Designs, Codes and Cryptography (2019) 87:679–683
https://doi.org/10.1007/s10623-018-00592-8

A new lower bound for the smallest complete (k, n)-arc in $\mathrm{PG}(2, q)$

S. Alabdullah[1] · J. W. P. Hirschfeld[1]

Received: 10 January 2018 / Revised: 27 November 2018 / Accepted: 4 December 2018 /
Published online: 21 December 2018

Abstract
In $\mathrm{PG}(2, q)$, the projective plane over the field $\mathbf{F}_q$ of q elements, a (k, n)-arc is a set $\mathcal{K}$ of k points with at most n points on any line of the plane. A fundamental question is to determine the values of k for which $\mathcal{K}$ is complete, that is, not contained in a $(k + 1, n)$-arc. In particular, what are the smallest and largest values of k for a complete $\mathcal{K}$, denoted by $t_n(2, q)$ and $m_n(2, q)$? Here, a new lower bound for $t_n(2, q)$ is established and compared to known values for small q.

Keywords Finite projective plane · Arc · Lower bound

Mathematics Subject Classification 51E20

1 Introduction and background

A projective plane of order q consists of a set of $q^2 + q + 1$ points and a set of $q^2 + q + 1$ lines, where each line contains exactly $q + 1$ points and two distinct points lie on exactly one line. It follows from the definition that each point is contained in exactly $q + 1$ lines and two distinct lines have exactly one common point.

The main focus of this paper is to find a lower bound for k of a (k, n)-arc in $\mathrm{PG}(2, q)$. First, some basic constants and their properties are summarised. See [8, Chap. 12] or [7, Chap. 12].

Definition 1.1 A (k, n)-*arc* in $\mathrm{PG}(2, q)$ is a set $\mathcal{K}$ of k points, no $n + 1$ of which are collinear, but with at least one set of n points collinear. When $n = 2$, a $(k, 2)$-arc is a k-*arc*.

Definition 1.2 A (k, n)-arc is *complete* if it is not contained in a $(k, n + 1)$-arc.

This is one of several papers published in *Designs, Codes and Cryptography* comprising the "Special Issue on Coding and Cryptography".

✉ J. W. P. Hirschfeld
jwph@sussex.ac.uk

S. Alabdullah
S.Alabdullah@sussex.ac.uk

1 Department of Mathematics, University of Sussex, Brighton BN1 9QH, UK

Notation 1.3 The maximum value of k for a (k, n)-arc to exist is denoted by $m_n(2, q)$.

Definition 1.4 A line ℓ is an i-*secant* of $\mathcal{K}$ if $|\ell \cap \mathcal{K}| = i$.

Notation 1.5 For a (k, n)-arc $\mathcal{K}$ in $\mathrm{PG}(2, q)$, let

$$\begin{aligned} \tau_i &= \text{the total number of } i\text{-secants of} \mathcal{K}, \\ \rho_i = \rho_i(P) &= \text{the number of } i\text{-secants through a point } P \text{ of} \mathcal{K}, \\ \sigma_i = \sigma_i(Q) &= \text{the number of} i - \text{secants through a point} Q \text{of } \mathrm{PG}(2, q) \backslash \mathcal{K}. \end{aligned}$$

Lemma 1.6 *For a (k, n)-arc $\mathcal{K}$, the following equations hold*:

$$\sum_{i=0}^{n} \tau_i = q^2 + q + 1; \tag{1.1}$$

$$\sum_{i=1}^{n} i\tau_i = k(q + 1); \tag{1.2}$$

$$\sum_{i=2}^{n} \tfrac{1}{2} i(i - 1)\tau_i = \tfrac{1}{2} k(k - 1). \tag{1.3}$$

Proof See [8, Chap. 12]. □

The constants ρ_i, σ_i are useful in investigations of the properties of (k, n)-arcs, but are not required here.

Theorem 1.7

$$m_2(2, q) = \begin{cases} q + 2, & for\, q\, even; \\ q + 1, & for\, q\, odd. \end{cases}$$

Proof See [8, Chap. 8]. □

Theorem 1.8 *(1)*

$$m_n(2, q) \begin{cases} = (n - 1)q + n, & for\, q\, even\, and\, n \mid q; \\ < (n - 1)q + n, & for\, q\, odd. \end{cases}$$

(2) A (k, n)-arc $\mathcal{K}$ is maximal if and only if every line in $\mathrm{PG}(2, q)$ *is either an n-secant or a* 0-*secant.*

Proof See [8, Chap. 12]. □

Lemma 1.9 *If $\mathcal{K}$ is a complete (k, n)-arc, then $(q + 1 - n)\tau_n \geq q^2 + q + 1 - k$, with equality if and only if $\sigma_n = 1$ for all Q in* $\mathrm{PG}(2, q) \backslash \mathcal{K}$.

Proof See [8, Chap. 12]. □

Definition 1.10 The *type* of a point P in $\mathrm{PG}(2, q)$ for a (k, n)-arc is the $(n + 1)$-tuple $(\rho_0, \rho_1, \ldots, \rho_n)$.

Table 1 Bounds for complete k-arcs for $4 \le q \le 23$

q	4	5	7	8	9	11	13	16	17	19	23
$b_2(2, q)$	5	5	5	5	6	6	6	7	7	7	8
$t_2(2, q)$	6	6	6	6	6	7	8	9	10	10	10
$m_2(2, q)$	6	6	8	10	10	12	14	18	20	20	25

Table 2 Bounds for complete $(k, 3)$-arcs for $4 \le q \le 16$

q	4	5	7	8	9	11	13	16
$b_3(2, q)$	7	8	9	9	9	10	11	12
$t_3(2, q)$	7	9	9	11	12	13	15	15
$m_3(2, q)$	9	11	15	15	17	21	23	28

2 New lower bound

A lower bound for the smallest complete (k, n)-arcs $\mathcal{K}$ is established below.

Theorem 2.1 *In* PG$(2, q)$, *a complete* (k, n)-*arc does not exist for* $k \le n^*$, *where*

$$n^* = \frac{(q+1-n^2) + \sqrt{(q+1-n^2)^2 + 4(n^2-n)(q+1-n)(q^2+q+1)}}{2(q+1-n)}.$$

Proof Let $\mathcal{K}$ be a complete (k, n)-arc. The number of n-secants through a point P in $\mathcal{K}$ is at most $(k-1)/(n-1)$. Then, counting the set $\{(P, \ell)\}$, where ℓ is an n-secant and P is a point of $\mathcal{K}$ incident with ℓ gives that

$$\tau_n \le \frac{k(k-1)}{n(n-1)}. \tag{2.1}$$

On the other hand, Lemma 1.9 implies that

$$\tau_n \ge \frac{q^2+q+1-k}{q+1-n} \tag{2.2}$$

Now, from Eqs. (2.1) and (2.2),

$$\frac{k^2-k}{n^2-n} = \frac{q^2+q+1-k}{q+1-n}.$$

Hence

$$\begin{aligned}
&(q+1-n)k^2 - (q+1-n)k = (n^2-n)(q^2+q+1) - (n^2-n)k,\\
&(q+1-n)k^2 - (q+1-n-n^2+n)k - (n^2-n)(q^2+q+1) = 0,\\
&(q+1-n)k^2 - (q+1-n^2)k - (n^2-n)(q^2+q+1) = 0. \qquad (2.3)
\end{aligned}$$

Now, Eq. (2.3) implies that $k = n^* > 0$. □

This can be applied to k-arcs and $(k, 3)$-arcs, as in Tables 1 and 2, with the notation $n^* = b_n(2, q)$ and $n = 2, 3$.

Table 3 Lower bounds for complete $(k, 3)$-arcs for $4 \leq q \leq 16$

k	[14]	[9]	Theorem	Exact result
4	7	6	7	7
5	8	6	8	9
7	9	7	9	9
8	9	8	9	11
9	10	8	10	12
11	10	9	10	13
13	11	10	11	15
16	12	11	12	15

3 Comparison with known results

Table 3 gives the comparison, for $(k, 3)$-arcs, between [9,14] and Theorem 2.1 for the values of q with $4 \leq q \leq 16$.

Acknowledgements Salam Alabdullah obtained a Ph.D. studentship funded by the Ministry of Higher Education and Scientific Research of the Government of Iraq via the University of Basra.

References

1. Bartoli D., Marcugini S., Pambianco F.: The maximum and the minimum sizes of complete $(n, 3)$-arcs in $PG(2, 16)$. In: Thirteenth International Workshop on Algebraic and Combinatorial Coding Theory, Pomorie, Bulgaria, pp. 77–82, 15–21 June 2012.
2. Bartoli D., Giulietti M., Zini G.: Complete $(k, 3)$-arcs from quartic curves. Des. Codes Cryptogr. **79**, 487–505 (2015).
3. Bierbrauer J.: The maximal size of a 3-arc in PG$(2, 8)$. J. Comb. Math. Comb. Comput. **45**, 145–161 (2003).
4. Coolsaet K., Sticker H.: A full classification of the complete k-arcs in $PG(2, 23)$ and $PG(2, 25)$. J. Comb. Des. **17**, 459–477 (2009).
5. Coolsaet K., Sticker H.: The complete k-arcs of PG$(2, 27)$ and PG$(2, 29)$. J. Comb. Des. **19**, 111–130 (2011).
6. Coolsaet K., Sticker H.: The complete $(k, 3)$-arcs of PG$(2, q)$, $q \leq 13$. J. Comb. Des. **20**, 89–111 (2012).
7. Hirschfeld J.W.P.: Projective Geometries Over Finite Fields, pp. xii+474. Clarendon Press, Oxford (1979).
8. Hirschfeld J.W.P.: Projective Geometries Over Finite Fields, 2nd edn, pp. xiv+555. Oxford University Press, Oxford (1998).
9. Hirschfeld J.W.P., Pichanick E.V.D.: Bounds for arcs of arbitrary degree in finite Desargusian planes. J. Comb. Des. **24**, 184–196 (2016).
10. Marcugini S., Milani A., Pambianco F.: A computer based classification of the $(n, 3)$-arcs in $PG(2, 7)$. Rapporto Tecnico n.7 (1998).
11. Marcugini S., Milani A., Pambianco F.: A computer based classification of the $(n, 3)$-arcs in $PG(2, 8)$. Rapporto Tecnico n.8 (1998).
12. Marcugini S., Milani A., Pambianco F.: A computer based classification of the $(n, 3)$-arcs in $PG(2, 9)$. Rapporto Tecnico n.9 (1998).
13. Marcugini S., Milani A., Pambianco F.: Maximal $(n, 3)$-arcs in $PG(2, 11)$. Discret. Math. **208**(209), 421–426 (1999).
14. Marcugini S., Milani A., Pambianco F.: Maximal $(n, 3)$-arcs in $PG(2, 13)$. Discret. Math. **294**, 139–145 (2005).

15. Marcugini S., Milani A., Pambianco F.: Complete arcs in $PG(2, 25)$: the spectrum of size and the classification of the smallest complete arcs. Discret. Math. **307**, 739–747 (2007).

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.