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Sphericity Test in Nested Repeated Measures Model of Gabbara

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Abstract

The sphericity test in nested repeated measures model of Gabbara is given as an application of generalized sphericity test of Al-Mouel.
Key words : Sphericity Test, Likelihood Ratio Criterion, Nested Repeated Measures Model, Generalized Sphericity Test.

S1- Introduction

In many statistical analyses that considered univariate, the assumption is made that a set of random variables are independent and have a common variance. Several researchers consider a test of these assumptions based on repeated set of observations. More precisely, they used a sample of p-component vectors $Y_1,\ldots,Y_n$ from $N(\mu,\Sigma)$ to test the hypothesis

$$\mathrm{H}:\Sigma=\sigma^2\,\mathrm{I}, \tag{1}$$

where σ^2 is not specified and I is the identity matrix (see Anderson (1984) [2], Muirhead (1982) [6], Mauchly (1940) [5], Timm (2002) [7]). The null hypothesis in (1) is called the hypothesis of sphericity. Al-Mouel (2004) [1] considers testing problem which is a generalization of this problem and the test is given by letting $Y_1,\ldots,Y_n$ be independent of each other, and identically distributed $N_p(\mu,\Sigma)$ and considering the partition

$Y_i=[Y_{i1},Y_{i2},\ldots,Y_{ik}]'$, $\mu=[\mu_1,\mu_2,\ldots,\mu_k]'$ and

$$\Sigma=\begin{bmatrix}\Sigma_{11} & \Sigma_{12} & \ldots & \Sigma_{1k}\\ \Sigma_{21} & \Sigma_{22} & \ldots & \Sigma_{2k}\\ \vdots & \vdots & & \vdots\\ \Sigma_{k1} & \Sigma_{k2} & \ldots & \Sigma_{kk}\end{bmatrix},$$

where Y_{ir} and μ_k are $p_r\times 1$ vectors and Σ_{rr} is $p_r\times p_r$ matrices $(r=1,2,\ldots,k)$ with $\sum_{r=1}^{k} p_r = p$. He tests the null hypothesis

$$\mathrm{H}_0:\Sigma=\begin{bmatrix}\mathrm{I}_{q_1}\otimes\Lambda_{11} & 0 & \ldots & 0\\ 0 & \mathrm{I}_{q_2}\otimes\Lambda_{22} & \ldots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \ldots & \mathrm{I}_{q_k}\otimes\Lambda_{kk}\end{bmatrix}, \tag{2}$$

where Λ_{rr} is $m_r\times m_r$ matrices with $q_r\times m_r=p_r, r=1,2,\ldots,k$, I_s denote the $s\times s$ identity matrix and $\otimes$ be the Kroncker product between two matrices. And he shows the criterion for H_0 is

$$\Lambda = \frac{|A|^{\frac{n}{2}}}{\prod_{r=1}^{k} \left(\frac{|B_r|}{q_r^{m_r}}\right)^{\frac{n q_r}{2}}} , \qquad (3)$$

Where

$$A = \sum_{i=1}^{n} (Y_i - \overline{Y})(Y_i - \overline{Y})' = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix}, \qquad (4)$$

A_{rr} are $p_r \times p_r$ matrices, $B_r = \sum_{i=1}^{q_r} A_{rr,ii}$, and

$$A_{rr} = \sum_{i=1}^{n} (Y_{ir} - \overline{Y_r})(Y_{ir} - \overline{Y_r})' = \begin{bmatrix} A_{rr,11} & A_{rr,12} & \cdots & A_{rr,1q} \\ A_{rr,21} & A_{rr,22} & \cdots & A_{rr,2q} \\ \vdots & \vdots & & \vdots \\ A_{rr,q_r 1} & A_{rr,q_r 2} & \cdots & A_{rr,q_r q_r} \end{bmatrix}, \qquad (5)$$

for $r = 1,2,\ldots,k$. In this paper we study the sphericity test in nested repeated measures model (NRMM) of Gabbara (1985) [4] as an application of generalized sphericity test .

S2- Nested Repeated Measures Model (NRMM) of Gabbara (1985) [4]

In this section, we state the NRMM of Gabbara (1985) [4], which is given below. Gabbara considered the NRMM, which occurs in the analysis of variance (ANOVA) when a particular individual (person, rat, field, etc.) has a number of subindividuals (children, offspring, subfields, etc.) and each subindividual receives several treatments. He assumed that each individual has the same number, d, of subindividuals and each subindividual receives the same number r of treatments. He supposed that Y_{ijk} be the k^{th} observation on the j^{th} sub-individual from i^{th} individual, for $i = 1,\ldots,m$, $j = 1,\ldots,d$ and $k = 1,\ldots,r$, and $Y_{ij} = (Y_{ij1},\ldots,Y_{ijr})'$ be the vector of observations on the j^{th} sub-individual from the i^{th} individual and $Y_i = (Y_{i1},\ldots,Y_{id})'$ be the vector of observations on the sub-individuals of the i^{th} individual. Let $\mu_{ijk} = E(Y_{ijk})$, $\mu_{ij} = E(Y_{ij})$ and $\mu_i = E(Y_i)$. It is assumed that Y_i are independently normally distributed with mean μ_i and common covariance Σ, which is positive definite matrix. He assumed that all the measurements have the same variance σ^2, every pair of measurements that come from the same subindividual have the same covariance $\sigma^2\rho_2$; every pair of measurements that come from the same individual but different subindividuals have the same covariance $\sigma^2\rho_1$, and every pair of measurements that come from different individuals have covariance zero. In symbols

$$COV(Y_{ijk}, Y_{i'j'k'}) = \begin{cases} \sigma^2 & if \;\; i = i', j = j', k = k' \\ \sigma^2 \rho_2 & if \;\; i = i', j = j', k \neq k' \\ \sigma^2 \rho_1 & if \;\; i = i', j \neq j' \\ 0 & if \;\; i \neq i' \end{cases} \tag{6}$$

He assumed that

$$\mu_i = \delta_i \; j_d \otimes j_r + \gamma_i \otimes j_r + \eta_i \tag{7}$$

where δ_i is a scalar, $\gamma_i = (\gamma_{i1}, \ldots, \gamma_{id})'$ is a $d \times 1$ vector orthogonal to j_d , $\eta_i = (\eta_{i11}, \ldots, \eta_{idr})'$ is a $dr \times 1$ vector orthogonal to every column of the matrix $\mathrm{I}_d \otimes j_r$ and j_s is the $s \times 1$ vector of one's. Let $Y_1, \ldots, Y_m$ be independent dr-dimensional normal random vectors such that

$$Y_i \sim N_{dr}(\mu_i, \Sigma), \; i = 1, \ldots, m, \tag{8}$$

where μ_i is given in (7) and Σ is defined in (6).

Then he showed that

$$\Sigma = \sigma^2 [(1 - \rho_2) \mathrm{I}_{dr} + (\rho_2 - \rho_1) \; \mathrm{I}_d \otimes J_r + \rho_1 \, J_{dr}], \tag{9}$$

The model defined by (6)-(9) is called the NRMM.

S3- Transforming the NRMM (Gabbara (1985) [4])

In this section, we use the transformation of the NRMM, which is given by Gabbara (1985) [4]. This transformation is given below.

Let U_* be an $dr \times dr$ orthogonal matrix given in the following form

$$U_* = \begin{bmatrix} (dr)^{-\frac{1}{2}} \; j'_d \otimes j'_r \\ r^{-\frac{1}{2}} U'_d \otimes j'_r \\ U^*_d \otimes U'_r \\ \; \end{bmatrix}, \tag{10}$$

where U'_s be $(s-1)\times s$ matrix such that $U'_s U_s = \mathrm{I}_{s-1}$, $U_s U'_s = \mathrm{I}_s - (\frac{1}{s}) J_s$,

$U'_s\ j_{s=0}$, $j_s U'_s = 0$, and U^*_s be $s\times s$ orthogonal matrix defined as :

$$U^*_s = \begin{bmatrix} s^{-\frac{1}{2}}\ j'_s \\ U'_s \end{bmatrix} \qquad (11)$$

Let $$Y^*_i = \begin{bmatrix} Y^*_{i1} \\ Y^*_{i2} \\ Y^*_{i3} \end{bmatrix} = U_* Y_i = \begin{bmatrix} (dr)^{-\frac{1}{2}}\ j'_d \otimes j'_r \\ r^{-\frac{1}{2}} U'_d \otimes j'_r \\ U^*_d \otimes U'_r \end{bmatrix} Y_i\ , \qquad (12)$$

where $Y^*_{i1}, Y^*_{i2}, Y^*_{i3}$ are $1\times 1, (d-1)\times 1, d(r-1)\times 1$ respectively.

Since U_* is an invertible matrix and does not depend on any unknown parameters, observing $Y_1,\ldots,Y_m$ is equivalent to observing Y^*_{i1}, Y^*_{i2} and Y^*_{i3}. Then the Y^*_i are independent. Also

$$Y^*_i \sim N_{dr}\,(U_*\,\mu_i\,, U_*\,\Sigma U'_*) \qquad (13)$$

Now $$U_*\,\mu_i = \begin{bmatrix} (dr)^{-\frac{1}{2}}\ j'_d \otimes j'_r \\ r^{-\frac{1}{2}} U'_d \otimes j'_r \\ U^*_d \otimes U'_r \end{bmatrix} \mu_i = \begin{bmatrix} (dr)^{\frac{1}{2}}\,\delta_i \\ r^{\frac{1}{2}} U'_d\,\gamma_i \\ (U^*_d \otimes U'_r)\,\eta_i \end{bmatrix}, \qquad (14)$$

where μ_i is given in (7), and

$$U_*\Sigma U'_* = \sigma^2 \begin{bmatrix} 1+(r-1)\rho_2 + r(d-1)\rho_1 & 0 & 0 \\ 0 & [1+(r-1)\rho_2 - r\rho_1]\mathrm{I}_{d-1} & 0 \\ 0 & 0 & (1-\rho_2)\mathrm{I}_{d(r-1)} \end{bmatrix}, \qquad (15)$$

or

$$\Sigma^* = U_* \Sigma U_*' = \begin{bmatrix} \tau_1^2 & 0 & 0 \\ 0 & \tau_2^2 \, I_{d-1} & 0 \\ 0 & 0 & \tau_3^2 \, I_{d(r-1)} \end{bmatrix}, \quad (16)$$

Where $\tau_1^2 = \sigma^2 [1 + (r-1)\rho_2 + r(d-1)\rho_1],$

$$\tau_2^2 = \sigma^2 [1 + (r-1)\rho_2 - r\rho_1],$$

$$\tau_3^2 = \sigma^2 [1 - \rho_2], \quad (17)$$

$(\tau_1^2, \tau_2^2, \tau_3^2)$ is just an invertible function of $(\sigma^2, \rho_1, \rho_2)$ which is a reparametrization. Hence Y_{i1}^*, Y_{i2}^* and Y_{i3}^* are independent and $Y_{i1}^* \sim N_1(\sqrt{dr}\,\delta_i, \tau_1^2)$, $Y_{i2}^* \sim N_{d-1}(\sqrt{r}\,U_d'\,\gamma_i, \tau_2^2 \, I_{d-1})$, $Y_{i3}^* \sim N_{d(r-1)}([U_d^* \otimes U_r']\eta_i, \tau_3^2 \, I_{d(r-1)})$

S4- The Sphericity Test in NRMM

We consider the covariance structure in NRMM of Gabbara (1985) [4]. We wish to test the null hypothesis

$$H_0 : \Sigma = \sigma^2 [(1-\rho_2) I_{dr} + (\rho_2 - \rho_1) \, I_d \otimes J_r + \rho_1 J_{dr}], \quad (18)$$

which is based on the sample $Y_1, \ldots, Y_m$. Since the observing $Y_1, \ldots, Y_m$ is equivalent to observing Y_{i1}^*, Y_{i2}^* and Y_{i3}^*, and Σ is equivalent to Σ^*, where Σ^* is given in (16), then testing the null hypothesis (18) is equivalent to testing the null hypothesis

$$H_0 : \Sigma^* = U_* \Sigma U_*' = \begin{bmatrix} \tau_1^2 & 0 & 0 \\ 0 & \tau_2^2 \, I_{d-1} & 0 \\ 0 & 0 & \tau_3^2 \, I_{d(r-1)} \end{bmatrix}, \quad (19)$$

which is based on the sample $Y_1^*, \ldots, Y_m^*$. We see that (19) is a special case of the form (2). Then we can apply the generalized sphericity test of Al-Mouel (2004) [1] .

Hence, the likelihood ratio criterion for H_0 is :

$$\Lambda = \frac{|A|^{\frac{m}{2}}}{\prod_{g=1}^{3} |B_g|^{\frac{m}{2}}}, \quad (20)$$

where $A = \sum_{i=1}^{m} (Y_i^* - \overline{Y^*})(Y_i^* - \overline{Y^*})' = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad (21)$

A_{11}, A_{22}, A_{33} are $1\times1, (d-1)\times(d-1), d(r-1)\times d(r-1)$,

$$B_g = trace(A_{gg}), g = 1,2,3, \text{ and} \tag{22}$$

$$A_{gg} = \sum_{i=1}^{m} (Y_{ig}^* - \overline{Y_g^*})(Y_{ig}^* - \overline{Y_g^*})', g = 1,2,3, \tag{23}$$

and $\overline{Y^*}$ be the sample mean vector formed from a sample observations on Y_i^* that means $\overline{Y^*}$ partition as : $\overline{Y^*}' = (\overline{Y_1^*}', \overline{Y_2^*}', \overline{Y_3^*}')$, where $\overline{Y_1^*}', \overline{Y_2^*}', \overline{Y_3^*}'$ are $1\times1, (d-1)\times1, d(r-1)\times1$ respectively.

Conclusion

The likelihood ratio criterion, for H_0 (19) which is based on the sample $Y_1^*, \ldots, Y_m^*$, is

$$\Lambda = \frac{|A|^{\frac{m}{2}}}{\prod_{g=1}^{3} |B_g|^{\frac{m}{2}}},$$

where A and B_g are given in (21) and (22) respectively.

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الاختبار الكروي لنموذج القياسات المتكررة المتداخل لكبارة

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المستخلص

تمت دراسة الاختبار الكروي لنموذج القياسات المتكررة المتداخل لـ (كباره) بوصفه تطبيقاً للاختبار الكروي العام للمويّل.