On a general class of hybrid weakly Picard mappings

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Abstract

The concept of weak contraction from the case of multi-valued maps is extended to hybrid pair of single valued maps and multi-valued maps and then corresponding convergence theorems for Picard iteration associated to a hybrid weak contraction are obtained our work extend and improve recent results due to Daffer et al .(1995) , Berinde (2004) and Berinde et al.(2007) .

Keywords: Metric space; Hausdorff distance; Multi-hybrid mapping; Fixed point& coincidence point; Weak contraction; Weakly Picard operator

1-Introduction

Hybrid fixed point theory for non-linear single-valued and multi-valued maps is a new development in the domain of contraction-type multi-valued theory.

The study of such maps was initiated during 1980-83 by Mukherjee [10], Naimpaly et al. [11], Hadzic [12], Rhoades et al. [13] and Singh et. al [17-19, 21-22] .

Hybrid fixed point theory has potential application in functional inclusions, optimization theory ,Fractal graphics and discrete dynamic for set-valued operators.

In this paper, we obtain a few generalizations and extension of theorem 2.1 and other similar results (c .f [1] and [2]) where in continuity of maps is not needed ,completeness of the space is relaxed to the completeness of a subspace and the commutatively requirement is tight and minimal.

2-Preliminaries

be a metric space we shall follow the following notations and definitions (X,d) Let

$X\}$. is a non-empty subset of $P(X)=\{A;A$

$X\}$. is a non-empty closed subset of $CL(X)=\{A;A$

$X\}$. is a non-empty closed and bounded subset of $CB(X)=\{A;A$

$H(A,B)=max\{sup\{d(a,B);a\in A\},sup\{d(a,B);b\in B\}\}$. We consider

and X of B and A will denote the ordinary distance between subsets $d(A,B)$ Throughout,

. further , $\{a\}$ is the singleton A when $d(A,B)$ will stand for $d(a,B)$

and $f:X\to X$ the collection of coincidence points of $C(f,T)=\{z;fz\in Tz\}$

. Consider the T the collection of fixed points of $T=\{x;Tx=x\}$ .and Fix $T:X\to CL(X)$

the Hausdorff metric $(CL(X),H)$ $f:X\to X$ and $T;X\to CL(X)$ following conditions for

d. space induced by

$$(2.1)\ H(Tx,Ty)\le q\,d(fx,fy)$$

$$(2.2)\ H(Tx,Ty)\le q\max\{d(fx,fy),d(fx,Tx),d(fy,Ty),1/2[d(fx,Ty)+d(fy,Tx)]\}$$

$o<q<1$ where $x, y \in X$, For all

(2.3) $H(Tx, Ty) \leq \theta d(fx,fy) + L\min\{d(fx,Tx),d(fy,Ty),d(fx,Ty), d(fy,Tx)\}$

(2.4) $H(Tx,Ty) \leq \alpha\ (d(fx, fy)d(fx, fy)+L\min\{d(fx,Tx),d(fy,Ty),d(fx,Ty), d(fy,Tx)\}$

for every $\lim\sup_{r\to t^+} \alpha(r) < 1$ satisfying $\alpha:[0.\infty] \to [0,1)$ and $x, y \in X$ For all $t \in [0, \infty)$.

Notice that (2.1) is induced in (2.2)

These condition when T is a single-valued map and $f = Id$ (the identity map) on X , are compared in [14]. Further, (2.3) with $f = Id$ (the identity map) is called multi-valued weak contraction [1].

-weak (θ, L) The condition (2.4) with f=identity is called a generalized multi-valued contraction and (2.3) is induced in (2.4)

the Hausdorff metric space $(CL(X),H)$ be a metric space and X **Theorem 2.1** [20]. Let . Let X is the collection of all nonempty closed subset of $CL(X)$, where d induced by and$T(X) \subseteq f(X)$ be such that $f: X \to X$ and $T:X \to CL(X)$

$H(Tx, Ty) \leq q\max\{d(fx,fy),d(fx,Tx),d(fy,Ty),[d(fx,Ty)+d(fy,Tx)]/2\}$

T , then X is a complete subspace of $f(X)$. If $0 \leq q < 1$, where $x,y \in X$ For every $fz \in Tz$ such that $z \in X$ have a coincidence, i.e., there exists a point f and

We remark that under the conditions of theorem 2.1, T and f need not have a common fixed point even if f and T are commuting and continuous as the following example.

$Tx = [1+x, \infty)$ be endowed with the usual metric , $X = [0, \infty)$ **Example 2.1** ([11]). Let . Further, $T(X) \subseteq f(X)=X$ .Then $fx=2x$ and

$H(Tx,Ty) \leq q\, d(fx, fy), \quad x, y \in X, 1/2 \leq q < 1.$

satisfy all the requirements of theorem2.1, since (2.2) lies (2.1). f and T Thus

Evidently, T and f have a coincidence point $z\ (\geq 1)$, i.e., $fz \in Tz$ for any $z \geq 1$. Notice that T and f have no common fixed points .To prove the main results in this paper, we shall need the following lemma and definitions.

. Then, $q > 1$ and $A,B \subseteq X$ be a metric space. Let (X,d) **Lemma 2.1**,[4]. Let $d(a,b) \leq q\,H(A,B)$. such that $b \in B$, There exists $a \in A$ For every

$T:X \to CB(X)$. **Definition 2.1** [7]. Let

$ffx \in Tfx$. if $x \in X$ is said to be T-weakly commuting at $f : X \to X$ The map

Definition 2.2 [1]. Let (X,d) be a metric space and $T:X \to P(X)$ be a multi-valued operator . T is said to be weakly Picard operator iff for each $x \in X$ and any $y \in Tx$,there exists a sequence $\{x_n\}$ such that

$x_o = x, x_1 = y$ (i)

$n = 0,1,2,....;$ for all $x_{n+1} \in Tx_n$ (ii)

$T.$ is convergent and its limit is a fixed point of $\{x_n\}$ (iii)The sequence

a multi-$T:X \rightarrow CB(X).$ be a complete metric space and (X,d) **Theorem 2.2 ,**[1]. Let weak contraction then.(θ, L)- valued

;$Fix(T) \neq \phi$ (1)

that converges to a x_o at the point T of $\{x_n\}$,There exists an orbit $x_o \in X$ (2) for any ,for which the following estimates hold T of u fixed point

$$d(x_n,u) \leq \frac{h^n}{1-h} d(x_o,x_1), \qquad n=1,2,.....,$$

$$d(x_n,u) \leq \frac{h}{1-h} d(x_{n-1},x_n), \qquad n=1,2,.....,$$

$h<1.$ For certain constant

3-Main Results

We shall introduce the following definition

Definition 3.1

$.f:X \rightarrow X$ and $T:X \rightarrow P(X)$ be a metric space , (X,d) Let

$x \in X$ -weakly Picard operator iff for each (θ,f) is said to be hybrid (T,f) The pair

such that$\{fx_n\}$ there exists a sequence $y=f(x) \in Tx$ and any

$f(x_o)=y_o, f(x_1)=y_1$ (i)

$n=0,1,2,.....;$ for all $fx_{n+1} \in Tx_n$ (ii)

and T that is $fz \in Tz,$, and $z \in X$ for some fz converges to $\{fx_n\}$ (iii)The sequence

$.z$ have a coincidence at f

The next theorem is the main result of this paper; it basically shows that any hybrid weak-contraction is a hybrid weakly Picard operator.

be a metric space.(X,d) **Theorem 3.1**: Let

and (2.3) holds for all $T(X) \subseteq f(X)$ be such that $f:X \rightarrow X$ and $T:X \rightarrow CB(X)$ Let

, then$X$ is a complete subspace of TX or fX if $x,y \in X$

$. \neq \phi \; C(f,T)$ (1)

,and $z \in X$ for some fz converge to $\{fx_n\}$ there exist $x_o \in X$ (2) for any

$.z$ have a coincidence at f and T ,that is. , $fz \in Tz$

For which following estimates hold:

$$(3.1)\; d(fx_n,fz) \leq \frac{h^n}{1-h} d(fx_o,fx_1), \qquad n=1,2,.....,$$

$$d(fx_n,fz) \leq \frac{h}{1-h} d(fx_{n-1},fx_n), \qquad n=1,2,....., \qquad (3.2)$$

For a certain constant $h<1.$

(3) f and T have a common fixed point provided that f is T-weakly commuting at z and $ffz = fz$ for $z \in C(f, T)$.

such $x_1 \in X$,we can find a point $TX \subseteq fX$ since $x_o \in X$ Let $q > 1$. **Proof** : Let
. $y_1 = fx_1 \in Tx_o$ that
. By $C(f, T) \neq \phi$ which implies $fx_1 \in Tx_1$, i.e., $Tx_o = Tx_1$ then $H(Tx_o, Tx_1) = 0$ If
such that $y_2 = fx_2 \in Tx_1$ Lemma (2.1) , there exists

$$d(y_1, y_2) \leq q\, H(Tx_o, Tx_1).$$

$$d(y_1, y_2) \leq q\, [\theta d(fx_o, fx_1) + Ld(fx_1, Tx_o)].$$

$$\leq q\, [\theta d(fx_o, fx_1)]$$

$$\leq q\, \theta d(y_1, y_2),$$

. $fx_2 \in Tx_2$,i.e., $Tx_1 = Tx_2$ then $H(Tx_1, Tx_2) = 0$ If
$H(Tx_1, Tx_2) \neq 0$ Let
such that $y_3 = f(x_3) \in Tx_2$ By Lemma (2.1) , there exist

$$d(y_2, y_3) \leq q\, [\theta d(fx_1, fx_2) + Ld(fx_2, Tx_1)].$$

$$\leq q\, [\theta d(fx_1, fx_2)]$$

$$\leq q\, \theta(y_1, y_2),$$

$q > 1 \quad \ni \quad h = q\,\theta < 1$ We take
satisfying $\{ y_n = fx_n \}$ In this manner, we obtain a sequence

$$(3.3) \quad d(y_n, y_{n+1}) \leq h\, d(y_{n-1}, y_n), \quad n = 1, 2, \ldots\ldots$$

By (3.3) we inductively obtain

$$(3.4) \quad d(y_n, y_{n+1}) \leq h^n\, d(y_o, y_n),$$

and ,respectively ,

$$(3.5) \quad d(y_{n+k}, y_{n+k+1}) \leq h^{k+1}\, d(y_{n-1}, y_n), \quad k \in N, \quad n \geq 1$$

By (3.4) we then obtain

$$(3.6) \quad d(y_n, y_{n+p}) \leq \frac{h^n(1 - h^p)}{1 - h} d(y_o, y_1), \; n, p \in N$$

is a complete metric subspace fX is a Cauchy sequence .since $\{ y_n \}$ $0 < h < 1$, Since
, it follows that X of
u call it fX have a limit in $\{ y_{2n} \}$ and its subsequence $\{ y_n \}$ Therefore,
. $u = fz$. then $z = f^{-1}u$ Let
. u also converges to $\{ y_{2n+1} \}$ Notice that the sub sequence

$$d(Tz, y_{2n+1}) \leq H(Tz, Tx_{2n}).$$

$$\leq \theta\, d(fz, fx_{2n}) + Ld(fx_{2n}, Tz)$$

this obtain $n \to \infty$ Making
$u = fz \in Tz$. ,This implies $n \to \infty$ as $d(Tz, fz) \to 0$
in (3.6), By (3.5) we get similarly to (3.6) $p \to \infty$ To obtain (3.1) we let

$$(3.7)\ d(fx_n, fx_{n+p}) \le \frac{h(1-h^p)}{1-h} d(fx_{n-1}, fx_n), \quad p \in N, n \ge 1$$

in (3.7) we obtain (3.2). $p \to 0$ Making

$ffz \in Tfz$ and $ffz = fz$ Furthermore , by virtue of condition (3) ,we obtain .$u = fz$ have a common fixed point T and f i.e $u = fu \in Tu$ Thus

Remarks:

1- Generally , $C(f, T)$ the coincidence point set of a hybrid weak contraction contains more than one coincidence point ,since in the case of multi-valued weak contraction their fixed point set is not a singleton for more details see [1] and [2]

2- Theorem (3.1) with $f = Id$ (the identity map) is the main result Theorem 3 in [2]

be a metric space.(X,d) **Theorem 3.2**: Let

- weak contraction $(\alpha - L)$. A generalized hybrid $f: X \to X$ and $T: X \to CB(X)$

. Then X is a complete subspace of TX or fX . IF $TX \subseteq fX$ i.e, (2.4) holds and

1- $C(F, T) \ne \phi$.

2- For any $x_o \in X$, The Picard iteration $\{fx_n\}$ defined by $x_o \in X$ and $fx_{n+1} \in Tx_n$, $n = 0, 1, 2, \ldots$ converges to a coincidence point of f and T .

3- f and T have a common fixed point provided that f is T-weakly commuting at z and $ffz = fz = u$, $u \in C(f, T)$.

Proof: Let $x_o \in X$,since $TX \subseteq fX$,we can find a point $x_1 \in X$ such that $fx_1 \in Tx_o$ .select a positive integer n_1 such that

$$\alpha^{n_1}(d(fx_0, fx_1)) \le [1 - \alpha(d(fx_0, fx_1))]\, d(fx_0, fx_1))] \tag{3.8}$$

, using the definition of the Hausdorff metric , so that $fx_2 \in Tx_1$ We may select

$$d(fx_2, fx_1) \le H(Tx_1, Tx_0) + \alpha^{n_1}(d(fx_0, fx_1)).$$

We than have

$$d(fx_2, fx_1) \le \alpha(d(fx_1, fx_0))\, d(fx_1, fx_0) + LD(fx_1, Tx_0)$$
$$+ \alpha^{n_1}(d(fx_0, fx_1)) < d(fx_1, fx_0)$$

so that $n_2 < n_1$ Now choose a positive integer

$$\alpha^{n_2}(d(fx_2, fx_1)) < [1 - \alpha(d(fx_2, fx_1))]\, d(fx_2, fx_1).$$

so that $fx_3 \in Tx_2$, select $Tx_2 \in CB(X),$ since

$$d(fx_3, fx_2) \le H(Tx_2, Tx_1) + \alpha^{n_2}(d(fx_2, fx_1)$$

By
$$\le \alpha(d(fx_2, fx_1))\, d(fx_2, fx_1) + Ld(fx_2, Tx_1) + \alpha^{n_2}(d(fx_2, fx_1))$$
$$< d(fx_2, fx_1).$$

such that n_k Repeating this process, we may select a positive integer

$$\alpha^{n_k}(d(fx_k, fx_{k-1})) < [1-\alpha(d(fx_k, fx_{k-1}))]\, d(fx_k, fx_{k-1}).$$

so that $fx_{k+1} \in Tx_k$ Now select

$$(3.9)\quad d(fx_{k+1}, fx_k) \le H(Tx_k, Tx_{k-1}) + \alpha^{n_k}(d(fx_k, fx_{k-1})).$$

$d(fx_{k+1}, fx_k)) < d(fx_k, fx_{k-1}).$ Then

is a monotone non-increasing sequence of nonnegative $\{d_k = (fx_k, fx_{k-1})\}.$ So that satisfies the following inequality $\{d_n\}$ numbers. By (3.8) we deduce the sequence

$$(3.10)\quad d_{k+1} \le \alpha(d_k) + \alpha^{n_k}(d_k), \quad k = 1,2,3,.....$$

By induction ,from (3.10) and using the relevant part of the proof of Theorem (2.1) [4, p.657] call it fX has a limit in $\{fx_{2k}\}$ is a Cauchy sequence and its subsequence $\{fx_k\}$, we get $.fz = u$ .Then $z \in f^{-1}u$ .Let u

$u.$ also converges to $\{fx_{2k+1}\}$ Notice that the subsequence

$$d(Tz, fx_{2k+1}) \le H(Tz, Tx_{2k})$$
$$\le \alpha(d(fz, fx_{2k}))d(fz, fx_{2k}) + Ld(fx_{2k}, Tz)$$

. we obtain $k \to 0$ Making

$d(Tz, fz) \to 0$ as $n \to \infty$

This implies $u = fz \in Tz$

We can use the same argument in theorem (3.1) for the rest of the proof .

Corollary 3.1 [1]**:** Let (X,d) be a complete metric space and $T: X \to CB(X)$ a generalized multi-valued (α, L)- weak contraction, i.e., a mapping for which there exists a function $\alpha: [0,\infty) \to [0,1)$ satisfying $\limsup_{r\to t^+} \alpha(r) < 1$, for every $t \in [0,\infty),$ such that $H(Tx, Ty) \le \alpha d(x,y)(d(x,y)) + LD(y, Tx),$ for all $x, y \in X.$
has at least one fixed point.T Then

$T: X \to CB(X)$ be a metric space (X,d) **Corollary 3.2**[1] : Let a monotone $\alpha: [0,\infty] \to [0,1]$ A generalized multi-valued - weak contraction , with .If (2.4) is satisfied $t \in [0,\infty)$,for each $0 \le \alpha(t) < 1,$ increasing function satisfying has at least one fixed point. T , then $f = Id$ with

Remarks :

1- Theorem 3.2 generalize the main results of Berinde and Berinde [1] also generalize Theorem 1.2 in [5] and corollary 2.2 in [5] to more general contractive condition (2.4) .
2- If $L = 0$ in the condition (2.3) and $f = Id$ (the identity map)then by theorem (3.2) we obtain theorem 2.1 in [4] and other related results in [8] and [9] .
3- In the case of single –valued mappings the results in [3] ,[4] and [24] which are obtained as particular cases of theorem(3.1) and (3.2) ,these two theorems also extend ,partially or totally many fixed point theorem for multi-valued and hybrid mapping in [6-15-16] and [21-23].

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