\ \ i=i',\ j\neq j',k=k' \\ \sigma^2\rho_1 & if\ \ i=i',\ j\neq j',k\neq k' \\ 0 & if\ \ i\neq i'\end{cases} \tag{5}$$

Let

$$\mu_i=\delta_i\ j_{ab}+\alpha_i\otimes j_b+j_a\otimes\beta_i+\eta_i\ , \tag{6}$$

where

δ_i is a scalar,

$\alpha_i=(\alpha_{i1},\ldots,\alpha_{ia})'$ is an $a\times 1$ vector orthogonal to j_a,

$\beta_i=(\beta_{i1},\ldots,\beta_{ib})'$ is a $b\times 1$ vector orthogonal to j_b,

$\eta_i = (\eta_{i11}, \dots, \eta_{iab})'$ is an $ab \times 1$ vector orthogonal to every column of the matrix $\mathrm{I}_a \otimes j_b$ and every column of the matrix $j_a \otimes \mathrm{I}_b$, and j_s is the $s \times 1$ vector of one's.

Let $Y_1, \dots, Y_m$ be independent ab-dimensional normal random vectors such that

$$Y_i \sim N_{ab}(\mu_i, \Sigma), i = 1, \dots, m, \tag{7}$$

where μ_i is given in (6) and Σ is defined in (5).

Then, he showed that

$$\Sigma = \sigma^2[(1 - \rho_3 - \rho_2 + \rho_1)\mathrm{I}_{ab} + (\rho_3 - \rho_1)\mathrm{I}_a \otimes J_b + (\rho_2 - \rho_1) J_a \otimes \mathrm{I}_b + \rho_1 J_{ab}], \tag{8}$$

The model defined by (5)- (8) is called the RMM.

3- Transforming the RMM (Gabbara (1994))

In this section, we use the transformation of the RMM , which is given by Gabbara (1994) [4].

Let U_* be an $ab \times ab$ orthogonal matrix given in the following form

$$U_* = \begin{bmatrix} (ba)^{-\frac{1}{2}} (j_b' \otimes j_a') \\ b^{-\frac{1}{2}} (U_a' \otimes j_a') \\ a^{-\frac{1}{2}} (j_a' \otimes U_b') \\ (U_a' \otimes U_b') \end{bmatrix}, \tag{9}$$

where U_s' be $(s-1) \times s$ matrix such that $U_s' U_s = \mathrm{I}_{s-1}$, $U_s U_s' = \mathrm{I}_s - (\frac{1}{s}) J_s$, $U_s' j_s = 0$, $j_s U_s' = 0$, and J_s is the $s \times s$ matrix of one's .

Let

$$Y_i^* = \begin{bmatrix} Y_{i1}^* \\ Y_{i2}^* \\ Y_{i3}^* \\ Y_{i4}^* \end{bmatrix} = U_* Y_i = \begin{bmatrix} (ba)^{-\frac{1}{2}} j_a' \otimes j_b' \\ b^{-\frac{1}{2}} U_a' \otimes j_b' \\ a^{-\frac{1}{2}} j_a' \otimes U_b' \\ U_a' \otimes U_b' \end{bmatrix} Y_i , \tag{10}$$

where $Y_{i1}^*, Y_{i2}^*, Y_{i3}^*, Y_{i4}^*$ are $1\times 1, (a-1)\times 1, (b-1)\times 1, (a-1)(b-1)\times 1$, respectively. Since U_* is an invertible matrix and does not depend on any unknown parameters, observing $Y_1,\ldots,Y_m$ is equivalent to observing Y_{i1}^*, Y_{i2}^*, Y_{i3}^* and Y_{i4}^*. Thus

$$Y_i^* \sim N_{ab}(U_*\mu_i, U_*\Sigma U_*') . \tag{11}$$

Now

$$U_* \mu_i = \begin{bmatrix} (ba)^{-\frac{1}{2}} j_a' \otimes j_b' \\ b^{-\frac{1}{2}} U_a' \otimes j_b' \\ a^{-\frac{1}{2}} j_a' \otimes U_b' \\ U_a' \otimes U_b' \end{bmatrix} \mu_i = \begin{bmatrix} \sqrt{ba}\ \delta_i \\ \sqrt{b}\ U_a' \alpha_i \\ \sqrt{a}\ U_b' \beta_i \\ (U_a' \otimes U_b')\ \eta_i \end{bmatrix}, \tag{12}$$

where μ_i is given in (6), and

$$\Sigma^* = U_* \Sigma U_*' = \begin{bmatrix} \tau_1^2 & 0 & 0 & 0 \\ 0 & \tau_2^2 I_{a-1} & 0 & 0 \\ 0 & 0 & \tau_3^2 I_{b-1} & 0 \\ 0 & 0 & 0 & \tau_4^2 I_{(a-1)(b-1)} \end{bmatrix}, \tag{13}$$

where

$$\tau_1^2 = \sigma^2[1+(b-1)\rho_3 + (a-1)\rho_2 + (b-1)(a-1)\rho_1],$$

$$\tau_2^2 = \sigma^2[1+(b-1)\rho_3 - \rho_2 - (b-1)\rho_1],$$

$$\tau_3^2 = \sigma^2[1-\rho_3 + (a-1)\rho_2 - (a-1)\rho_1],$$

$$\tau_4^2 = \sigma^2[1-\rho_3 - \rho_2 + \rho_1], \tag{14}$$

$(\tau_1^2, \tau_2^2, \tau_3^2, \tau_4^2)$ is just an invertible function of $(\sigma^2, \rho_1, \rho_2, \rho_3)$ which is a reparametrization.

Hence Y_{i1}^*, Y_{i2}^*, Y_{i3}^* and Y_{i4}^* are independent, and

$$Y_{i1}^* \sim N_1(\sqrt{ba}\ \delta_i, \tau_1^2),$$

$$Y_{i2}^* \sim N_{a-1}(\sqrt{b}\ U_a' \alpha_i, \tau_2^2 \mathrm{I}_{a-1}),$$

$$Y_{i3}^* \sim N_{b-1}(\sqrt{a}\ U_b' \beta_i, \tau_3^2 \mathrm{I}_{b-1}), \text{ and}$$

$$Y_{i4}^* \sim N_{(a-1)(b-1)}([U_a' \otimes U_b']\eta_i, \tau_4^2 \mathrm{I}_{(a-1)(b-1)}). \qquad (15)$$

4: The Sphericity Test in RMM

We consider the covariance structure in RMM of Gabbara (1994) [4] . We wish to test the null hypothesis

$$\mathrm{H}_0 : \Sigma = \sigma^2[(1-\rho_3-\rho_2+\rho_1)\mathrm{I}_{ab} + (\rho_3-\rho_1)\mathrm{I}_a \otimes J_b + (\rho_2-\rho_1) J_a \otimes \mathrm{I}_b + \rho_1 J_{ab}], \qquad (16)$$

which is based on the sample $Y_1, \dots, Y_m$. Since the observing $Y_1, \dots, Y_m$ is equivalent to the observing Y_{i1}^*, Y_{i2}^*, Y_{i3}^* and Y_{i4}^*, and Σ is equivalent to Σ^*, where Σ^* is given in (13), then testing the null hypothesis (16) is equivalent to testing the null hypothesis

$$\mathrm{H}_0 : \Sigma^* = U_* \Sigma U_*' = \begin{bmatrix} \tau_1^2 & 0 & 0 & 0 \\ 0 & \tau_2^2 \mathrm{I}_{a-1} & 0 & 0 \\ 0 & 0 & \tau_3^2 \mathrm{I}_{b-1} & 0 \\ 0 & 0 & 0 & \tau_4^2 \mathrm{I}_{(a-1)(b-1)} \end{bmatrix}, \qquad (17)$$

which is based on the sample $Y_1^*, \dots, Y_m^*$, or

$$\mathrm{H}_0 : \Sigma^* = diag(\tau_1^2, \tau_2^2 \mathrm{I}_{a-1}, \tau_3^2 \mathrm{I}_{b-1}, \tau_4^2 \mathrm{I}_{(a-1)(b-1)}). \qquad (18)$$

We see that (17) is a special case of the form (1). Then we can applied the generalized sphericity test of Al-Mouel (2004) [1] .

Hence, the likelihood ratio criterion for H_0 is :

$$\Lambda = \frac{|A|^{\frac{m}{2}}}{\prod_{g=1}^{4} \left|B_g\right|^{\frac{m}{2}}}, \qquad (19)$$

where

$$A = \sum_{i=1}^{m} (Y_i^* - \overline{Y^*})(Y_i^* - \overline{Y^*})' = mS, \qquad (20)$$

$$B_g = trace(A_{gg}), g = 1,2,3,4, \text{ and} \qquad (21)$$

$$A_{gg} = \sum_{i=1}^{m} (Y_{ig}^* - \overline{Y_g^*})(Y_{ig}^* - \overline{Y_g^*})', g = 1,2,3,4, \qquad (22)$$

where $\overline{Y^*}$ and S be respectively the sample mean vector and covariance matrix formed from a sample observations on Y_i^* that means $\overline{Y^*}$ and A partition as :

$$\overline{Y^*}' = (\overline{Y_1^*}', \overline{Y_2^*}', \overline{Y_3^*}', \overline{Y_4^*}'), \text{ and } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}, \qquad (23)$$

where $\overline{Y_1^*}', \overline{Y_2^*}', \overline{Y_3^*}', \overline{Y_4^*}'$ are $1\times 1, (a-1)\times 1, (b-1)\times 1, (a-1)(b-1)\times 1$ and $A_{11}, A_{22}, A_{33}, A_{44}$ are $1\times 1, (a-1)\times(a-1), (b-1)\times(b-1),$ $(a-1)(b-1)\times(a-1)(b-1)$ respectively.

5 : Conclusion

The likelihood ratio criterion, for H_0 (17) which is based on the sample $Y_1^*, \ldots, Y_m^*$, is

$$\Lambda = \frac{|A|^{\frac{m}{2}}}{\prod_{g=1}^{3} \left|B_g\right|^{\frac{m}{2}}},$$

where A and B_g are given in (20) and (21) respectively.

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الاختبار الكروي لنموذج القياسات المتكررة

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المستخلص

لقد تمت دراسة الاختبار الكروي لنموذج القياسات المتكررة و حساب معيار نسبة الترجيح الاعظم لهذا الاختبار.