Continuous dependence of double diffusive convection in a porous medium with temperature-dependent density

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Abstract

The structural stability of a double diffusive convection in a porous medium of the Forchheimer type was studied, when the density of fluid depends on temperature and concentration as a cubic and linear function, respectively. It has been shown that for this problem, with thermal convection in a plane infinite layer, the resonance can occur between the internal layers that arise. The main parameter is the internal heat source and its presence may lead to oscillatory convection in linear instability inducing resonance. Thus, in this study, the structural stability problem of continuous dependence on the heat source itself for a model of nonisothermal flow in a porous medium of Forchheimer type was analyzed. Furthermore, the continuous dependence of the solution on changes in the Forchheimer coefficients has been shown.

Keywords: Structural stability, Double diffusive, Darcy's law, Forchheimer theory, Cubic density.

1. Introduction

The problem of double diffusive convection in a horizontal layer of porous material saturated with an incompressible fluid has attracted the attention of many writers, see cf. Straughan [1]. An important category of such problem is the structural stability in porous media, or continuous dependence on the model itself. In general, in the field of continuum mechanics, or in partial differential equations, structural stability is prominent, cf. Hirsch and Smale [2]. The continuous dependence on modelling, for the elasticity field, was initiated in a seminal paper of Knops and Payne [3], and these authors have produced improved results in Knops and Payne [4]. Payne [5-7] also developed the field of structural stability, and since then many papers have emerged. References to these can be found in the field of porous media in chapter 2 of the book by Straughan [1], with recent contributions from Aulisa et al. [8], Ciarletta et al. [9], Hoang and Ibragimov [10], Harfash [11-13], Liu [14], Liu et al. [15, 16].

In this paper, we continue with Straughan [17] and Gentile and Straughan [18] who studied the continuous dependence on the heat source in a penetrative convection model in a Forchheimer porous medium when the density depends in quadratic and cubic manner on the temperature field, respectively. For many applications, the quadratic dependence is insufficient and a cubic dependence is necessary, cf. McKay and Straughan [19], Straughan ([20], pp. 143-144). Moreover, in a situation where fluid flow is not small, it is possible to introduce Forchheimer coefficients in the Darcy equations (see [21, 22]), with the idea being that the pressure gradient is no longer proportional to the velocity itself, cf. Straughan ([1], p. 12), Néel [23]. Here we deal with the Forchheimer model with quadratic degree. We first analyse the continuous dependence of the solution on changes in the heat source. Then we check the continuous dependence on Forchheimer coefficients. A separate analysis is provided for each of the parameters, which is necessary because the bounds obtained are different in each case.

For Forchheimer theory, Straughan [24] showed that with heat source and nonlinear density, penetrative convection may occur simultaneously in different layers in a porous medium, resulting a resonance phenomenon. The oscillatory convection results from an interaction between the effects of nonlinear density and heat source. It is therefore important to demonstrate the continuous dependence on the heat source. Since the model we are studying is highly nonlinear, the analysis, as shown here, is non trivial.

2. Basic Equations

We take the momentum equation in a saturated material of Forchheimer type to have the form,

$$v_i + a|\mathbf{v}|v_i + b|\mathbf{v}|^2 v_i = -\pi_{,i} + g_i T + h_i T^2 + L_i T^3 + I_i C, \tag{1}$$

where v_i , T and π are velocity, temperature and pressure, a and b are Forchheimer coefficients, and g_i, h_i, L_i and I_i are vectors incorporating the gravity field which, without loss of generality, we take such that $|\mathbf{g}| \leq 1$, $|\mathbf{h}| \leq 1$, $|\mathbf{L}| \leq 1$ and $|\mathbf{I}| \leq 1$. The standard indicial notation is assumed throughout, with, for example, subscript,i denoting $\partial/\partial x_i$, and subscript, t denoting $\partial/\partial t$. The balance of mass equation for an incompressible fluid is

$$v_{i,i} = 0, \tag{2}$$

while the temperature and concentration equations have, respectively, the following forms

$$T_i + v_i T_{,i} = \Delta T + Q, \tag{3}$$

$$C_{,t} + v_i C_{,i} = \Delta C. \tag{4}$$

Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary Γ smooth enough to allow application of the divergence theorem. Then, Equations (1)-(3) are defined on $\Omega \times (0, \mathcal{T}]$, for $\mathcal{T} < \infty$ a fixed time. The boundary conditions we employ are that

$$v_i n_i = 0, \qquad \text{on} \qquad \Gamma \times [0, \mathcal{T}], \tag{5}$$

and

$$T(\mathbf{x}, t) = h(\mathbf{x}, t), \quad C(\mathbf{x}, t) = k(\mathbf{x}, t), \quad \mathbf{x} \in \Gamma, \quad t \in [0, \mathcal{T}], \tag{6}$$

where h and k are prescribed functions and n is the unit outward normal to Γ. The initial condition is

$$T(\mathbf{x}, 0) = T_0(\mathbf{x}) \qquad \text{and} \qquad C(x, 0) = C_0(\mathbf{x}), \tag{7}$$

where T_0 and C_0 are prescribed functions. Let the boundary-initial value problem comprised of Equations (1)-(3) together with conditions (5)-(7) be denoted by $\mathcal{P}$.

3. A priori estimates

Firstly, we need to find some a priori estimates for various norms of T and C which are important to derive the continuous dependence of a solution of problem $\mathcal{P}$ on the heat source Q and Forchheimer coefficients a and b. To find these estimates, the functions $G(\mathbf{x}, t), K(\mathbf{x}, t), I(\mathbf{x}, t), F(\mathbf{x}, t)$ and $H(\mathbf{x}, t)$ have been introduced as solutions to the following boundary value problems

$$\begin{aligned} \Delta G &= 0, && \text{in } \Omega, \\ G &= h(\mathbf{x}, t), && \text{on } \Gamma, \end{aligned} \tag{8}$$

$$\begin{aligned} \Delta K &= 0, && \text{in } \Omega, \\ K &= k(\mathbf{x}, t), && \text{on } \Gamma, \end{aligned} \tag{9}$$

$$\begin{aligned} &\Delta I = 0, && \text{in } \Omega, \\ &I = h^3(\mathbf{x}, t), && \text{on } \Gamma, \end{aligned} \tag{10}$$

$$\begin{aligned} &\Delta F = 0, && \text{in } \Omega, \\ &F = h^5(\mathbf{x}, t), && \text{on } \Gamma, \end{aligned} \tag{11}$$

and then last step is

$$\begin{aligned} &\Delta H = 0, && \text{on } \Omega, \\ &H = h^{2p-1}(\mathbf{x}, t), && \text{in } \Gamma, \end{aligned} \tag{12}$$

where p is a positive integer to be specified later. Now, multiply Equation (1) by v_i integrate over Ω, and apply the Cauchy-Schwarz and arithmetic-geometric mean inequalities, to see that

$$\begin{aligned} \| \mathbf{v} \|^2 + a \| \mathbf{v} \|_3^3 + b \| \mathbf{v} \|_4^4 &= (g_i T, v_i) + (h_i T^2, v_i) + (L_i T^3, v_i) + (I_i C, v_i) \\ &\leq (T, v_i) + (T^2, v_i) + (T^3, v_i) + (C, v_i) \\ &\leq \frac{1}{6} \| \mathbf{v} \|^2 + \frac{3}{2} \| T \|^2 + \frac{1}{6} \| v \|^2 + \frac{3}{2} \| T \|_4^4 \\ &+ \frac{1}{6} \| \mathbf{v} \|^2 + \frac{3}{2} \| T \|_6^6 + \frac{1}{6} \| v \|^2 + \frac{3}{2} \| C \|^2 \\ &= \frac{2}{3} \| \mathbf{v} \|^2 + \frac{3}{2} (\| T \|^2 + \| T \|_4^4 + \| T \|_6^6 + \| C \|^2), \end{aligned} \tag{13}$$

Thus, from inequality (13), it follows that

$$\frac{1}{3} \| \mathbf{v} \|^2 + a \| \mathbf{v} \|_3^3 + b \| \mathbf{v} \|_4^4 \leq \frac{3}{2} (\| T \|^2 + \| T \|_4^4 + \| T \|_6^6 + \| C \|^2), \tag{14}$$

which leads to the following bounds

$$\begin{aligned} \| \mathbf{v} \|^2 &\leq \frac{9}{2} (\| T \|^2 + \| T \|_4^4 + \| T \|_6^6 + \| C \|^2), \\ \| \mathbf{v} \|_3^3 &\leq \frac{3}{2a} (\| T \|^2 + \| T \|_4^4 + \| T \|_6^6 + \| C \|^2), \\ \| \mathbf{v} \|_4^4 &\leq \frac{3}{2b} (\| T \|^2 + \| T \|_4^4 + \| T \|_6^6 + \| C \|^2). \end{aligned} \tag{15}$$

We now form the expressions

$$\int_0^t \int_\Omega (T - G)(T_{,s} + v_i T_{,i} - \Delta T - Q) d\mathbf{x} ds = 0, \tag{16}$$

$$\int_0^t \int_\Omega (C - K)(C_{,s} + v_i C_{,i} - \Delta C) d\mathbf{x} ds = 0, \tag{17}$$

$$\int_0^t \int_\Omega (T^3 - I)(T_{,s} + v_i T_{,i} - \Delta T - Q) d\mathbf{x} ds = 0, \tag{18}$$

and

$$\int_0^t \int_\Omega (T^5 - F)(T_{,s} + v_i T_{,i} - \Delta T - Q) d\mathbf{x} ds = 0, \tag{19}$$

where t is some number such that $0 < t \leq \mathcal{T}$. Next, integrate by parts in (16) and employ the boundary condition $(8)_2$ to see that

$$\begin{aligned} \frac{1}{2} \| T \|^2 + \int_0^t \| \nabla T \|^2 ds &\leq \frac{1}{2} \| T_0 \|^2 + \int_0^t (T, Q) ds + (T, G) + |(T_0, G_0)| \\ &+ \left| \int_0^t (G_{,s}, T) ds \right| + \int_0^t \int_\Omega G v_i T_{,i} d\mathbf{x} ds + \int_0^t \oint_\Gamma h\left(\frac{\partial G}{\partial n}\right) dA ds + \left| \int_0^t (G, Q) ds \right|. \end{aligned} \tag{20}$$

We next bound the cubic term on the right of (20) as follows, where G_m is the maximum of G on $\Gamma \times [0, T]$,

$$\int_0^t \int_\Omega G v_i T_{,i} d\mathbf{x}ds \le G_m \sqrt{\int_0^t \parallel \mathbf{v} \parallel^2 ds \int_0^t \parallel \nabla T \parallel^2 ds}$$
$$\le \frac{3G_m}{\sqrt{2}} \sqrt{\int_0^t (\parallel T \parallel^2 + \parallel C \parallel^2 + \parallel T \parallel_4^4 + \parallel T \parallel_6^6) ds \int_0^t \parallel \nabla T \parallel^2 ds} \quad (21)$$
$$\le \frac{9G_m^2 \alpha_1}{4} \int_0^t (\parallel T \parallel^2 + \parallel C \parallel^2 + \parallel T \parallel_4^4 + \parallel T \parallel_6^6) ds + \frac{1}{2\alpha_1} \int_0^t \parallel \nabla T \parallel^2 ds,$$

where we have used the Cauchy-Schwarz and arithmetic-geometric mean inequalities and where $\alpha_1 > 0$ is a constant to be chosen. Then, by using the Cauchy-Schwarz and arithmetic-geometric mean inequalities and with $\alpha_1 = 2$, we arrive at

$$\frac{1}{2} \parallel T \parallel^2 + \frac{3}{4} \int_0^t \parallel \nabla T \parallel^2 ds \le 2 \parallel G \parallel^2 + \frac{1}{8} \parallel T \parallel^2 + \parallel T_0 \parallel^2 + \frac{1}{2} \parallel G_0 \parallel^2 + \frac{1}{2} \int_0^t \parallel G_{,s} \parallel^2 ds$$
$$+ (1 + \frac{9G_m}{2}) \int_0^t \parallel T \parallel^2 ds + \frac{9G_m^2}{2} \int_0^t (\parallel C \parallel^2 + \parallel T \parallel_4^4 + \parallel T \parallel_6^6) ds \quad (22)$$
$$+ \frac{1}{2} \int_0^t \oint_\Gamma h^2 dA ds + \frac{1}{2} \int_0^t \oint_\Gamma (\frac{\partial G}{\partial n})^2 dA ds + \frac{1}{2} \int_0^t \parallel G \parallel^2 ds + \int_0^t \parallel Q \parallel^2 ds.$$

We now return to Equation (17) and perform various integrations by parts to find with the aid of the Cauchy-Schwarz and arithmetic-geometric mean inequalities,

$$\frac{1}{2} \parallel C \parallel^2 + \int_0^t \parallel \nabla C \parallel^2 ds \le \frac{4}{3} \parallel K \parallel^2 + \frac{3}{16} \parallel C \parallel^2 + \parallel C_0 \parallel^2 + \frac{1}{2} \parallel K_0 \parallel^2$$
$$+ \frac{1}{2\alpha_2} \int_0^t \parallel K_{,s} \parallel^2 ds + \frac{\alpha_2}{2} \int_0^t \parallel C \parallel^2 ds + \frac{1}{4} \int_0^t \parallel \nabla C \parallel^2 ds \quad (23)$$
$$+ K_m^2 \int_0^t \parallel \mathbf{v} \parallel^2 ds + \frac{1}{2} \int_0^t \oint_\Gamma k^2 dA ds + \frac{1}{2} \int_0^t \oint_\Gamma (\frac{\partial K}{\partial n})^2 dA ds,$$

where K_m is the maximum value of K on $\Gamma \times [0, T]$. Inequality $(15)_1$ is next employed in (23) and then with further use of the arithmetic-geometric mean inequality in (23) we may show that

$$\frac{5}{16} \parallel C \parallel^2 + \frac{3}{4} \int_0^t \parallel \nabla C \parallel^2 ds \le \frac{4}{3} \parallel K \parallel^2 + \parallel C_0 \parallel^2 + \frac{1}{2} \parallel K_0 \parallel^2 + \frac{1}{2} \int_0^t \parallel K_{,s} \parallel^2 ds$$
$$+ (\frac{1}{2} + \frac{9K_m^2}{2}) \int_0^t \parallel C \parallel^2 ds + \frac{9K_m^2}{2} \int_0^t (\parallel T \parallel^2 + \parallel T \parallel_4^4 + \parallel T \parallel_6^6) ds \quad (24)$$
$$+ \frac{1}{2} \int_0^t \oint_\Gamma k^2 dA ds + \frac{1}{2} \int_0^t \oint_\Gamma (\frac{\partial K}{\partial n})^2 dA ds.$$

Next, after integration by parts in (18) and some rearrangement we may produce

$$\frac{1}{4} \parallel T \parallel_4^4 - \frac{1}{4} \parallel T_0 \parallel_4^4 + \frac{3}{4} \int_0^t \parallel \nabla T^2 \parallel^2 ds - \int_0^t (T^3, Q) ds - (I, T) + (I_0, T_0)$$
$$+ \int_0^t (T, I_{,s}) ds - \int_0^t \int_\Omega I v_i T_{,i} d\mathbf{x} ds + \int_0^t (I, Q) ds - \int_0^t \oint_\Gamma h (\frac{\partial I}{\partial n}) dA ds = 0. \quad (25)$$

Hence, with further use of the Cauchy-Schwarz and the arithmetic-geometric mean inequalities, from (25) we deduce

$$\frac{1}{4} \parallel T \parallel_4^4 + \frac{3}{4} \int_0^t \parallel \nabla T^2 \parallel^2 ds \le \frac{1}{2} \int_0^t \parallel I_{,s} \parallel^2 ds + (\frac{1}{2} + \frac{9I_m^2}{2}) \int_0^t \parallel T \parallel^2 ds$$
$$+ (\frac{3}{4} + \frac{9I_m^2}{2}) \int_0^t \parallel T \parallel_4^4 ds + \frac{1}{4} \parallel T_0 \parallel_4^4 + 4 \parallel I \parallel^2 + \frac{1}{2} \parallel I_0 \parallel^2 + \frac{1}{2} \parallel T_0 \parallel^2$$
$$+ \frac{1}{4} \int_0^t \parallel \nabla T \parallel^2 ds + \frac{9I_m^2}{2} \int_0^t (\parallel C \parallel^2 + \parallel T \parallel_6^6) ds + \frac{1}{2} \int_0^t \oint_\Gamma h^2 dA ds \quad (26)$$
$$+ \frac{1}{2} \int_0^t \oint_\Gamma (\frac{\partial I}{\partial n})^2 dA ds + \frac{1}{2} \int_0^t \parallel I \parallel^2 ds + \frac{1}{2} \int_0^t \parallel Q \parallel^2 ds + \frac{1}{4} \int_0^t \parallel Q \parallel_4^4 ds.$$

Form the identity (19), by integrations by parts one then finds

$$\frac{1}{6} \parallel T \parallel_6^6 - \frac{1}{6} \parallel T_0 \parallel_6^6 + \frac{5}{9} \int_0^t \parallel \nabla T^3 \parallel^2 ds - \int_0^t \oint_\Gamma h^5 (\frac{\partial T}{\partial n}) dA ds - \int_0^t (T^5, Q) ds$$
$$+ (F_0, T_0) + \int_0^t (F_{,s}, T) ds - \int_0^t \int_\Omega F v_i T_{,i} d\mathbf{x} ds + \int_0^t \oint_\Gamma h^5 (\frac{\partial T}{\partial n}) dA ds \quad (27)$$
$$+ \int_0^t (F, Q) ds - \int_0^t \oint_\Gamma h (\frac{\partial F}{\partial n}) dA ds = 0.$$

Next, making further use of the arithmetic-geometric mean inequality in (27) we may derive

$$\begin{aligned}&\frac{1}{6}\parallel T\parallel_6^6+\frac{5}{9}\int_0^t\parallel\nabla T^3\parallel^2 ds\le\frac{1}{6}\parallel T_0\parallel_6^6+\frac{\beta}{2}\parallel F\parallel^2+\frac{1}{2\beta}\parallel T\parallel^2+\frac{5}{6}\int_0^t\parallel T\parallel_6^6 ds\\&+\frac{1}{6}\int_0^t\parallel Q\parallel_6^6 ds+\frac{1}{2}\parallel F_0\parallel^2+\frac{1}{2}\parallel T_0\parallel^2+\frac{1}{2}\int_0^t\parallel F_{,s}\parallel^2 ds+\frac{1}{2}\int_0^t\parallel T\parallel^2 ds\\&+\frac{9F_m^2}{2}\int_0^t(\parallel T\parallel^2+\parallel C\parallel^2+\parallel T\parallel_4^4+\parallel T\parallel_6^6)ds+\frac{1}{4}\int_0^t\parallel\nabla T\parallel^2 ds\\&+\frac{1}{2}\int_0^t\oint_\Gamma h^2dAds+\frac{1}{2}\int_0^t\oint_\Gamma(\frac{\partial F}{\partial n})^2dAds+\frac{1}{2}\int_0^t\parallel F\parallel^2 ds+\frac{1}{2}\int_0^t\parallel Q\parallel^2 ds.\end{aligned}\qquad(28)$$

Next, add (22), (24), (26) and (28) to obtain

$$\begin{aligned}&\frac{1}{4}\parallel T\parallel^2+\frac{5}{16}\parallel C\parallel^2+\frac{1}{4}\parallel T\parallel_4^4+\frac{1}{6}\parallel T\parallel_6^6+\frac{1}{4}\int_0^t\parallel\nabla T\parallel^2 ds\\&+\frac{3}{4}\int_0^t\parallel\nabla C\parallel^2 ds+\frac{3}{4}\int_0^t\parallel\nabla T^2\parallel^2 ds+\frac{5}{9}\int_0^t\parallel\nabla T^3\parallel^2 ds\\&\le[2+\frac{9G_m^2}{2}+\frac{9K_m^2}{2}+\frac{9I_m^2}{2}+\frac{9F_m^2}{2}]\int_0^t\parallel T\parallel^2 ds\\&+[\frac{9G_m^2}{2}+\frac{1}{2}+\frac{9K_m^2}{2}+\frac{9I_m^2}{2}+\frac{9F_m^2}{2}]\int_0^t\parallel C\parallel^2 ds\\&+[\frac{9G_m^2}{2}+\frac{9K_m^2}{2}+\frac{3}{4}+\frac{9I_m^2}{2}+\frac{9F_m^2}{2}]\int_0^t\parallel T\parallel_4^4 ds\\&+[\frac{9G_m^2}{2}+\frac{9K_m^2}{2}+\frac{9I_m^2}{2}+\frac{9F_m^2}{2}+\frac{5}{6}]\int_0^t\parallel T\parallel_6^6 ds+E(t),\end{aligned}\qquad(29)$$

where $E(t)$ is a term we will show is bounded by data and is defined by

$$\begin{aligned}E(t)&=2\parallel G\parallel^2+\frac{4}{3}\parallel K\parallel^2+4\parallel F\parallel^2+4\parallel I\parallel^2+2\parallel T_0\parallel^2+2\int_0^t\parallel Q\parallel^2 ds+\frac{1}{2}\parallel G_0\parallel^2\\&+\frac{1}{2}\parallel K_0\parallel^2+\frac{1}{2}\parallel I_0\parallel^2+\frac{1}{2}\parallel F_0\parallel^2+\frac{1}{6}\parallel T_0\parallel_6^6+\parallel C_0\parallel^2+\frac{1}{4}\parallel T_0\parallel_4^4\\&+\frac{1}{2}\int_0^t\parallel G_{,s}\parallel^2 ds+\frac{1}{2}\int_0^t\parallel K_{,s}\parallel^2 ds+\frac{1}{2}\int_0^t\parallel I_{,s}\parallel^2 ds+\frac{1}{2}\int_0^t\parallel F_{,s}\parallel^2 ds\\&+\frac{1}{2}\int_0^t\oint_\Gamma k^2dAds+\frac{3}{2}\int_0^t\oint_\Gamma h^2dAds+\frac{1}{2}\int_0^t\oint_\Gamma(\frac{\partial G}{\partial n})^2dAds\\&+\frac{1}{2}\int_0^t\oint_\Gamma(\frac{\partial K}{\partial n})^2dAds+\frac{1}{2}\int_0^t\oint_\Gamma(\frac{\partial I}{\partial n})^2dAds+\frac{1}{2}\int_0^t\oint_\Gamma(\frac{\partial F}{\partial n})^2dAds\\&+\frac{1}{2}\int_0^t\parallel G\parallel^2 ds+\frac{1}{2}\int_0^t\parallel I\parallel^2 ds+\frac{1}{2}\int_0^t\parallel F\parallel^2 ds+\frac{1}{6}\int_0^t\parallel Q\parallel_6^6 ds+\frac{1}{4}\int_0^t\parallel Q\parallel_4^4 ds.\end{aligned}\qquad(30)$$

Payne and Straughan [25] show that for a function ϕ satisfying

$$\begin{aligned}&\Delta\phi=0,\quad\text{in}\ \ \Omega,\\&\phi=M,\quad\text{on}\ \ \Gamma,\end{aligned}\qquad(31)$$

then one may use a Rellich identity, cf. Payne and Weinberger [26], to determine constants c_1, c_2 such that

$$\parallel\nabla\phi\parallel^2+c_1\oint_\Gamma(\frac{\partial\phi}{\partial n})^2dA\le\oint_\Gamma c_2|\nabla_sM|^2dA,\qquad(32)$$

where ∇_s denotes the surface gradient over the boundary. They also show that

$$2(\psi\nabla\phi,\nabla\phi)+\parallel\phi\parallel^2\le\psi_1\oint_\Gamma M^2dA,\qquad(33)$$

where

$$\psi_1=\max_\Gamma|\frac{\partial\psi}{\partial n}|,$$

with ψ solving the boundary value problem,

$$\begin{aligned}&\Delta\psi=-1,\quad\text{in}\ \ \Omega,\\&\psi=0,\quad\text{on}\ \ \Gamma.\end{aligned}$$

Thus, inequalities (33) and (32) lead to bounds for $E(t)$ in terms of data. In fact, one may show

$$E(t) \leq D(t), \tag{34}$$

where

$$\begin{aligned} D(t) &= 2 \parallel T_0 \parallel^2 + \frac{1}{4} \parallel T_0 \parallel_4^4 + \frac{1}{6} \parallel T_0 \parallel_6^6 + \parallel K_0 \parallel^2 + 2\int_0^t \parallel Q \parallel^2 ds \\ &+ \frac{1}{4}\int_0^t \parallel Q \parallel_4^4 ds + \frac{1}{6}\int_0^t \parallel Q \parallel_6^6 ds + (\frac{3}{2}+\psi_1)\int_0^t \oint_\Gamma h^2 dAds \\ &+ \frac{1}{2}\int_0^t \oint_\Gamma k^2 dAds + 2\psi_1 \oint_\Gamma h^2 dA + \frac{\psi_1}{2}\oint_\Gamma h_0^2 dA + 4\psi_1 \oint_\Gamma h^6 dA \\ &+ \frac{\psi_1}{2}\oint_\Gamma h_0^6 dA + 4\psi_1 \oint_\Gamma h^{10} dA + \frac{\psi_1}{2}\oint_\Gamma h_0^{10} dA + \frac{4}{3}\psi_1 \oint_\Gamma k^2 dA + \frac{\psi_1}{2}\oint_\Gamma h_0^2 dA \\ &+ \frac{c_2}{2c_1}\int_0^t \oint_\Gamma |\nabla_s h|^2 dAds + \frac{c_2}{2c_1}\int_0^t \oint_\Gamma |\nabla_s h^3|^2 dAds + \frac{c_2}{2c_1}\int_0^t \oint_\Gamma |\nabla_s h^5|^2 dAds \\ &+ \frac{c_2}{2c_1}\int_0^t \oint_\Gamma |\nabla_s k|^2 dAds + \frac{\psi_1}{2}\int_0^t \oint_\Gamma h^4 h_{,s}^2 dAds + \frac{\psi_1}{2}\int_0^t \oint_\Gamma h^8 h_{,s}^2 dAds \\ &+ \frac{\psi_1}{2}\int_0^t \oint_\Gamma k_{,s}^2 dAds + \frac{\psi_1}{2}\int_0^t \oint_\Gamma h^6 dAds + \frac{\psi_1}{2}\int_0^t \oint_\Gamma h^{10} dAds + \parallel C_0 \parallel^2. \end{aligned} \tag{35}$$

We put

$$\begin{aligned} \mathcal{K} = \max\{&4(2+\frac{9G_m^2}{2}+\frac{9K_m^2}{2}+\frac{9I_m^2}{2}+\frac{9F_m^2}{2}), \frac{16}{5}(\frac{9G_m^2}{2}+\frac{1}{2}+\frac{9K_m^2}{2}+\frac{9I_m^2}{2}+\frac{9F_m^2}{2}), \\ &4(\frac{9G_m^2}{2}+\frac{9K_m^2}{2}+\frac{3}{4}+\frac{9I_m^2}{2}+\frac{9F_m^2}{2}), 6(\frac{9G_m^2}{2}+\frac{9K_m^2}{2}+\frac{9I_m^2}{2}+\frac{9F_m^2}{2}+\frac{5}{6})\}, \end{aligned} \tag{36}$$

and then from (29) we may derive

$$J' - \mathcal{K}J \leq D(t), \tag{37}$$

where we have introduced the function $J(t)$ defined by

$$J(t) = \int_0^t (\frac{1}{4} \parallel T \parallel^2 + \frac{5}{16} \parallel C \parallel^2 + \frac{1}{4} \parallel T \parallel_4^4 + \frac{1}{6} \parallel T \parallel_6^6) ds. \tag{38}$$

Upon setting

$$D_1 = \int_0^t D(s)\exp(\mathcal{K}(t-s))ds, \tag{39}$$

one integrates (37) to show

$$J(t) \leq D_1. \tag{40}$$

Upon further setting $D_2(t) = \mathcal{K}D_1 + D$, one uses (39) to find

$$\frac{1}{4} \parallel T \parallel^2 + \frac{5}{16} \parallel c \parallel^2 + \frac{1}{4} \parallel T \parallel_4^4 + \frac{1}{6} \parallel T \parallel_6^6 \leq D_2(t). \tag{41}$$

Thus, (41) and (40) yield

$$\begin{aligned} &\parallel T \parallel^2 \leq 4D_2, \quad \parallel C \parallel^2 \leq \frac{16}{5}D_2, \quad \parallel T \parallel_4^4 \leq 4D_2, \quad \parallel T \parallel_6^6 \leq 6D_2, \\ &\int_0^t \parallel T \parallel^2 \leq 4D_1, \quad \int_0^t \parallel C \parallel^2 \leq \frac{16}{5}D_1, \quad \int_0^t \parallel T \parallel_4^4 \leq 4D_1, \quad \int_0^t \parallel T \parallel_6^6 \leq 6D_1. \end{aligned} \tag{42}$$

Furthermore, from inequality (29) we then find

$$\begin{aligned} &\int_0^t \parallel \nabla T \parallel^2 \leq 4D_2, \quad \int_0^t \parallel \nabla C \parallel^2 \leq \frac{4}{3}D_2, \\ &\int_0^t \parallel \nabla T^2 \parallel^2 \leq \frac{4}{3}D_2, \quad \int_0^t \parallel \nabla T^3 \parallel^2 \leq \frac{9}{5}D_2. \end{aligned} \tag{43}$$

The next step is to derive a bound for $\sup_{\Omega\times[0,T]} |T|$. To this end, we form the combination

$$\int_0^t \int_\Omega (T^{2p-1} - H)(T_{,s} + v_i T_i - \Delta T - Q) d\mathbf{x}ds = 0. \tag{44}$$

After some integrations by parts we then show

$$\begin{aligned} &\frac{1}{2p}\int_0^t \int_\Omega \frac{d}{ds}(T^{2p}) d\mathbf{x}ds - \int_0^t \int_\Omega T^{2p-1}\Delta T d\mathbf{x}ds - \int_0^t \int_\Omega T^{2p-1} Q d\mathbf{x}ds - (H,T) + (H_0,T_0) \\ &- \int_0^t \int_\Omega H v_i T_{,i} d\mathbf{x}ds + \int_0^t (H_{,s},T)ds + \int_0^t \int_\Omega H\Delta T d\mathbf{x}ds + \int_0^t \int_\Omega HQ d\mathbf{x}ds = 0 \\ &, \end{aligned} \tag{45}$$

then

$$\int_\Omega T^{2p}d\mathbf{x}+\frac{2(2p-1)}{p}\int_0^t\int_\Gamma \nabla(T^p)\nabla(T^p)d\mathbf{x}ds=\int_\Omega T_0^{2p}d\mathbf{x}+2p(H,T)$$
$$-2p(H_0,T_0)-2p\int_0^t(H_{,s},T)ds+2p\int_0^t\int_\Gamma Hv_iT_{,i}d\mathbf{x}ds+2p\int_0^t\oint_\Gamma h\frac{\partial H}{\partial n}dAds \quad (46)$$
$$+2p\int_0^t\int_\Omega T^{2p-1}Qd\mathbf{x}ds-2p\int_0^t\int_\Omega HQd\mathbf{x}ds.$$

Now, by using the maximum principle, arithmetic-geometric mean inequality and bound $(15)_1$ we have

$$2p\int_0^t\int_\Omega Hv_iT_{,i}d\mathbf{x}ds\le 2pH_m\sqrt{\int_0^t\|\mathbf{v}\|^2ds\int_0^t\|\nabla T\|^2ds}$$
$$\le 2p\frac{3H_m}{\sqrt{2}}(\sqrt{\int_0^t(\|T\|^2+\|C\|^2+\|T\|_4^4+\|T\|_6^6)ds})\sqrt{\int_0^t\|\nabla T\|^2ds}, \quad (47)$$

where H_m denotes the maximum value of H on Γ. Then with the aid of the Cauchy-Schwarz and the arithmetic-geometric mean inequalities one sees that

$$-2p\int_0^t(H_{,s},T)ds\le 2p\sqrt{\int_0^t\|H_{,s}\|^2ds\int_0^t\|T\|^2ds}, \quad (48)$$

$$-2p\int_0^t\oint_\Gamma h\frac{\partial H}{\partial n}dAds\le 2p\sqrt{\int_0^t\oint_\Gamma h^2dAds\int_0^t\oint_\Gamma(\frac{\partial H}{\partial n})^2dAds}, \quad (49)$$

$$2p\int_0^t\int_\Omega T^{2p-1}Qd\mathbf{x}ds\le\int_0^t\int_\Omega Q^{2p}ds+(2p-1)\int_0^t\int_\Omega T^{2p}d\mathbf{x}ds, \quad (50)$$

and

$$2p\int_0^t\int_\Omega HQd\mathbf{x}ds\le\int_0^t\int_\Omega Q^{2p}ds+(2p-1)\int_0^t\int_\Omega H^{2p/(2p-1)}d\mathbf{x}ds. \quad (51)$$

Then use of (47)-(51) in (46) leads to

$$\int_\Omega T^{2p}d\mathbf{x}\le\int_\Omega T_0^{2p}d\mathbf{x}+2p(\|T\|\|H\|+\|T_0\|\|H_0\|)+2p\sqrt{\int_0^t\|H_{,s}\|^2ds\int_0^t\|T\|^2ds}$$
$$+\frac{3}{\sqrt{2}}ph_m^{2p-1}\sqrt{\int_0^t\|T\|^2ds+\int_0^t\|C\|^2ds+\int_0^t\|T\|_4^4ds+\int_0^t\|T\|_6^6ds}\sqrt{\int_0^t\|\nabla T\|^2ds}$$
$$+2p\sqrt{\int_0^t\oint_\Gamma h^2dAds\int_0^t\oint_\Gamma(\frac{\partial H}{\partial n})^2dAds}+2\int_0^t\int_\Omega Q^{2p}d\mathbf{x}ds$$
$$+(2p-1)\int_0^t\int_\Omega T^{2p}d\mathbf{x}ds+(2p-1)\int_0^t\int_\Omega H^{2p/(2p-1)}d\mathbf{x}ds. \quad (52)$$

An application of inequalities (33) and (32) together with (42) and (43) yields

$$\int_\Omega T^{2p}d\mathbf{x}\le\int_\Omega T_0^{2p}d\mathbf{x}+2p(\sqrt{4D_2}+\|T_0\|)\psi_1^{1/2}(\oint_\Gamma h^{4p-2}dA)^{1/2}$$
$$+2p\sqrt{4D_1\psi_1\int_0^t\oint_\Gamma[(h^{2p-1})_{,s}]^2dAds}+\frac{3}{2}ph_m^{2p-1}\sqrt{(22D_1+4D_2)}$$
$$+2p\sqrt{\int_0^t\oint_\Gamma h^2dAds(\frac{c_2}{c_1})\int_0^t\oint_\Gamma|\nabla_sh^{2p-1}|^2dAds}+2\int_0^t\|Q\|_{2p}^{2p}ds \quad (53)$$
$$+(2p-1)\int_0^t\int_\Omega T^{2p}d\mathbf{x}ds+\psi_1(2p-1)\int_0^t\int_\Omega h^{2p}d\mathbf{x}ds.$$

Then, a further application of the Cauchy-Schwarz leads to

$$(\oint_\Gamma h^{4p-2}dA)^{1/2}\le h_m^{2p-1}(\oint_\Gamma dA)^{1/2}=\frac{h_m^{2p}}{h_m}[m(\Gamma)]^{1/2}, \quad (54)$$

$$(\int_0^t \oint_\Gamma h^{4p-4} h_{,s}^2 dAds)^{1/2} \le h_m^{2p-2} (\int_0^t \oint_\Gamma h_{,s}^2 dAds)^{1/2} = \frac{h_m^{2p}}{h_m^2} (\int_0^t \oint_\Gamma h_{,s}^2 dAds)^{1/2}, \quad (55)$$

and

$$(\int_0^t \oint_\Gamma h^{4p-4} |\nabla_s h|^2 dAds)^{1/2} \le h_m^{2p-2} (\int_0^t \oint_\Gamma |\nabla_s h|^2 dAds)^{1/2} = \frac{h_m^{2p}}{h_m^2} (\int_0^t \oint_\Gamma |\nabla_s h|^2 dAds)^{1/2}, \quad (56)$$

where $m(\Gamma)$ is the surface measure of Γ. By employing (54)-(56) in (53), we thus derive

$$\begin{aligned} \int_\Omega T^{2p} d\mathbf{x} &\le \int_\Omega T_0^{2p} d\mathbf{x} + 2\int_0^t \parallel Q \parallel^{2p} ds + (2p-1)\int_0^t \int_\Omega T^{2p} d\mathbf{x} ds \\ &+ \frac{2p}{h_m}(\sqrt{4D_1} + \parallel T_0 \parallel)\psi_1^{1/2} h_m^{2p} [m(\Gamma)]^{1/2} + \frac{2p}{h_m}(4D_1\psi_1)^{1/2}(2p-1) h_m^{2p} (\int_0^t \oint_\Gamma h_{,s}^2 dAds)^{1/2} \\ &+ \frac{3p(22D_1+4D_2)^{1/2}}{2h_m} h_m^{2p} + \psi_1 (2p-1) t \; m(\Gamma) h_m^{2p} \\ &+ \frac{2p(2p-1)}{h_m^2} (\frac{c_2}{c_1})^{\frac{1}{2}} (\int_0^t \oint_\Gamma |\nabla_s h|^2 dAds)^{\frac{1}{2}} (\int_0^t \oint_\Gamma h^2 dAds)^{\frac{1}{2}} h_m^{2p}. \end{aligned} \quad (57)$$

Denote the coefficients of the first to sixth terms in parentheses, on the right of (58), by $r_1(p)$, . . . , $r_6(p)$, then

$$\begin{aligned} \int_\Omega T^{2p} d\mathbf{x} &\le \int_\Omega T_0^{2p} d\mathbf{x} + (r_2 + r_3 + r_4 + r_5 + r_6) h_m^{2p} \\ &+ 2\int_0^t \parallel Q \parallel_{2p}^{2p} ds + \psi_1 t \; m(\Gamma) h_m^{2p} (2p-1) + (2p-1) \int_0^t \oint_\Gamma T^{2p} d\mathbf{x} ds. \end{aligned} \quad (58)$$

Let $\mathcal{B}(p)$ has the form

$$\mathcal{B}(p) = \parallel T_0 \parallel_{2p}^{2p} + (r_2 + r_3 + r_4 + r_5 + r_6) h_m^{2p} + 2\int_0^t \parallel Q \parallel_{2p}^{2p} ds + \psi_1 t \; m(\Gamma) h_m^{2p} (2p-1), \quad (59)$$

then we find

$$\int_\Omega T^{2p} d\mathbf{x} \le (2p-1)\int_0^t \oint_\Gamma T^{2p} d\mathbf{x} ds + \mathcal{B}(p). \quad (60)$$

Inequality (60) is now integrated, and then we take the $1/2p$ power, to find

$$[R'(t)]^{1/2p} \le [\int_0^t \mathcal{B}(p) e^{(2p-1)(t-s)} ds]^{1/2p}. \quad (61)$$

Let now $p \to \infty$ and then (61) leads to

$$\sup_{\Omega \times [o,\mathcal{T}]} |T| \le \max\{|T_0|_m, \sup_{[o,\mathcal{T}]} |Q|_m, \sup_{[o,\mathcal{T}]} |h_m|\}. \quad (62)$$

Lemma 3.1 *If $C(x,0) \in L^\infty(\Omega)$, then*

$$\sup_{\Omega \times [o,\mathcal{T}]} |C| \le \max[|C_0|_m]. \quad (63)$$

Proof: Multiply (4) by c^{p-1} for $p > 1$ (where we assume the concentration is scaled to be non-negative, otherwise p is chosen as an even integer). Thus,

$$\frac{d}{dt}\int_\Omega C^p d\mathbf{x} = -p(p-1)\int_\Omega C^{p-2} |\nabla C|^2 d\mathbf{x} - K_1 p \int_\Omega C^p d\mathbf{x}. \quad (64)$$

We may integrate this and drop non-positive terms on the right to deduce

$$[\int_\Omega C^p d\mathbf{x}]^{1/p} \le [\int_\Omega C_0^p d\mathbf{x}]^{1/p}. \quad (65)$$

Let now $p \to \infty$ in (65) to find the desired result. ■

Inequality (62) and (63) are a priori estimates we are seeking for T and C, respectively. We henceforth denote the right-hand side of (62) and (63) by T_m and C_m, respectively.

4. Continuous dependence on Q

Let $\{u_i, S, \varphi, \pi_1\}$ and $\{v_i, T, C, \pi_2\}$ be solutions to $\mathcal{P}$ for the same values of h, k, T_0 and C_0 but for different heat source functions $R(\mathbf{x}, t)$ and $Q(\mathbf{x}, t)$. Define the difference variables

w_i, θ, ϕ, q and Π by

$$w_i = u_i - v_i, \theta = S - T, \phi = \varphi - C, q = R - Q, \Pi = \pi_1 - \pi_2.$$

It is easily verified that $\{w_i, \theta, \phi, \pi\}$ satisfies the boundary-initial value problem

$$\begin{aligned} & w_i + a[|\mathbf{u}|u_i - |\mathbf{v}|v_i] + b[|\mathbf{u}|^2 u_i - |\mathbf{v}|^2 v_i] = -\Pi_{,i} + g_i\theta + h_i(S+T)\theta \\ & + L_i\theta(S^2 + ST + T^2) + I_i\phi, \\ & w_{i,i} = 0, \\ & \theta_{,t} + w_i T_{,i} + u_i\theta_{,i} = \Delta\theta + q, \\ & \phi_{,t} + w_i C_{,i} + u_i\phi_{,i} = \Delta\phi. \end{aligned} \tag{66}$$

on $\Omega \times (0, T]$, together with

$$\begin{aligned} & w_i n_i = 0, \ \ \theta = 0, \ \ \phi = 0, \qquad \text{on } \ \Gamma \times (0, T], \\ & \theta(\mathbf{x}, 0) = 0, \ \ \phi(\mathbf{x}, 0) = 0, \qquad \mathbf{x} \in \Omega, \end{aligned} \tag{67}$$

Multiply $(66)_1$ by w_i and integrate over Ω. Using the Cauchy-Schwarz inequality and arithmetic-geometric mean inequality one obtains

$$\begin{aligned} & \| \mathbf{w} \|^2 + a\int_\Omega [|\mathbf{u}|u_i - |\mathbf{v}|v_i] w_i d\mathbf{x} + +b\int_\Omega [|\mathbf{u}|^2 u_i - |\mathbf{v}|^2 v_i] w_i d\mathbf{x} \\ & = (\theta g_i, w_i) + (h_i(S+T)\theta, w_i) + (L_i\theta(S^2 + TS + T^2)\phi, w_i) + (I_i\phi, w_i) \\ & \le (\theta, w_i) + (S_m + T_m)(\theta, w_i) + (S_m^2 + T_m S_m + T_m^2)(\theta, w_i) + (\phi, w_i) \\ & = (\theta, w_i) + X_m(\theta, w_i) + Y_m(\theta, w_i) + (\phi, w_i) \\ & \le \tfrac{1}{6} \| \mathbf{w} \|^2 + \tfrac{3}{2} \| \theta \|^2 + \tfrac{1}{6} \| \mathbf{w} \|^2 + \tfrac{3}{2} X_m^2 \| \theta \|^2 + \tfrac{1}{6} \| \mathbf{w} \|^2 + \tfrac{3}{2} Y_m^2 \| \theta \|^2 \\ & + \tfrac{1}{6} \| \mathbf{w} \|^2 + \tfrac{3}{2} \| \phi \|^2 \\ & \le \tfrac{2}{3} \| \mathbf{w} \|^2 + \tfrac{3}{2}(1 + X_m^2 + Y_m^2)[\| \theta \|^2 + \| \phi \|^2], \end{aligned} \tag{68}$$

where $X_m = S_m + T_m$, and $Y_m = S_m^2 + T_m S_m + T_m^2$ with S_m being the maximum value of S, analogous to T_m. Now, we observe with the aid of the triangle inequality, that

$$\begin{aligned} & a\int_\Omega [|\mathbf{u}|u_i - |\mathbf{v}|v_i] w_i d\mathbf{x} = \tfrac{a}{2}\int_\Omega [|\mathbf{u}| + |\mathbf{v}|] w_i w_i d\mathbf{x} + \tfrac{a}{2}\int_\Omega [|\mathbf{u}| - |\mathbf{v}|]^2 (|\mathbf{u}| + |\mathbf{v}|) d\mathbf{x} \\ & \ge \tfrac{a}{2}\int_\Omega [|\mathbf{u}| + |\mathbf{v}|] w_i w_i d\mathbf{x} \ge \tfrac{a}{2}\int_\Omega |\mathbf{u} - \mathbf{v}| w_i w_i d\mathbf{x} \\ & \ge \tfrac{a}{2}\int_\Omega |w_i| w_i w_i d\mathbf{x} = \tfrac{a}{2} \| \mathbf{w} \|_3^3, \end{aligned} \tag{69}$$

and, similarly

$$\begin{aligned} & b\int_\Omega [|\mathbf{u}|^2 u_i - |\mathbf{v}|^2 v_i] w_i d\mathbf{x} = \tfrac{b}{2}\int_\Omega [|\mathbf{u}|^2 + |\mathbf{v}|^2] w_i w_i d\mathbf{x} + \tfrac{b}{2}\int_\Omega [|\mathbf{u}|^2 - |\mathbf{v}|^2]^2 (|\mathbf{u}| + |\mathbf{v}|) d\mathbf{x} \\ & \ge \tfrac{b}{2}\int_\Omega [|\mathbf{u}|^2 + |\mathbf{v}|^2] w_i w_i d\mathbf{x} \ge \tfrac{b}{4}\int_\Omega [|\mathbf{u}| + |\mathbf{v}|]^2 w_i w_i d\mathbf{x} \\ & \ge \tfrac{b}{4}\int_\Omega |u_i - v_i|^2 w_i w_i d\mathbf{x} = \tfrac{b}{4} \| \mathbf{w} \|_4^4. \end{aligned} \tag{70}$$

Next, we put (69) and (70) in (68), to find

$$\| \mathbf{w} \|^2 + \frac{3a}{2} \| \mathbf{w} \|_3^3 + \frac{3b}{4} \| \mathbf{w} \|_4^4 \le \frac{9}{2}(1 + X_m^2 + Y_m^2)[\| \theta \|^2 + \| \phi \|^2]. \tag{71}$$

Furthermore, multiply (3) and(4) by θ and ϕ, respectively, and integrate over Ω to see after integrated by parts, discard the negative terms, and used a little rearrangement that

$$\frac{d}{dt} \| \theta \|^2 \le \| q \|^2 + \frac{T_m^2}{2} \| \mathbf{w} \|^2, \tag{72}$$

$$\frac{d}{dt} \| \phi \|^2 \le \frac{C_m^2}{2} \| \mathbf{w} \|^2, \tag{73}$$

where C_m being the maximum value of C. From (72) and (73), we have

$$\frac{d}{dt}[\| \theta \|^2 + \| \phi \|^2] \le \| q \|^2 + \frac{1}{2}(C_m^2 + T_m^2) \| \mathbf{w} \|^2. \tag{74}$$

Upon insertion of (71) in (74) we see that

$$\frac{d}{dt}[\| \theta \|^2 + \| \phi \|^2] \leq \| q \|^2 + \frac{9}{4}(C_m^2 + T_m^2)(1 + X_m^2 + Y_m^2)[\| \theta \|^2 + \| \phi \|^2]. \tag{75}$$

Let

$$\mathbf{K} = \frac{9}{4}(C_m^2 + T_m^2)(1 + X_m^2 + Y_m^2),$$

then, upon integration of (75) we find

$$\| \theta \|^2 + \| \phi \|^2 \leq \| q \|^2 \left[\frac{\exp(\mathbf{K}t) - 1}{\mathbf{K}}\right]. \tag{76}$$

Thus, we have established continuous dependence on Q in the L^2 measure of θ and ϕ. Additionally, from inequalities (76) in(71) we then derive

$$\| \mathbf{w} \|^2 + \frac{3a}{2} \| \mathbf{w} \|_3^3 + \frac{3b}{4} \| \mathbf{w} \|_4^4 \leq \frac{9}{2}(1 + X_m^2 + Y_m^2) \| q \|^2 \left[\frac{\exp(\mathbf{K}t) - 1}{\mathbf{K}}\right]. \tag{77}$$

Therfore, we also have continuous dependence on Q in the L^2, L^3 and L^4 measures of $\mathbf{w}$.

5. Continuous dependence on a

Now, to investigate continuous dependence on a, we let (u_i, S, φ, π_1) and (v_i, T, C, π_2) be solutions to (1)-(4) for the same boundary-initial-value problems for different coefficients a_1 and a_2 . Define the difference variables and constant as

$$w_i = u_i - v_i,\ \theta = S - T,\ \phi = \varphi - C,\ \Pi = \pi_1 - \pi_2,\ a = a_1 - a_2. \tag{78}$$

Then (w_i, θ, ϕ, Π) solves the boundary-initial-value problem

$$w_i + [a_1|\mathbf{u}|u_i - a_2|\mathbf{v}|v_i] + b[|\mathbf{u}|^2 u_i - |\mathbf{v}|^2 v_i] = -\Pi_{,i} + g_i\theta + h_i(S + T)\theta + L_i\theta(S^2 + ST + T^2) + I_i\phi, \tag{79}$$

$$w_{i,i} = 0, \tag{80}$$

$$\theta_{,t} + w_i T_{,i} + u_i \theta_{,i} = \Delta\theta, \tag{81}$$

$$\phi_{,t} + w_i C_{,i} + u_i \phi_{,i} = \Delta\phi. \tag{82}$$

To establish continuous dependence on a we rearrange the a_1 and a_2 terms as

$$a_1|\mathbf{u}|u_i - a_2|\mathbf{v}|v_i = \frac{a}{2}[|\mathbf{u}|u_i + |\mathbf{v}|v_i] + \tilde{a}[|\mathbf{u}|u_i - |\mathbf{v}|v_i], \tag{83}$$

where $\tilde{a} = \frac{a_1 + a_2}{2}$. Hence, from (83) and (79) we arrive at

$$\begin{aligned} &w_i + \frac{a}{2}[|\mathbf{u}|u_i + |\mathbf{v}|v_i] + \tilde{a}[|\mathbf{u}|u_i - |\mathbf{v}|v_i] + b[|\mathbf{u}|^2 u_i - |\mathbf{v}|^2 v_i] \\ &= -\Pi_{,i} + g_i\theta + h_i(S + T)\theta + L_i\theta(S^2 + ST + T^2) + I_i\phi. \end{aligned} \tag{84}$$

Furthermore, from (84), the Cauchy-Schwarz inequality and arithmetic-geometric mean inequality we may derive

$$\begin{aligned} &\| \mathbf{w} \|^2 + \frac{a}{2}\int_\Omega [|\mathbf{u}|u_i + |\mathbf{v}|v_i] w_i d\mathbf{x} + \tilde{a}\int_\Omega [|\mathbf{u}|u_i - |\mathbf{v}|v_i] w_i d\mathbf{x} \\ &+ b\int_\Omega [|\mathbf{u}|^2 u_i - |\mathbf{v}|^2 v_i] w_i d\mathbf{x} \\ &= (g_i\theta, w_i) + (h_i(S + T)\theta, w_i) + (L_i\theta(S^2 + ST + T^2), w_i) + (I_i\phi, w_i) \\ &\leq \frac{2}{3} \| w_i \|^2 + \frac{3}{2}(1 + X_m^2 + Y_m^2)[\| \theta \|^2 + \| \phi \|^2], \end{aligned} \tag{85}$$

since, from (69) we have

$$\tilde{a}\int_\Omega [|\mathbf{u}|u_i - |\mathbf{v}|v_i] w_i d\mathbf{x} = \frac{\tilde{a}}{2}\int_\Omega [|\mathbf{u}| + |\mathbf{v}|] w_i w_i d\mathbf{x} + \frac{\tilde{a}}{2}\int_\Omega (|\mathbf{u}|^2 - |\mathbf{v}|)^2 [|\mathbf{u}| + |\mathbf{v}|] d\mathbf{x}, \tag{86}$$

and from (70), we obtain

$$b\int_\Omega [|\mathbf{u}|^2 u_i - |\mathbf{v}|^2 v_i] w_i d\mathbf{x} \geq \frac{b}{2}\int_\Omega [|\mathbf{u}|^2 + |\mathbf{v}|^2] w_i w_i d\mathbf{x} + \frac{b}{2}\int_\Omega [|\mathbf{u}|^2 - |\mathbf{v}|^2]^2 d\mathbf{x}, \tag{87}$$

then, we use (86) and (87) in (85), to find

$$\frac{1}{3}\parallel \mathbf{w}\parallel^2+\frac{\tilde{a}}{2}\int_\Omega(|\mathbf{u}|+|\mathbf{v}|)w_iw_id\mathbf{x}+\frac{\tilde{a}}{2}\int_\Omega(|\mathbf{u}|-|\mathbf{v}|)^2(|\mathbf{u}|+|\mathbf{v}|)d\mathbf{x}$$
$$+\frac{b}{2}\int_\Omega[|\mathbf{u}|^2+|\mathbf{v}|^2]w_iw_id\mathbf{x}+\frac{b}{2}\int_\Omega[|\mathbf{u}|^2-|\mathbf{v}|^2]^2d\mathbf{x} \quad (88)$$
$$\leq\frac{3}{2}(1+X_m^2+Y_m^2)[\parallel\theta\parallel^2+\parallel\phi\parallel^2]-\frac{a}{2}\int_\Omega[|\mathbf{u}|u_iw_i+|\mathbf{v}|v_iw_i]d\mathbf{x}.$$

From this point the proof splits into two parts depending on whether $b>0$ or $b=0$. Suppose $b=0$, we then use the Cauchy-Schwarz and arithmetic-geometric mean inequalities to see that

$$\frac{a}{2}\left|\int_\Omega[|\mathbf{u}|u_iw_i+|\mathbf{v}|v_iw_i]d\mathbf{x}\right|\leq\frac{a^2}{8b^2}\int_\Omega(u_iu_i+v_iv_i)d\mathbf{x}+\frac{b}{2}\int_\Omega[|\mathbf{u}|^2+|\mathbf{v}|^2]w_iw_id\mathbf{x}. \quad (89)$$

If we use (89) in (88), we find

$$\frac{1}{3}\parallel\mathbf{w}\parallel^2\leq\frac{3}{2}(1+X_m^2+Y_m^2)[\parallel\theta\parallel^2+\parallel\phi\parallel^2]+\frac{a^2}{8b^2}\int_\Omega[|\mathbf{u}|^2+|\mathbf{v}|^2]d\mathbf{x}$$
$$=\mathbf{K}_1[\parallel\theta\parallel^2+\parallel\phi\parallel^2]+\frac{a^2}{8b^2}\parallel\mathbf{u}\parallel^2+\frac{a^2}{8b^2}\parallel\mathbf{v}\parallel^2, \quad (90)$$

where

$$\mathbf{K}_1=\frac{3}{2}(1+X_m^2+Y_m^2).$$

Now, we will find bounds for $\parallel\mathbf{u}\parallel^2$ and $\parallel\mathbf{v}\parallel^2$. By multiplying (1) by v_i and and integrating over Ω, we then obtain

$$\parallel\mathbf{v}\parallel^2+a\parallel\mathbf{v}\parallel_3^3+b\parallel\mathbf{v}\parallel_4^4\leq(T,v_i)+(T^2,v_i)+(T^3,v_i)+(C,v_i)$$
$$\leq\frac{1}{6}\parallel\mathbf{v}\parallel^2+\frac{3}{2}\parallel T\parallel^2+\frac{1}{6}\parallel\mathbf{v}\parallel^2+\frac{3}{2}\parallel T\parallel_4^4+\frac{1}{6}\parallel\mathbf{v}\parallel^2+\frac{3}{2}\parallel T\parallel_6^6+\frac{1}{6}\parallel\mathbf{v}\parallel^2+\frac{3}{2}\parallel C\parallel^2, \quad (91)$$

which lead to the following bound

$$\frac{1}{3}\parallel\mathbf{v}\parallel^2+a\parallel\mathbf{v}\parallel_3^3+b\parallel\mathbf{v}\parallel_4^4\leq\mathbf{K}_2, \quad (92)$$

where $\mathbf{K}_2=\frac{3}{2}\parallel T\parallel_4^4+\frac{1}{6}\parallel\mathbf{v}\parallel^2+\frac{3}{2}\parallel T\parallel_6^6+\frac{1}{6}\parallel\mathbf{v}\parallel^2+\frac{3}{2}\parallel C\parallel^2$. Thus, we have the following estimates

$$\parallel\mathbf{v}\parallel^2\leq3\mathbf{K}_2=\mathbf{K}_3,\quad\parallel\mathbf{v}\parallel_3^3\leq\frac{\mathbf{K}_2}{a}=\mathbf{K}_4,\quad\parallel\mathbf{v}\parallel_4^4\leq\frac{\mathbf{K}_2}{b}=\mathbf{K}_5, \quad (93)$$

and similarly we have

$$\parallel\mathbf{u}\parallel^2\leq3\mathbf{K}_2=\mathbf{K}_3,\quad\parallel\mathbf{u}\parallel_3^3\leq\frac{\mathbf{K}_2}{a}=\mathbf{K}_4,\quad\parallel\mathbf{u}\parallel_4^4\leq\frac{\mathbf{K}_2}{b}=\mathbf{K}_5. \quad (94)$$

Then, from (90), (93) and (94) we find

$$\frac{1}{3}\parallel\mathbf{w}\parallel^2\leq\mathbf{K_1}[\parallel\theta\parallel^2+\parallel\phi\parallel^2]+a^2\mathbf{K}_6, \quad (95)$$

where $\mathbf{K}_6=\frac{\mathbf{K}_3}{4b^2}$.

Suppose now that $b=0$, we then have from (88) that

$$\frac{1}{3}\parallel\mathbf{w}\parallel^2+\frac{\tilde{a}}{2}\int_\Omega(|\mathbf{u}|+|\mathbf{v}|)w_iw_id\mathbf{x}+\frac{\tilde{a}}{2}\int_\Omega(|\mathbf{u}|-|\mathbf{v}|)^2(|\mathbf{u}|+|\mathbf{v}|)d\mathbf{x}$$
$$\leq\mathbf{K}_1[\parallel\theta\parallel^2+\parallel\phi\parallel^2]-\frac{a}{2}\int_\Omega[|\mathbf{u}|u_iw_i+|\mathbf{v}|v_iw_i]d\mathbf{x}. \quad (96)$$

We then use the Cauchy-Schwarz and arithmetic-geometric mean inequalities as follows:

$$\frac{a}{2}\left|\int_\Omega[|\mathbf{u}|u_iw_i+|\mathbf{v}|v_iw_i]d\mathbf{x}\right|\leq\frac{a^2}{8\tilde{a}}\int_\Omega(|\mathbf{u}|^3+|\mathbf{v}|^3)d\mathbf{x}+\frac{\tilde{a}}{2}\int_\Omega(|\mathbf{u}|+|\mathbf{v}|)w_iw_id\mathbf{x}. \quad (97)$$

We now employ (97) in (96) to find, after discard the positive term, that

$$\frac{1}{3}\parallel\mathbf{w}\parallel^2\leq\mathbf{K}_1[\parallel\theta\parallel^2+\parallel\phi\parallel^2]+\frac{a^2}{8\tilde{a}}\int_\Omega(|\mathbf{u}|^3+|\mathbf{v}|^3)d\mathbf{x}. \quad (98)$$

By inserting the estimates (93) and (94) in (98), we find

$$\frac{1}{3}\parallel\mathbf{w}\parallel^2\leq\mathbf{K}_1[\parallel\theta\parallel^2+\parallel\phi\parallel^2]+a^2\mathbf{K}_7,$$

where $\mathbf{K}_7 = \frac{\mathbf{K}_4}{4\tilde{a}}$. Thus, we have for both cases $b > 0$ and $b = 0$, that

$$\frac{1}{3} \parallel \mathbf{w} \parallel^2 \leq \mathbf{K}_1[\parallel \theta \parallel^2 + \parallel \phi \parallel^2] + a^2\mathbf{K}_8, \quad (99)$$

where $\mathbf{K}_8 = \max\{\mathbf{K}_6, \mathbf{K}_7\}$. Thus, we find

$$\parallel \mathbf{w} \parallel^2 \leq \mathbf{K}_1[\parallel \theta \parallel^2 + \parallel \phi \parallel^2] + 3a^2\mathbf{K}_8. \quad (100)$$

Also, it is easy to show that

$$\begin{aligned} &\frac{d}{dt} \parallel \theta \parallel^2 \leq \frac{T_m^2}{2} \parallel \mathbf{w} \parallel^2, \\ &\frac{d}{dt} \parallel \phi \parallel^2 \leq \frac{C_m^2}{2} \parallel \mathbf{w} \parallel^2, \end{aligned} \quad (101)$$

which lead to

$$\frac{d}{dt}[\parallel \theta \parallel^2 + \parallel \phi \parallel^2] \leq \frac{1}{2}(T_m^2 + C_m^2) \parallel \mathbf{w} \parallel^2. \quad (102)$$

Use of (100) in (102) and then integration the resulted equation allows us to see that

$$\parallel \theta \parallel^2 + \parallel \phi \parallel^2 \leq a^2\mathbf{K}_{10}[\frac{\exp(\mathbf{K}_9 t) - 1}{\mathbf{K}_9}], \quad (103)$$

where

$$\mathbf{K}_9 = \frac{3}{2}(T_m^2 + C_m^2)\mathbf{K}_1,$$

and

$$\mathbf{K}_{10} = \frac{3}{2}(T_m^2 + C_m^2)\mathbf{K}_8,$$

Inequality (103) establishes continuous dependence of θ and ϕ in L^2 on the Forchheimer coefficient a. A similar continuous dependence estimate for w_i may then be established with the help of (100) and (103).

6. Continuous dependence on b

We commence with a study of continuous dependence on the coefficient b. Therefore, let (u_i, S, φ, π_1) and (v_i, T, C, π_2) be solutions to Equations (1)-(4) for the same boundary and initial conditions, but for different coefficient b_1 and b_2 . Define the difference variables u_i, θ ,ϕ and Π and constant b by

$$w_i = u_i - v_i, \quad \theta = S - T, \quad \phi = E - c, \quad \Pi = \pi_1 - \pi_2, \quad b = b_1 - b_2, \quad (104)$$

and then we find that (w_i, θ, ϕ, π) satisfy the boundary-initial value problem

$$\begin{aligned} &w_i + a[|\mathbf{u}|u_i - |\mathbf{v}|v_i] + [b_1|\mathbf{u}|^2u_i - b_2|\mathbf{v}|^2v_i] = -\Pi_{,i} + g_i\theta + h_i(S+T)\theta \\ &+L_i\theta(S^2 + ST + T^2) + I_i\phi, \\ &w_{i,i} = 0, \\ &\theta_{,t} + w_iT_{,i} + u_i\theta_{,i} = \Delta\theta, \\ &\phi_{,t} + w_iC_{,i} + u_i\phi_{,i} = \Delta\theta. \end{aligned} \quad (105)$$

We write

$$b_1|\mathbf{u}|^2u_i - b_2|\mathbf{v}|^2v_i = \frac{b}{2}[|\mathbf{u}|^2u_i + |\mathbf{v}|^2v_i] + \tilde{b}[|\mathbf{u}|u_i - |\mathbf{v}|v_i], \quad (106)$$

where $\tilde{b} = ((b_1 + b_2)/2$. Multiply $(105)_1$ by w_i and integrate over Ω, and then use (106) and (69) in the resulting equation from $(105)_1$ to find

$$\begin{aligned} &\parallel \mathbf{w} \parallel^2 + \frac{a}{2}\int_\Omega (|\mathbf{u}| + |\mathbf{v}|)w_iw_i d\mathbf{x} + \frac{a}{2}\int_\Omega (|\mathbf{u}|^2 - |\mathbf{v}|^2)(|\mathbf{u}| + |\mathbf{v}|)d\mathbf{x} \\ &+\frac{b}{2}\int_\Omega [|\mathbf{u}|^2u_iw_i + |\mathbf{v}|^2v_iw_i]d\mathbf{x} + \tilde{b}\int_\Omega [|\mathbf{u}|^2u_i - |\mathbf{v}|^2v_i]w_i d\mathbf{x} \\ &\leq \frac{2}{3} \parallel \mathbf{w} \parallel^2 + \frac{3}{2}(1 + X_m^2 + Y_m^2)[\parallel \theta \parallel^2 + \parallel \phi \parallel^2]. \end{aligned} \quad (107)$$

We use (87) for $\tilde{b}$ in (107) then discard the a terms from the resulting equation to find

$$\frac{1}{3}\parallel \mathbf{w} \parallel^2 \leq \frac{3}{2}(1+X_m^2+Y_m^2)[\parallel \theta \parallel^2 + \parallel \phi \parallel^2] - \frac{b}{2}\int_\Omega [|\mathbf{u}|u_i w_i + |\mathbf{v}|v_i w_i] d\mathbf{x} - \frac{\tilde{b}}{2}\int_\Omega [|\mathbf{u}|^2 + |\mathbf{v}|^2] w_i w_i d\mathbf{x}. \tag{108}$$

Next, the Cauchy-Schwarz and arithmetic-geometric mean inequalities are employed to see that

$$\frac{b}{2}\left|\int_\Omega [|\mathbf{u}|^2 u_i w_i + |\mathbf{v}|^2 v_i w_i] d\mathbf{x}\right| \leq \frac{b^2}{8\tilde{b}}\int_\Omega (|\mathbf{u}|^4 + |\mathbf{v}|^4) d\mathbf{x} + \frac{\tilde{b}}{2}\int_\Omega [|\mathbf{u}|^2 + |\mathbf{v}|^2] w_i w_i d\mathbf{x}. \tag{109}$$

Substituting (109) in (108) to obtain

$$\frac{1}{3}\parallel \mathbf{w} \parallel^2 \leq \frac{3}{2}(1+X_m^2+Y_m^2)[\parallel \theta \parallel^2 + \parallel \phi \parallel^2] + \frac{b^2}{8\tilde{b}}\int_\Omega (|\mathbf{u}|^4 + |\mathbf{v}|^4) d\mathbf{x} \leq \mathbf{K_1}[\parallel \theta \parallel^2 + \parallel \phi \parallel^2] + \frac{b^2}{8\tilde{b}}\int_\Omega (|\mathbf{u}|^4 + |\mathbf{v}|^4) d\mathbf{x}. \tag{110}$$

We next employ the estimates (93) and (94) in (110) to arrive at

$$\frac{1}{3}\parallel \mathbf{w} \parallel^2 \leq 3\mathbf{K}_1[\parallel \theta \parallel^2 + \parallel \phi \parallel^2] + 3b^2\mathbf{K}_{10}, \tag{111}$$

where $\mathbf{K}_{10} = \frac{1}{4\tilde{b}}\mathbf{K}_5$. Next, by using the above bound in (102), and after integration we find

$$\parallel \theta \parallel^2 + \parallel \phi \parallel^2 \leq b^2\mathbf{K}_{12}\left[\frac{\exp(\mathbf{K}_{11}t)-1}{\mathbf{K}_{11}}\right], \tag{112}$$

where $\mathbf{K}_{11} = \frac{3}{2}(T_m^2 + C_m^2)\mathbf{K}_1$ and $\mathbf{K}_{12} = \frac{3}{2}(T_m^2 + C_m^2)\mathbf{K}_{10}$. The continuous dependence for w_i on the Forchheimer coefficient b follows directly from (111) and (112).

7. Conclusions

The equations for double diffusive convection in a porous medium of Forchheimer's law are analysed when the density of fluid depends on temperature and concentration as a cubic and linear function. The question of continuous dependence of the solution on the heat source and Forchheimer coefficients is one which belongs to the general class of structural stability problems. Structural stability (or continuous dependence on the model itself) is one of major importance and it may be argued that it is as or more important than the widely accepted notion of stability as continuous dependence upon the initial data. Therefore, we establish rigorous a priori bounds with coefficients which depend only on boundary data and initial data and which demonstrate continuous dependence of the solution on changes in Q, a and b.

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الاستمرارية المعتمدة للحل لمسألة الحمل المزدوج الانتشار في وسط مسامي عندما تكون الكثافة معتمدة على درجة الحرارة

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المستخلص

تمت دراسة الاستقرارية الهيكلية لمسالة الحمل المزدوج الانتشار في وسط مسامي من نوع فورشايمر ، عندما تعتمد كثافة السائل والتركيز على درجة الحرارة كدوال تكعيبية وخطية ، على التوالي. وقد تبين أن لهذه المشكلة ، مع الحمل الحراري فقط وفي طبقة لانهائية، فان الرنين بين الطبقات الداخلية للمائع يمكن أن يحدث. المشكلة الرئيسية هي مصدر الحرارة الداخلي وقد يؤدي وجوده إلى حصول الحمل المتذبذب في حالة عدم الاستقرار الخطي مسببا حالة الرنين. لذا، في هذه الدراسة ، تم تحليل مشكلة الاستقرار الهيكلي للمسألة من حيث دراسة الاعتماد المستمر للحل على مصدر الحرارة نفسه لنموذج الحمل المزدوج الانتشار في وسط مسامي من نوع فورشايمر. علاوة على ذلك ، تم إظهار الاعتماد المستمر للحل على التغييرات في معاملات فورشايمر.