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Article

Alternating Direction Implicit Formulation of the Differential Quadrature Method for Solving Burger Equations

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Abstract: In this paper, a new development of differential quadrature method was proposed. It is known alternating direction implicit formulation of the differential quadrature method (ADI-DQM) for computing the numerical solutions of the two-dimension Burger equations. The results confirm that this method has a high accuracy, good convergence and less workload comparing with the other numerical methods.

Keywords: Differential quadrature method, Burger equation, ADI, accuracy

Mathematics Subject Classification (2000): 65N22, 65M12, 76A05.

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1. Introduction

Consider the two-dimensional Burger equations:

$$\frac{\partial u_1}{\partial t}+\alpha u_1\frac{\partial u_1}{\partial x}+\alpha u_2\frac{\partial u_1}{\partial y}-\frac{1}{Re}\left(\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}\right)=0 \quad (x,y)\in\Omega\ ,\ \ t>0 \qquad (1a)$$

$$\frac{\partial u_2}{\partial t}+\alpha u_1\frac{\partial u_2}{\partial x}+\alpha u_2\frac{\partial u_2}{\partial y}-\frac{1}{Re}\left(\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}\right)=0 \quad (x,y)\in\Omega\ ,\ \ t>0 \qquad (1b)$$

with the initial conditions

$$u_1(x,y,0)=\emptyset_1(x,y) \quad , \quad u_2(x,y,0)=\emptyset_2(x,y) \qquad (1c)$$

and the boundary conditions

$$\left.\begin{aligned} u_1(x,y,t) &= f(x,y,t) \\ u_2(x,y,t) &= g(x,y,t) \end{aligned}\right\} (x,y) \in \partial\,\Omega\ , \quad \mathrm{t} > 0 \qquad (1d)$$

where Ω is the computational domain with the boundary $\partial\Omega$, Re is the Reynolds number , α constant, u_1 and u_2 are velocity components and $\emptyset_1, \emptyset_2, f$ and g are the known functions.

Burgers' equation is a fundamental nonlinear partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics, such as modeling of dynamics, heat conduction, shock waves, and acoustic waves [3,9]. Equations $(1a)$ and $(1b)$ are a special form of incompressible Navier-Stokes equations without having pressure term and continuity equation. The first attempt to solve Burgers' equation analytically was given by Bateman, who derived the steady solution for a simple one-dimensional Burgers' equation, which was used by Burger to model turbulence. In the past several years, numerical solutions to one-dimensional Burgers' equation and system of multidimensional Burgers' equations have attracted a lot of attention of the researchers [13]. Many researchers use the Equations ($1_{a\text{-}d}$) and mentioned in [1,3,4,9,13,14] .We compare the numerical results of ADI-DQM for solving problem(1) with the results of other numerical methods such as the differential quadrature method (DQM), the radial basis function (RBF) [1] and the finite element method (FEM) [4] .The purpose of this paper is to introduce and apply our new improvement of DQM that is known the alternating direction implicit formulation of the differential quadrature method for solving two-dimensional Burger equation. The results that we obtain from our proposed method will be saved and compared to prove the efficiency of the method in accuracy and stability. The advantages of this work are that the ADI-DQM reduces the computational workload and improvement with regard to its accuracy and rapid convergence.

2. Differential Quadrature Method

The differential quadrature is a numerical technique used to solve the initial and boundary value problems. This method was proposed by Bellman in the early 1970s[2]. The essence of the method is that the partial (ordinary) derivatives of a function with respect to variable in governing equation, are approximated by a weighted linear sum of function values at all discrete points in that direction (here, let $h = \Delta x = \Delta y$ denote the step size of spatial space and Δt is the step size with respect to time), then the equation can be transformed into a set of ordinary differential equations or algebraic equations. According to the DQM, the r^{th} -order partial derivatives $\frac{\partial^r u_1}{\partial x^r}$ of a function $u_1(x,y)$ at a point (x_i, y_j) and the s^{th}-order partial derivatives $\frac{\partial^s u_1}{\partial y^s}$ of a function $u_1(x,y)$ at a point (x_i, y_j) , can be approximated by the same formula given in [11],as:

$$\left.\frac{\partial^r u_1}{\partial x^r}\right|_{x=x_i} = \sum_{k=1}^{N} A_{ik}^{(r)}\, u_1(x_k, y) \qquad , \qquad i = 1,2, \dots, N \tag{2}$$

$$\left.\frac{\partial^s u_1}{\partial y^s}\right|_{y=y_j} = \sum_{l=1}^{M} B_{jl}^{(s)}\, u_1(x, y_l) \qquad , \qquad j = 1,2, \dots, M \tag{3}$$

where$A_{ik}^{(r)}$, $B_{jl}^{(s)}$ are the respective weighting coefficients for the r^{th} -order and s^{th}-order derivatives with respect to x and y respectively. Bellman et al. [2] proposed two approaches to compute the weighting coefficients $A_{ik}^{(r)}$, $B_{jl}^{(s)}$. To improve Bellman's approaches in computing the weighting coefficients, many attempts have been made by researchers. Quan and Chang [7, 8] introduced one of the most valuable attempts. After that, Shu's [11] introduced a general approach, which was inspired from Bellman's approach, was made available in the literature. Shu's[11] give Shu's recurrence formulation for higher order derivatives as:

$$A_{ik}^{(r)} = r\left(A_{ii}^{(r-1)} A_{ik}^{(1)} - \frac{A_{ik}^{(r-1)}}{(x_i - x_k)}\right) , \qquad k, i = 1, \dots, N, \qquad 2 \le r \le N-1 \ , i \ne k \tag{4}$$

and

$$A_{ii}^{(r)} = -\sum_{k=1}^{N} A_{ik}^{(r)} \qquad , \qquad 1 \le r \le N-1 \qquad , \ i \ne k \qquad , i = 1,2, \dots, N \tag{5}$$

where $A_{ik}^{(1)}$ are the weighting coefficients of the first order derivative given below

$$A_{ik}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_k) M^{(1)}(x_k)} \qquad\qquad for \quad i \ne k$$

where $M(x) = (x - x_1)(x - x_2) \dots (x - x_N)$ and $M^{(1)}(x_i) = \prod_{j=1}^{N}(x_i - x_j)$ $i \ne j$

The same formulas can be obtained for weighting coefficients of the high order derivatives with respect to y. By using equations (2) and (3), we can approximate the partial derivatives of the equation $(1a)$ to obtain the system of ordinary differential equations as:

$$\left.\frac{\partial u_1}{\partial t}\right|_{ij}^{n} + \sum_{k=1}^{N} \alpha {u_1}_{ij} A_{ik}^{(1)} {u_1}_{kj} + \sum_{l=1}^{M} \alpha {u_2}_{ij} B_{jl}^{(1)}\, {u_1}_{il} = \frac{1}{Re}\left(\sum_{k=1}^{N} A_{ik}^{(2)} {u_1}_{kj} + \sum_{l=1}^{M} B_{jl}^{(2)} {u_1}_{il}\right) \tag{6}$$

Approximating the first-order derivative with respect to the temporal variable by using the forward differences and then arrangement the terms of equation (6), we obtain the system of algebraic equations as:

$$\frac{u_{1_{ij}}^{n+1}-u_{1_{ij}}^{n}}{\Delta t}+\sum_{k=1}^{N}\left(\alpha u_{1_{ij}}^{n}A_{ik}^{(1)}-\frac{1}{Re}A_{ik}^{(2)}\right)u_{1_{kj}}^{n}+\sum_{l=1}^{M}(\alpha u_{2_{ij}}^{n}B_{jl}^{(1)}-\frac{1}{Re}B_{jl}^{(2)})u_{1_{il}}^{n}=0 \tag{7}$$

The similar formulas can be obtained for equation $(1b)$

3. Alternating Direction Formulation of the DQM

Peaceman and Rachford [6] introduced the alternating direction implicit technique in the mid-50s for solving the system of algebraic equations, which results from finite difference discretization of partial differential equations (PDEs). From iterative method's perspective, ADI method can be considered as a special relaxation method, where a big system is simplified into a number of smaller systems such that each of them can be solved efficiently and the solution of the whole system is gotten from the solutions of the sub-systems in an iterative method. Using alternating direction implicit method into equation (7), we get the following two systems of algebraic equations in the form:

$$\frac{u_{1_{ij}}^{n+\frac{1}{2}}-u_{1_{ij}}^{n}}{\frac{\Delta t}{2}}+\sum_{k=1}^{N}\left(\alpha u_{1_{ij}}^{n}A_{ik}^{(1)}-\frac{1}{Re}A_{ik}^{(2)}\right)u_{1_{kj}}^{n+\frac{1}{2}}+\sum_{l=1}^{M}(\alpha u_{2_{ij}}^{n}B_{jl}^{(1)}-\frac{1}{Re}B_{jl}^{(2)})u_{1_{il}}^{n}=0 \tag{8}$$

$$\frac{u_{1_{ij}}^{n+1}-u_{1_{ij}}^{n+\frac{1}{2}}}{\frac{\Delta t}{2}}+\sum_{k=1}^{N}\left(\alpha u_{1_{ij}}^{n+\frac{1}{2}}A_{ik}^{(1)}-\frac{1}{Re}A_{ik}^{(2)}\right)u_{1_{kj}}^{n+\frac{1}{2}}+\sum_{l=1}^{M}(\alpha u_{2_{ij}}^{n+\frac{1}{2}}B_{jl}^{(1)}-\frac{1}{Re}B_{jl}^{(2)})u_{1_{il}}^{n+1}=0 \tag{9}$$

Formula (8) is used to compute function values at all interval mesh points along rows and is known as a horizontal traverse or x −sweep. While, Formula (9) is used to compute function values at all interval mesh points along columns and is known as a vertical traverse or y −sweep.

In the same procedure we approximate equation $(1b)$ by using ADI-DQM to obtain the systems of algebraic equations.

4. Numerical Experiments and Discussion

In this section, we apply ADI-DQM on three test problems to demonstrate the efficiency of the ADI-DQM. Other researchers also considered these problems.

Problem 1.(Ali A. [1])

We consider Burger equation (1) with $\alpha=1,\ Re=100$, $L=1$ and initial conditions in the following forms:

$$u_1(x,y,0)=\frac{3}{4}-\frac{1}{4(1+\mathrm{e}^{\mathrm{Re}(y-x)/8})}\,,\qquad u_2(x,y,0)=\frac{3}{4}+\frac{1}{4(1+\mathrm{e}^{\mathrm{Re}(y-x)/8})} \tag{10}$$

The exact solutions are given by

$$u_1(x,y,t) = \frac{3}{4} - \frac{1}{4(1+\mathrm{e}^{\mathrm{Re}(4y-4x-t)/32})}, \qquad u_2(x,y,t) = \frac{3}{4} + \frac{1}{4(1+\mathrm{e}^{\mathrm{Re}(4y-4x-t)/32})} \tag{11}$$

The boundary conditions can be obtained easily from (11) by using $x, y = 0, 1$. In this problem, we found numerical results for u_1 and u_2 and use equally spaced grid points. In Table 1, we show the errors obtained in solving problem 1 with the ADI-DQM and DQM at $t = 0.01$, $\Delta t = 0.001$, $Re = 100$ and $(x, y) \in [0, 1]$ for different values of h. In Fig. 1, we show the exact and approximate solutions of the problem 1. The results confirm that ADI-DQM has a high accuracy, good convergence comparing with DQM.

Table 1. Errors obtained for problem 1 with $t = 0.01$, $\Delta t = 0.001$ and $Re = 100$

	Error norms for the u_1		*Error norms for the* u_2	
h	*Max* $\lvert error\rvert$ of DQM	*Max* $\lvert error\rvert$ of ADI-DQM	*Max* $\lvert error\rvert$ of DQM	*Max* $\lvert error\rvert$ of ADI-DQM
0.2	5.298339E-06	1.106499E-06	8.713128E-06	1.443198E-06
0.11	4.392205E-06	2.331511E-07	1.268978E-05	1.720921E-06
0.09	3.794236E-06	7.300147E-08	1.493134E-05	2.342516E-06

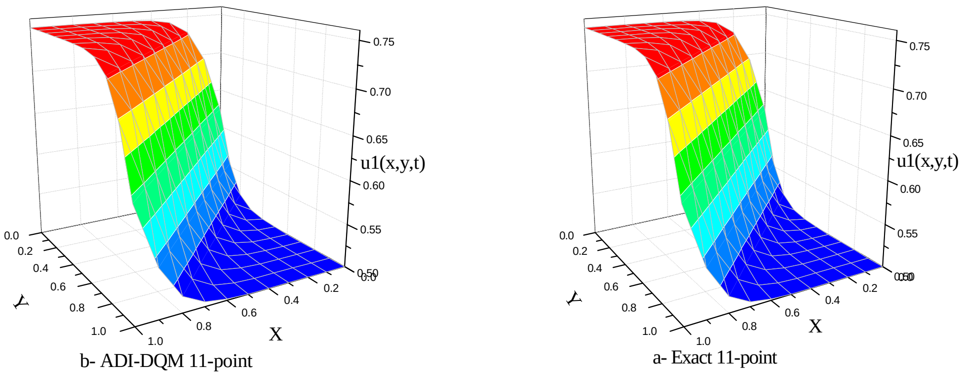

Fig. 1. Exact and approximate solution of the problem 1 with $t = 0.01$, $\Delta t = 0.001$ and $Re = 100$

Problem 2.(Zheng B. [14])

We consider Burger equation (1) with $\alpha = -2$, $Re = 1$, $L = 1$ and initial conditions in the following forms:

$$u_1(x,y,0) = \frac{1}{2} - \frac{x+y}{1+x+y}, \qquad u_2(x,y,0) = \frac{1}{2} + \frac{x+y}{1+x+y} \tag{12}$$

The exact solutions are given by

$$u_1(x,y,t) = \frac{1}{2} - \frac{x+y+t}{1+x+y+t}, \qquad u_2(x,y,t) = \frac{1}{2} + \frac{x+y+t}{1+x+y+t} \tag{13}$$

The boundary conditions can be obtained easily from (13) by using $x, y = 0, 1$. In this problem, we found numerical results for u_1 and u_2 and use equally spaced grid points. In Table 2, we show the errors obtained in solving problem 2 with the ADI-DQM and DQM at $t = 0.01$, $\Delta t = 0.0001$and $(x, y) \in [0, 1]$ for different values of h. In Fig. 2, we show the exact and approximate solutions of the problem 2. The results confirm that ADI-DQM has a high accuracy, good convergence comparing with DQM.

Table 2 . Errors obtained for problem 2 with $t = 0.01, \Delta t = 0.0001$

	Error norms for the u_1		*Error norms for the* u_2	
h	*Max* $\|error\|$ of DQM	*Max* $\|error\|$ of ADI-DQM	*Max* $\|error\|$ DQM	*Max* $\|error\|$ ADI-DQM
0.2	2.361438E-05	1.308369E-05	5.866786E-05	2.824992E-05
0.11	5.864640E-05	1.066213E-05	2.391282E-04	7.368749E-05
0.09	7.547103E-05	5.775655E-06	3.770356E-04	9.814879E-05

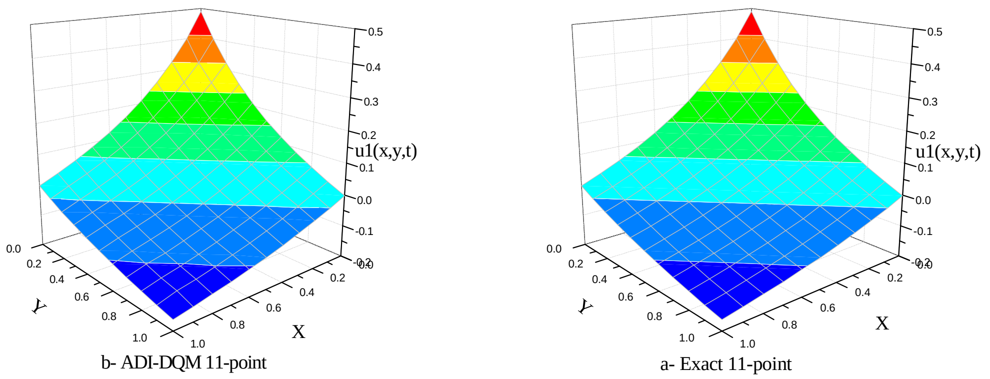

Fig. 2. Exact and approximate solution of the problem 2 with $t = 0.01$and $\Delta t = 0.0001$

Problem 3.(Ali A. [1])

We consider Burger equation (1) with $\alpha = 1$, $Re = 1$, $L = 1$ and initial conditions in the following forms:

$$u_1(x,y,0) = sin(\pi x)\sin(\pi y) \tag{14a}$$

$$u_2(x,y,0) = (sin(\pi x) + \sin(2\pi x))(sin(\pi y) + \sin(2\pi y)) \tag{14b}$$

The boundary conditions are given by

$$u_1(0,y,t) = u_1(1,y,t) = 0 \quad , \quad u_1(x,0,t) = u_1(x,1,t) = 0 \qquad (15a)$$

$$u_2(0,y,t) = u_2(0,y,t) = 0 \quad , \quad u_2(x,0,t) = u_2(x,1,t) = 0 \qquad (15b)$$

In this problem, no analytical solution is available for this system, but only several data points at time 0.01, we computed values of the velocity components u_1 and u_2 and use equally spaced grid points . In Table 3, we show the numerical results of the solving problem 3 with the ADI-DQM and DQM at $Re = 1$, $\Delta t = 0.001$and $(x,y) \in [0,1]$ for different values of points on grid points. The approximate solutions of the problem 3 are shown in Fig. 3. We are using grid point 12×12 withvelocity components. Numerical results of the problem 3 given at Tables 4 and 5.

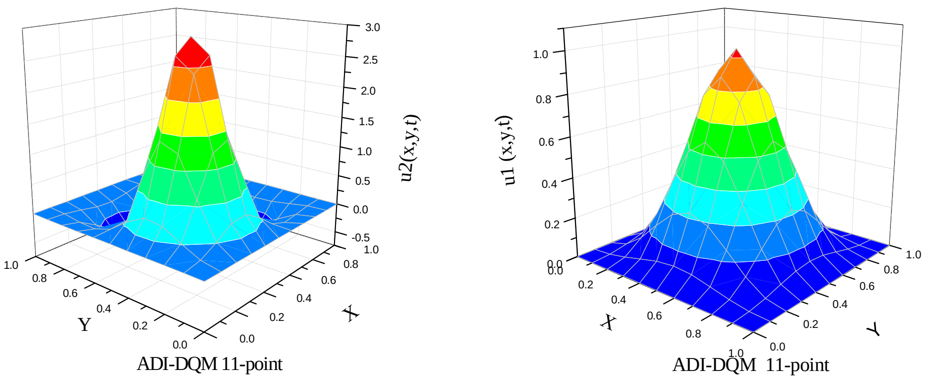

Fig.3. Approximate solution of the problem 3 with $t = 0.01$, $Re = 1$ and $\Delta t = 0.001$

5. Comparison with other Schemes

Comparison 1:

We compare the numerical results of problem 1 of ADI-DQM with the results of other numerical methods such as DQM, multi quadric RBF (MQ(RBF)) [1], Spline of degree seven RBF ((r^7)(RBF)) [1] and thin plate Spline RBF (TPS(RBF)) [1]. Table 4 shows the number of grid points and maximum absolute error in the numerical solutions resulted from using ADI-DQM with other methods. The error measurements resulted from ADI-DQM is more accurate than the methods DQM, MQ(RBF), (r^7)(RBF) and TPS(RBF). Moreover, the number of grid points by using ADI-DQM is less than the other methods.

Table 3. Comparison of the numerical results of the for problem 1 for different method at $t = 0.01$,$\Delta t = 0.001$ and $Re = 100$

Method	*Number of grid points*	$Max\|error\|$ of u_1	$Max\|error\|$of u_2
ADI-DQM	11×11	7.300E-08	2.343E-06
DQM	11×11	3.795E-06	1.493E-05
MQ(RBF) [1]	20×20	4.522E-07	3.590E-06
TPS(RBF) [1]	20×20	1.841E-05	1.003E-04
r^7 (RBF) [1]	20×20	6.926E-06	5.344E-05

Comparison 2:

We compare the numerical results of problem 3 of ADI-DQM with the results of other numerical methods such as DQM , MQ(RBF) [1], TPS(RBF) [1], and FEM [4]. Tables 4 and 5 show the number of grid points andnumerical solutions resulted from using ADI-DQM with other methods.

Table 4. Comparison of the numerical results of the values of u_1 for different method at $t = 0.01$ and $Re = 1$**,** for problem 3.

Method	Δt	Grid points	Numerical results of the values of u_1				
			(0.1,0.1)	(0.2,0.8)	(0.4,0.4)	(0.7,0.1)	(0.9,0.9)
ADI-DQM	0.001	12 x 12	0.07243	0.27741	0.71996	0.20482	0.07974
DQM	0.001	12 x 12	0. 07262	0.27692	0.71977	0.20402	0.07994
MQ[1]	0.0012	20 x 20	0.07251	0.27778	0.72174	0.20484	0.07944
TPS[1]	0.001	20 x 20	0.07238	0.27769	0.72173	0.20468	0.07931
FEM[4]	0.0006	20 x 20	0.07257	0.28842	0.72210	0.20117	0.07947

Table 5. Comparison of the numerical results of the values of u_2 for different method at $t = 0.01$ and $Re = 1$**,** for problem 3.

Method	Δt	Grid points	Numerical results of the values of u_2				
			(0.1,0.1)	(0.2,0.8)	(0.4,0.4)	(0.7,0.1)	(0.9,0.9)
ADI-DQM	0.001	12 x 12	0.42610	-0.12319	1.65214	0.06714	0.01320
DQM	0.001	12 x 12	0. 42543	-0.12276	1.65157	0.06657	0.01350
MQ[1]	0.0012	20 x 20	0.43087	-0.12410	1.65244	0.06705	0.01335
TPS[1]	0.001	20 x 20	0.42932	-0.12371	1.65240	0.06793	0.01305
FEM[4]	0.0006	20 x 20	0.44336	-0.12366	1.65499	0.06621	0.01367

6. Error Analysis and Stability of DQM

We can resolve another mission of the truncation error in the differential quadrature method. Depending on the DQM is identical to Lagrange polynomial interpolation of order $N-1$, Chen [5] has presented new formulas for the analysis of truncation error distribution of derivative in this method. The truncation error of the first-order derivative approximation by the DQM at the grid point x_i is given as;

$$\varepsilon^{(1)}(x_i) \leq \frac{K_1 M^{(1)}(x_i)}{N!} = K_1 e^{(1)}(x_i) \tag{16}$$

where $K_1 = Max\{|u^{(N)}(\xi)|\}$,ξ is unknown function of variable x, and $e^{(1)}(x_i)$ denotes the error distributions of the first- order derivative. For the truncation error of the second-order derivative approximation by DQM is given as,

$$\left|\varepsilon^{(2)}(x_i)\right| \leq 2K_2\left(1+\left|A_{ii}^{(1)}\right|\right)\frac{M^{(1)}(x_i)}{N!} = K_2 e^{(2)}(x_i) \tag{17}$$

where $K_2 = Max\{(\left|u^{(N)}(\xi)\right|, \left|\xi_x u^{(N+1)}(\xi)\right|)\}$, $e^{(2)}(x_i)$ denotes the error distributions of the second-order derivative. While the stability, from equation (6) we obtained the systems of algebraic equations in the form:

$$[A]\{u\} = \{b\} - \{s\} \tag{18}$$

where $[A]$ is the coefficient matrix containing the weighting coefficients, the dimension of the matrix $[A]$ is $(N-2)(M-2)\ by\ (N-2)(M-2)$,$\{u\}$ is a vector of unknown functional values at all the interior points ,$\{b\}$ is a vector still containing discretized time derivatives of u and $\{s\}$ vector contains known values of u at the boundary grid points. The stability analysis of this equation is based on the eigenvalue distribution of the DQ discretization matrix $[A]$. If $[A]$ has eigenvalues λ_i and corresponding eigenvector ξ_i, (i=1,2,…,K) K being the size of the matrix $[A]$, the similarity transformation reduces the system(18) of the from[1].

$$\frac{d\{U\}}{dt} = [D]\{U\} + \{S\} \tag{19}$$

where $[D] = [P]^{-1}[A][P]$,$\{U\} = [P]^{-1}\{u\}$ and $\{S\} = -[P]^{-1}\{s\}$

Since $[D]$is a diagonal matrix, Equation (19) is an uncoupled set of ordinary differential equations and [P] is a nonsingular matrix containing the eigenvectors as columns. Considering the i^{th}equation of (19)

$$\frac{dU_i}{dt} = \lambda_i U_i + S_i \tag{20}$$

This system has the solution

$$\{u\} = [P]\{U\} = \sum_{i=1}^{K} U_i\xi_i = \sum_{i=1}^{K}\left[U_i(0)e^{\lambda_i t} + \frac{S_i}{\lambda_i}(e^{\lambda_i t} - 1)\right]\xi_i$$

and this solution is stable as $t \to \infty$ if

$$R(\lambda_i) < 0 \qquad , \qquad i = 1,2,\dots,K \tag{21}$$

where $R(\lambda_i)$ denotes the real part of λ_i. This is the stability condition for the system (18). We explain the stability condition (21) in case using of grid points 4×4 on problems 1, 2 and 3. The eigenvalues of the matrix [A] are;

For problem1, $R(\lambda_1) = -0.105$, $R(\lambda_2) = -0.105$, $R(\lambda_3) = -0.044$ and, $R(\lambda_4) = -0.044$,
For problem 2, $R(\lambda_1) = -27.15$, $R(\lambda_2) = -43.8$, $R(\lambda_3) = -27.8$ and, $R(\lambda_4) = -43.1$,
For problem 3, $R(\lambda_1) = -35.08$, $R(\lambda_2) = -20.0$, $R(\lambda_3) = -50.31$ and, $R(\lambda_4) = -31.60$.

This means the stability condition (21) is hold. Zong and Lam (2002)[15] have shown that too large numbers of grid points may lead to instability. We conclude from the above discussion that accuracy requires large number of grid points, but stability requires the opposite. The accuracy and stability of the numerical solutions depend on the choice of grid points selected. Here, we use equally spaced types, which are introduced by Shu and Richards (1992) [10], Shu et al (2001) [12].

7. Conclusions

In this work, we employed the ADI-DQM to solve the Burger equation in two-dimension. The numerical results show that the ADI-DQM has the higher accuracy and convergence as well as the less computation workload by using few grid points. The results show that ADI-DQM has a good potential for solving Burger equations. Moreover, the efficiency of the method is proved in accuracy and stability.

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