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Research Article

New Branch of Intuitionistic Fuzzification in Algebras with Their Applications

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The intuitionistic fuzzification in $\rho-$algebras about the concepts of ideals and subalgebras given with several related characterizations is considered. Some new concepts like intuitionistic fuzzy $\rho-$ideal ($IF\rho i$), intuitionistic fuzzy $\rho-$subalgebra ($IF\rho s$), $\rho-$homomorphism, and intuitionistic fuzzy $\overline{\rho}-$ideal ($IF\overline{\rho}i$) are introduced and some of their descriptions are given in this work. Further, we show some applications on the family of all intuitionistic fuzzy $\rho-$subalgebras $IF\rho_S(\mathfrak{R})$ in $\rho-$algebra like the binary relations $\underset{\mu}{\approx}$, $\underset{\nu}{\approx}$ and Γ_r on $IF\rho_S(\mathfrak{R})$. Also, their equivalence classes are given and studied.

1. Introduction

The fuzzy set (FS) as suggested by Zadeh [1] in 1965 is a regulation to vagueness and encounter uncertainty. A FS maps each element of the universe of discourse to the interval $[0, 1]$. After the introduction of fuzzy sets theory by him, many mathematicians were conducted on the generalizations of the this concept and studied in the groups, algebras, and soft spaces (see [2–5]). By including a fuzzy set the degree of nonmembership, Atanassov [6] in 1986 suggested the intuitionistic fuzzy set (IFS), which seems more precise for provides opportunities and uncertainty quantification to accurately model a problem based on existing knowledge and monitoring. Also, this notion is discussed in different fields (see [7–11])

$BCK-$algebra, class of algebra of logic, was investigated by Imai and Iseki [12]. After that, the notion of $d-$algebras was investigated by Neggers and Kim [13]. In 2017, the concepts of $\rho-$algebra, $\overline{\rho}-$ideal, $\rho-$ideal, $\rho-$subalgebra, and permutation topological $\rho-$algebra were first proposed by Mahmood and Abud Alradha [14]. Next, they showed the notion of the soft $\rho-$algebra and soft edge $\rho-$algebra [15].

In the present work, the notions of intuitionistic fuzzy $\rho-$ideal ($IF\rho i$), intuitionistic fuzzy $\rho-$subalgebra ($IF\rho s$), $\rho-$homomorphism, and intuitionistic fuzzy $\overline{\rho}-$ideal ($IF\overline{\rho}i$) are introduced. Further, we show some applications on the family of all intuitionistic fuzzy $\rho-$subalgebras $IF\rho_S(\mathfrak{R})$ in $\rho-$algebra like the binary relations $\underset{\mu}{\approx}$, $\underset{\nu}{\approx}$ and Γ_r on $IF\rho_S(\mathfrak{R})$. Also, their equivalence classes are given and studied.

2. Preliminaries and Notations

We will recall basic definitions and results to obtain properties developed in this work.

Definition 1 (see [16]). An intuitionistic fuzzy set α (IFS, in short) over the universe $\mathfrak{R}$ is defined by $\alpha = \{\prec a, \mu_\alpha(a), \nu_\alpha(a) \succ \mid a \in \mathfrak{R}\}$, where $\mu_\alpha(a)\colon \mathfrak{R} \longrightarrow [0; 1]$, $\nu_\alpha(a)\colon \mathfrak{R} \longrightarrow [0; 1]$ with $0 \le \mu_\alpha(a) + \nu_\alpha(a) \le 1$, $\forall a \in \mathfrak{R}$. $\mu_\alpha(a)$ and $\nu_\alpha(a)$ are real numbers and their values represent the degree of membership and nonmembership of a to α, respectively.

Definition 2 (see [6]). The IF whole and empty sets of $\mathfrak{R}$ are defined by $\overline{1} = \{\prec a, (1, 0) \succ \mid a \in \mathfrak{R}\}$ and $\overline{0} = \{\prec a, (0, 1) \succ \mid a \in \mathfrak{R}\}$, respectively.

2.1. Basic Relations and Operations on Intuitionistic Fuzzy Sets [7]. Assume $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ\mid a \in \mathfrak{R}\}$ and $\beta = \{\prec a, (\mu_\beta(a), \nu_\beta(a)) \succ\mid a \in \mathfrak{R}\}$ are two IF sets of $\mathfrak{R}$. We deduced the following relations:

(1) [inclusion] $\alpha \subseteq \beta$ iff $\mu_\alpha(a) \le \mu_\beta(a)$ and $\nu_\alpha(a) \ge \nu_\beta(a)$, $\forall a \in \mathfrak{R}$,

(2) [equality] $\alpha = \beta$ iff $\alpha \subseteq \beta$ and $\beta \subseteq \alpha$,

(3) [intersection] $\alpha \tilde{\cap} \beta = \{(a, \min\{\mu_\alpha(a), \mu_\beta(a)\}, \max\{\nu_\alpha(a), \nu_\beta(a)\}) : a \in \mathfrak{R}\}$,

(4) [union] $\alpha \tilde{\cup} \beta = \{(a, \max\{\mu_\alpha(a), \mu_\beta(a)\}, \min\{\nu_\alpha(a), \nu_\beta(a)\}) : a \in \mathfrak{R}\}$,

(5) [complement] $\alpha^c = \{(a, \nu_\alpha(a), \mu_\alpha(a)), a \in \mathfrak{R}\}$.

Definition 3 (see [14]). We say $(\mathfrak{R}, \bullet, 0)$ is $\rho-$algebra if $(\bullet)$ is a binary operation on $\mathfrak{R}$ with a constant $0 \in \mathfrak{R}$ and such that

(1) $a \bullet a = 0$,

(2) $0 \bullet a = 0$,

(3) $a \bullet b = 0 = b \bullet a$ imply that $a = b$,

(4) For all $a \ne b \in \mathfrak{R} - \{0\}$ imply that $a \bullet b = b \bullet a \ne 0$.

Definition 4 (see [14]). Assume $(\mathfrak{R}, \bullet, 0)$ is a $\rho-$algebra and $\phi \ne K \subseteq \mathfrak{R}$. We say K is a $\rho-$subalgebra of $\mathfrak{R}$ if $a \bullet b \in K$, $\forall a, b \in K$.

Definition 5 (see [14]). Assume $(\mathfrak{R}, \bullet, 0)$ is $\rho-$algebra and $\phi \ne K \subseteq \mathfrak{R}$. We say K is $\rho-$ideal of $\mathfrak{R}$ if

(1) $a, b \in K$ imply $a \bullet b \in K$,

(2) $a \bullet b \in K$ and $b \in K$ imply $a \in K$, $\forall a, b \in \mathfrak{R}$.

Definition 6 (see [14]). Assume $(\mathfrak{R}, \bullet, 0)$ is a $\rho-$algebra and K subset of $\mathfrak{R}$. We say K is a $\overline{\rho}-$ideal of $\mathfrak{R}$ if

(1) $0 \in K$,

(2) $a \in K$ and $b \in \mathfrak{R} \longrightarrow a \bullet b \in K$, $\forall a, b \in \mathfrak{R}$.

Definition 7 (see [11]). Assume that $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ\mid a \in \mathfrak{R}\}$ is an IFS in $\mathfrak{R}$ and $r \in [0, 1]$. The set $W(\mu_\alpha, r) = \{a \in \mathfrak{R} \mid \mu_\alpha(a) \ge r\}$ (resp., $L(\nu_\alpha, r) = \{a \in \mathfrak{R} \mid \nu_\alpha(a) \le r\}$) is said to be *μ−level r−cut* (resp., *ν−level r−cut*) of α.

3. Intuitionistic Fuzzy $\rho-$Subalgebras in $\rho-$Algebras

In this section, we introduce some new concepts, such as $(IF\rho s)$, $(IF\rho i)$, $(IF\overline{\rho}i)$, and $\rho-$homomorphism which are introduced and discussed. Further, some binary relations $\underset{\mu}{\approx}$, $\underset{\nu}{\approx}$ and Γ_r on $IF\rho_S(\mathfrak{R})$ are given, and some basic properties are shown.

Definition 8. Assume $(\mathfrak{R}, \bullet, 0)$ is a $\rho-$algebra and $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ\mid a \in \mathfrak{R}\}$ is IFS of $\mathfrak{R}$. We say α is an $(IF\rho s)$ of $\mathfrak{R}$ if $\mu_\alpha(a \bullet b) \ge \min\{\mu_\alpha(a), \mu_\alpha(b)\}$ and $\nu_\alpha(a \bullet b) \le \max\{\nu_\alpha(a), \nu_\alpha(b)\}$, $\forall a, b \in \mathfrak{R}$.

Table 1

$*$	0	ω	∂	ℓ
0	0	0	0	0
ω	ω	0	∂	∂
∂	ω	∂	0	ω
ℓ	ℓ	∂	ω	0

Example 9. Let $\mathfrak{R} = \{0, \omega, \partial, \ell\}$ be $\rho-$algebra with Table 1.

Then, $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ\mid a \in \mathfrak{R}\} = \{(0, 0.9, 0.1), (\omega, 0.4, 0.3), (\partial, 0.7, 0.3), (\ell, 0.4, 0.2)\}$ is an $(IF\rho s)$ of $\mathfrak{R}$.

Definition 10. Assume $(\mathfrak{R}, \bullet, 0)$ is $\rho-$algebra and $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ\mid a \in \mathfrak{R}\}$ is IFS of $\mathfrak{R}$. We say α is $(IF\rho i)$ of $\mathfrak{R}$ if

(1) $\mu_\alpha(a \bullet b) \ge \min\{\mu_\alpha(a), \mu_\alpha(b)\}$ and $\nu_\alpha(a \bullet b) \le \max\{\nu_\alpha(a), \nu_\alpha(b)\}$,

(2) $\mu_\alpha(a) \ge \min\{\mu_\alpha(a \bullet b), \mu_\alpha(b)\}$ and $\nu_\alpha(a) \le \max\{\nu_\alpha(a \bullet b), \nu_\alpha(b)\}$, $\forall a, b \in \mathfrak{R}$.

Example 11. Let $(\mathfrak{R}, \bullet, 0)$ be $\rho-$algebra in Example 9 and let $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ\mid a \in \mathfrak{R}\} = \{(0, 0.8, 0.2), (\omega, 0.3, 0.4), (\partial, 0.2, 0.7), (\ell, 0.4, 0.3)\}$ be IFS of $\mathfrak{R}$. Then, α is $(IF\rho i)$ of $\mathfrak{R}$.

Definition 12. Assume $(\mathfrak{R}, \bullet, 0)$ is $\rho-$algebra and $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ\mid a \in \mathfrak{R}\}$ is IFS of $\mathfrak{R}$. We say α is $(IF\overline{\rho}i)$ of $\mathfrak{R}$ if

(1) $\mu_\alpha(0) \ge \mu_\alpha(\mathrm{a})$ and $\nu_\alpha(0) \le \nu_\alpha(a)$,

(2) $\mu_\alpha(a \bullet b) \ge \min\{\mu_\alpha(a), \mu_\alpha(b)\}$ and $\nu_\alpha(a \bullet b) \le \max\{\nu_\alpha(a), \nu_\alpha(b)\}$, $\forall a, b \in \mathfrak{R}$.

Example 13. Let $(\mathfrak{R}, \bullet, 0)$ be $\rho-$algebra in Example 9 and let $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ\mid a \in \mathfrak{R}\} = \{(0, 0.9, 0.1), (\omega, 0.4, 0.3), (\partial, 0.7, 0.3), (\ell, 0.4, 0.2)\}$ be IFS of $\mathfrak{R}$. Then, α is $(IF\overline{\rho}i)$ of $\mathfrak{R}$.

Remark 14.

(1) If $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ\mid a \in \mathfrak{R}\}$ is $(IF\rho i)$ of $\mathfrak{R}$, then α is $(IF\rho s)$.

(2) If $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ\mid a \in \mathfrak{R}\}$ is $(IF\overline{\rho}i)$ of $\mathfrak{R}$, then α is $(IF\rho s)$.

(3) If $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ\mid a \in \mathfrak{R}\}$ is $(IF\rho s)$ of $\mathfrak{R}$ and satisfies (2) in Definition 10, then α is $(IF\rho i)$.

(4) If $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ\mid a \in \mathfrak{R}\}$ is $(IF\rho s)$ of $\mathfrak{R}$ and satisfies (1) in Definition 12, then α is $(IF\overline{\rho}i)$.

Lemma 15. *If $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ\mid a \in \mathfrak{R}\}$ is $(IF\rho s)$ of $\mathfrak{R}$, then $\mu_\alpha(0) \ge \mu_\alpha(a)$ and $\nu_\alpha(0) \le \nu_\alpha(a)$, $\forall a \in \mathfrak{R}$.*

Proof. Let $a \in \mathfrak{R}$. Then $\mu_\alpha(0) = \mu_\alpha(a \bullet a) \ge \min\{\mu_\alpha(a), \mu_\alpha(a)\} = \mu_\alpha(a)$ and $\nu_\alpha(0) = \nu_\alpha(a \bullet a) \le \max\{\nu_\alpha(a), \nu_a(a)\} = \nu_\alpha(a)$. □

Theorem 16. *If $\{\alpha_i =\prec a,(\mu_{\alpha_i}(a),\nu_{\alpha_i}(a)) \succ\mid a\in\mathfrak{R}, i\in I\}$ is any family of (IFρs) of $\mathfrak{R}$, then $\widetilde{\bigcap}_{i\in I}\alpha_i$ is (IF$\overline{\rho}$i) of $\mathfrak{R}$, where $\widetilde{\bigcap}_{i\in I}\alpha_i = \{\prec a,(\min\{\mu_{\alpha_i}(a)\},\max\{\nu_{\alpha_i}(a)\})\succ\mid a\in\mathfrak{R}\}$.*

Proof. Let $a,b\in\mathfrak{R}$. Thus we consider that $\min\{\mu_{\alpha_i}(a\bullet b)\} \geq \min\{\min\{\mu_{\alpha_i}(a),\mu_{\alpha_i}(b)\}\} = \min\{\min\{\mu_{\alpha_i}(a)\},\min\{\mu_{\alpha_i}(b)\}\}$. Also $\max\{\nu_{\alpha_i}(a\bullet b)\} \leq \max\{\max\{\nu_{\alpha_i}(a),\nu_{\alpha_i}(b)\}\} = \max\{\max\{\nu_{\alpha_i}(a)\},\max\{\nu_{\alpha_i}(b)\}\}$.

Thus $\widetilde{\bigcap}_{i\in I}\alpha_i = \{\prec a,(\min\{\mu_{\alpha_i}(a)\},\max\{\nu_{\alpha_i}(a)\})\succ\mid a\in\mathfrak{R}\}$ satisfies condition (2) in Definition 12. Also, let $a\in\mathfrak{R}$. Hence, we consider that $\min\{\mu_{\alpha_i}(0)\} = \min\{\mu_{\alpha_i}(a\bullet a)\} \geq \min\{\mu_{\alpha_i}(a),\mu_{\alpha_i}(a)\} = \min\{\mu_{\alpha_i}(a)\}$. Furthermore, $\max\{\nu_{\alpha_i}(0)\} = \max\{\nu_{\alpha_i}(a\bullet a)\} \leq \max\{\nu_{\alpha_i}(a),\nu_{\alpha_i}(a)\} = \max\{\nu_{\alpha_i}(a)\}$. Then (1) in Definition 12 is held and hence $\widetilde{\bigcap}_{i\in I}\alpha_i$ is ($I\,\overline{\rho}i$) of $\mathfrak{R}$. □

Theorem 17. *If $\alpha = \{\prec a,(\mu_\alpha(a),\nu_\alpha(a))\succ\mid a\in\mathfrak{R}\}$ is (IFρi) of $\mathfrak{R}$, then $K=\prec a,\mu_\alpha(a),1-\mu_\alpha(a)\succ$ is (IFρi) of $\mathfrak{R}$.*

Proof. We need only to show that $1-\mu_\alpha(a)$ satisfies the first and second condition in Definition 10. Assume $\forall a,b\in\mathfrak{R}$. Then $1-\mu_\alpha(a\bullet b) \leq 1-\min\{\mu_\alpha(a),\mu_\alpha(b)\} = \max\{1-\mu_\alpha(a),1-\mu_\alpha(b)\}$. Furthermore, $1-\mu_\alpha(a) \leq 1-\min\{\mu_\alpha(a\bullet b),\mu_\alpha(b)\} = \max\{1-\mu_\alpha(a\bullet b),1-\mu_\alpha(b)\}$. Hence K is (*IFρi*) of $\mathfrak{R}$. □

Theorem 18. *If $\alpha = \{\prec a,(\mu_\alpha(a),\nu_\alpha(a))\succ\mid a\in\mathfrak{R}\}$ is (IFρs) of $\mathfrak{R}$, then the sets $T_\mu = \{a\in\mathfrak{R}\mid \mu_\alpha(a)=\mu_\alpha(0)\}$ and $T_\nu = \{a\in\mathfrak{R}\mid \nu_\alpha(a)=\nu_\alpha(0)\}$ are ρ−subalgebras of $\mathfrak{R}$.*

Proof. Let $a,b\in T_\mu$. Hence $\mu_\alpha(a)=\mu_\alpha(0)=\mu_\alpha(b)$, and $\mu_\alpha(a\bullet b)\geq\min\{\mu_\alpha(a),\mu_\alpha(b)\}=\mu_\alpha(0)$. By using Lemma 15, we consider that $\mu_\alpha(a\bullet b)=\mu_\alpha(0)$ or equivalently $a\bullet b\in T_\mu$. Now, let $a,b\in T_\nu$. This implies that $\nu_\alpha(a\bullet b)\leq\max\{\nu_\alpha(a),\nu_\alpha(b)\}=\nu_\alpha(0)$ and, by applying Lemma 15, we conclude that $\nu_\alpha(a\bullet b)=\nu_\alpha(0)$. Therefore $a\bullet b\in T_\nu$. □

Definition 19. Assume $\alpha = \{\prec a,(\mu_\alpha(a),\nu_\alpha(a))\succ\mid a\in\mathfrak{R}\}$ is (*IFρs*) of $\mathfrak{R}$. We say α has finite image, if each image of μ_α and ν_α is with finite cardinality (i.e., $\mathrm{Im}(\mu_\alpha)=\{\mu_\alpha(a)\mid a\in\mathfrak{R}\}$ and $\mathrm{Im}(\nu_\alpha)=\{\nu_\alpha(a)\mid a\in\mathfrak{R}\}$ such that $|\mathrm{Im}(\mu_\alpha)|<\infty$ and $|\mathrm{Im}(\nu_\alpha)|<\infty$).

Definition 20. Assume that $\alpha = \{\prec a,(\mu_\alpha(a),\nu_\alpha(a))\succ\mid a\in\mathfrak{R}\}$ is (*IFρs*) of $\mathfrak{R}$ and $r\in[0,1]$. The set $W(\mu_\alpha,r)=\{a\in\mathfrak{R}\mid \mu_\alpha(a)\geq r\}$ (resp., $L(\nu_\alpha,r)=\{a\in\mathfrak{R}\mid\nu_\alpha(a)\leq r\}$) is said to be *μ−level r−cut* (resp., *ν−level r−cut*) of α.

Theorem 21. *If $\alpha = \{\prec a,(\mu_\alpha(a),\nu_\alpha(a))\succ\mid a\in\mathfrak{R}\}$ is (IFρs) of $\mathfrak{R}$, then $W(\mu_\alpha,r)=\{a\in\mathfrak{R}\mid\mu_\alpha(a)\geq r\}$ and $L(\nu_\alpha,r)=\{a\in\mathfrak{R}\mid\nu_\alpha(a)\leq r\}$) of α are ρ−subalgebras of $\mathfrak{R}$.*

Proof. Let $a,b\in W(\mu_\alpha,r)$. Hence $\mu_\alpha(b)\geq r$ and $\mu_\alpha(b)\geq r$. This implies that $\mu_\alpha(a\bullet b)\geq\min\{\mu_\alpha(a),\mu_\alpha(b)\}\geq r$ so that $a\bullet b\in W(\mu_\alpha,r)$. Thus $W(\mu_\alpha,r)$ is ρ−subalgebra of $\mathfrak{R}$ Now let $a,b\in L(\nu_\alpha,r)$. Thus $\nu_\alpha(a\bullet b)\leq\max\{\nu_\alpha(a),\nu_\alpha(b)\}\leq r$ and $a\bullet b\in L(\nu_\alpha,r)$. Therefore $L(\nu_\alpha,r)$ is ρ−subalgebra of $\mathfrak{R}$. □

Theorem 22. *If $\alpha = \{\prec a,(\mu_\alpha(a),\nu_\alpha(a))\succ\mid a\in\mathfrak{R}\}$ is IFS of ρ − algebra $\mathfrak{R}$ such that the sets $W(\mu_\alpha,r)$ and $L(\nu_\alpha,r)$ are ρ−subalgebras of $\mathfrak{R}$, then α is an (IFρs) of $\mathfrak{R}$.*

Proof. Suppose that there are two members t_1 and t_2 in $\mathfrak{R}$ with $\mu_\alpha(t_1\bullet t_2)<\min\{\mu_\alpha(t_1),\mu_\alpha(t_2)\}$. Let $t=[\mu_\alpha(t_1\bullet t_2)+\min\{\mu_\alpha(t_1),\mu_\alpha(t_2)\}]/2$. Hence $\mu_\alpha(t_1\bullet t_2)<t<\min\{\mu_\alpha(t_1),\ \mu_\alpha(t_2)\}$ and so $t_1\bullet t_2\notin W(\mu_\alpha,t)$, but $t_1,t_2\in W(\mu_\alpha,t)$. This is a contradiction, and therefore $\mu_\alpha(a\bullet b)\geq\min\{\mu_\alpha(a),\mu_\alpha(b)\}$, $\forall a,b\in\mathfrak{R}$. Now assume that $\nu_\alpha(t_1\bullet t_2)>\min\{\nu_\alpha(t_1),\nu_\alpha(t_2)\}$ for some $t_1,t_2\in\mathfrak{R}$. Taking $k=[\nu_\alpha(t_1\bullet t_2)+\min\{\nu_\alpha(t_1),\nu_\alpha(t_2)\}]/2$, then we consider that $\nu_\alpha(t_1\bullet t_2)>k>\max\{\nu_\alpha(t_1),\nu_\alpha(t_2)\}$. It follows that $t_1,t_2\in L(\nu_\alpha,k)$ and $t_1\bullet t_2\notin L(\nu_\alpha,k)$. This is a contradiction. Therefore, we consider that $\nu_\alpha(a\bullet b)\leq\max\{\nu_\alpha(a),\nu_\alpha(b)\}$, $\forall a,b\in\mathfrak{R}$. Then A is (*IFρs*) of $\mathfrak{R}$. □

Theorem 23. *If H is ρ−subalgebra of $\mathfrak{R}$, then there exists (IFρs) α of $\mathfrak{R}$, where H satisfies both μ−level ρ−subalgebra and ν−level ρ−subalgebra of α in $\mathfrak{R}$.*

Proof. Assume H is ρ−subalgebra of $\mathfrak{R}$ and let μ_α and ν_α be fuzzy sets in $\mathfrak{R}$ defined by

$$\mu_\alpha(a)=\begin{cases}k, & \text{if } a\in H\\ 1, & \text{Otherwise,}\end{cases}\tag{1}$$

and

$$\nu_\alpha(a)=\begin{cases}m, & \text{if } \alpha\in H\\ 1, & \text{Otherwise,}\end{cases}\tag{2}$$

$\forall a\in\mathfrak{R}$, where $k,m\in(0,1)$ are fixed real numbers with $k+m<1$. Assume $a,b\in\mathfrak{R}$. Then $a\bullet b\in H$ whenever $a,b\in H$. This implies that $\mu_\alpha(a\bullet b)=\min\{\mu_\alpha(a),\mu_\alpha(b)\}$ and $\nu_\alpha(a\bullet b)\leq\max\{\nu_\alpha(a),\nu_\alpha(b)\}$. If at least one of a or b does not belong to H, then either $\mu_\alpha(a)=0$ or $\mu_\alpha(b)=0$ and hence either $\nu_\alpha(a)=1$ or $\nu_\alpha(b)=1$. It follows that $\mu_\alpha(a\bullet b)\geq 0=\min\{\mu_\alpha(a),\mu_\alpha(b)\}$, $\nu_\alpha(a\bullet b)\leq 1=\max\{\nu_\alpha(a),\nu_\alpha(b)\}$. Hence $\alpha = \{\prec a,(\mu_\alpha(a),\nu_\alpha(a))\succ\mid a\in\mathfrak{R}\}$ is (*IFρs*) of $\mathfrak{R}$. Obviously, $W(\mu_\alpha,k)=H=L(\nu_\alpha,m)$. □

Definition 24. Assume $\Theta:\mathfrak{R}\longrightarrow Y$ is a mapping of ρ−algebras. We say Θ is ρ−homomorphism if $\Theta(a\bullet b)=\Theta(a)\bullet\Theta(b)$, $\forall a,b\in\mathfrak{R}$. And $\Theta^{-1}(\beta)=\{\prec a,(\Theta^{-1}\mu_\beta(a),\Theta^{-1}\nu_\beta(a))\succ\mid a\in\mathfrak{R}\}$ is IFS in ρ−algebra $\mathfrak{R}$ for any IFS $\beta=\{\prec c,(\mu_\beta(c),\nu_\beta(c))\succ\mid c\in Y\}$ of ρ−algebra Y. Also, if $\alpha = \{\prec a,(\mu_\alpha(a),\nu_\alpha(a))\succ\mid a\in\mathfrak{R}\}$ is IFS in ρ−algebra $\mathfrak{R}$, then $\Theta(\alpha)$ is IFS in Y and defined by

$\Theta(\alpha)=\{\prec c,(\Theta_{\sup}\mu_\alpha(y),\Theta_{\inf}\nu_\alpha(c))\succ\mid c\in Y\}$, where

$$\Theta_{\sup}\mu_\alpha(c)=\begin{cases}\sup\{\mu_\alpha(a)\mid a\in\Theta^{-1}(c)\}, & \text{if } \Theta^{-1}(c)\neq 0,\\ 0, & \text{Otherwise,}\end{cases}\tag{3}$$

and

$$\Theta_{\inf}\nu_\alpha(c) = \begin{cases} \inf\{\nu_\alpha(a) \mid a \in \Theta^{-1}(c)\}, & \text{if } \Theta^{-1}(c) \neq 0, \\ 1, & \text{Otherwise}, \end{cases} \quad \forall c \in Y. \tag{4}$$

Theorem 25. *Let Θ be $\rho-$homomorphism of $\rho-$algebra $\mathfrak{R}$ into $\rho-$algebra Y and K be $(IF\rho s)$ of Y. Then $\Theta^{-1}(K)$ is $(IF\rho s)$ of $\mathfrak{R}$.*

Proof. Assuming $a, b \in \mathfrak{R}$, we have $\mu_{\Theta^{-1}(K)}(a \bullet b) = \mu_K(\Theta(a \bullet b)) = \mu_K(\Theta(a) \bullet \Theta(b)) \geq \min\{\mu_K(\Theta(a)), \mu_K(\Theta(b))\} = \min\{\mu_{\Theta^{-1}(K)}(a), \mu_{\Theta^{-1}(K)}(b)\}$ and $\nu_{\Theta^{-1}(K)}(a \bullet b) = \nu_K(\Theta(a \bullet b)) = \nu_K(\Theta(a) \bullet \Theta(b)) \leq \max\{\nu_K(\Theta(a)), \nu_K(\Theta(b))\} = \max\{\nu_{\Theta^{-1}(K)}(a), \nu_{\Theta^{-1}(K)}(b)\}$. Thus $\Theta^{-1}(K)$ is $(IF\rho s)$ of $\mathfrak{R}$. □

Theorem 26. *Assume $\Theta : \mathfrak{R} \longrightarrow Y$ is $\rho-$homomorphism of $\rho-$algebra $\mathfrak{R}$ into $\rho-$algebra Y and $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ \mid a \in \mathfrak{R}\}$ is $(IF\rho s)$ of $\mathfrak{R}$. Then $\Theta(\alpha) =\prec b, (\Theta_{\sup}(\mu_\alpha), \Theta_{\inf}(\nu_\alpha)) \succ$ is $(IF\rho s)$ of Y.*

Proof. Let $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ \mid a \in \mathfrak{R}\}$ be $(IF\rho s)$ of $\mathfrak{R}$ and let $t_1, t_2 \in Y$. Noticing that $\{a_1 \bullet a_2 \mid a_1 \in \Theta^{-1}(t_1)$ and $a_2 \in \Theta^{-1}(t_2)\} \subseteq \{a \in \mathfrak{R} \mid a \in \Theta^{-1}(t_1 \bullet t_2)\}$, we have $\Theta_{\sup}(\mu_\alpha)(t_1 \bullet t_2) = \sup\{\mu_\alpha(a) \mid a \in \Theta^{-1}(t_1 * t_2)\} \geq \sup\{\mu_\alpha(a_1 \bullet a_2) \mid a_1 \in \Theta^{-1}(t_1)$ and $a_2 \in \Theta^{-1}(t_2)\} \geq \sup\{\min\{\mu_\alpha(a_1), \mu_\alpha(a_2)\} \mid a_1 \in \Theta^{-1}(t_1)$ and $a_2 \in \Theta^{-1}(t_2)\} = \min\{\sup\{\mu_\alpha(a_1) \mid a_1 \in \Theta^{-1}(t_1)\}, \sup\{\mu_\alpha(a_2) \mid a_2 \in \Theta^{-1}(t_2)\}\} = \min\{\Theta_{\sup}(\mu_\alpha)(t_1), \Theta_{\sup}(\mu_\alpha)(t_2)\}$. Also, we consider that $\Theta_{\inf}(\nu_\alpha)(t_1 \bullet t_2) = \inf\{\nu_\alpha(a) \mid a \in \Theta^{-1}(t_1 \bullet t_2)\} \leq \inf\{\nu_\alpha(a_1 \bullet a_2) \mid a_1 \in \Theta^{-1}(t_1)$ and $a_2 \in \Theta^{-1}(t_2)\} \leq \inf\{\max\{\nu_\alpha(a_1), \nu_\alpha(a_2)\} \mid a_1 \in \Theta^{-1}(t_1)$ and $a_2 \in \Theta^{-1}(t_2)\} = \max\{\inf\{\nu_\alpha(a_1) \mid a_1 \in \Theta^{-1}(t_1)\}, \inf\{\nu_\alpha(a_2) \mid a_2 \in \Theta^{-1}(t_2)\}\} = \max\{\Theta_{\sup}(\nu_\alpha)(t_1), \Theta_{\sup}(\nu_\alpha)(t_2)\}$. Hence $\Theta(\alpha) =\prec b, (\Theta_{\sup}(\mu_\alpha), \Theta_{\inf}(\nu_\alpha)) \succ$ is $(IF\rho s)$ of Y. □

Theorem 27. *Assume $\Theta : \mathfrak{R} \longrightarrow Y$ is $\rho-$homomorphism of $\rho-$algebra $\mathfrak{R}$ into $\rho-$algebra Y and $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ \mid a \in \mathfrak{R}\}$ is $(IF\rho i)$ of $\mathfrak{R}$. Then $\Theta(\alpha) =\prec b, (\Theta_{\sup}(\mu_\alpha), \Theta_{\inf}(\nu_\alpha)) \succ$ is $(IF\rho i)$ of Y.*

Proof. Since $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ \mid a \in \mathfrak{R}\}$ is $(IF\rho i)$ of $\mathfrak{R}$, then by Theorem 26 and Remark 14 we have $\Theta(\alpha) =\prec b, (\Theta_{\sup}(\mu_\alpha), \Theta_{\inf}(\nu_\alpha)) \succ$ as $(IF\rho i)$ of Y. Hence condition (1) in Definition 10 is held. Since Θ is surjective, then for any $t_1, t_2 \in Y$, $\exists a_1, a_2 \in \mathfrak{R}$ such that $a_1 \in \Theta^{-1}\Theta(a_1) = \Theta^{-1}(t_1)$ and $a_2 \in \Theta^{-1}\Theta(a_2) = \Theta^{-1}(t_2)$. Also, $a_1 \bullet a_2 \in \Theta^{-1}(t_1) \bullet \Theta^{-1}(t_1) = \Theta^{-1}(t_1 \bullet t_1)$. Further, noticing that $\mu_\alpha(a_1) \geq \min\{\mu_\alpha(a_1 \bullet a_2), \mu_\alpha(a_2)\}$ and $\nu_\alpha(a_1) \leq \max\{\nu_\alpha(a_1 \bullet a_2), \nu_\alpha(a_2)\}$, for any $t_1, t_2 \in Y$, we have $\Theta_{\sup}(\mu_\alpha)(t_1) = \sup\{\mu_\alpha(a) \mid a \in \Theta^{-1}(t_1)\} \geq \sup\{\min\{\mu_\alpha(a_1 \bullet a_2), \mu_\alpha(a_2)\} \mid a_1 \bullet a_2 \in \Theta^{-1}(t_1 \bullet t_2)$ and $a_2 \in \Theta^{-1}(t_2)\} = \min\{\sup\{\mu_\alpha(a_1 \bullet a_2) \mid a_1 \bullet a_2 \in \Theta^{-1}(t_1 \bullet t_2)\}, \sup\{\mu_\alpha(a_2) \mid a_2 \in \Theta^{-1}(t_2)\}\} = \min\{\Theta_{\sup}(\mu_\alpha)(t_1 \bullet t_2), \Theta_{\sup}(\mu_\alpha)(t_2)\}$. Also, $\Theta_{\sup}(\nu_\alpha)(t_1) = \sup\{\nu_\alpha(a) \mid a \in \Theta^{-1}(t_1)\} \leq \sup\{\max\{\nu_\alpha(a_1 \bullet a_2), \nu_\alpha(a_2)\} \mid a_1 \bullet a_2 \in \Theta^{-1}(t_1 \bullet t_2)$ and $a_2 \in \Theta^{-1}(t_2)\} = \max\{\sup\{\nu_\alpha(a_1 \bullet a_2) \mid a_1 \bullet a_2 \in \Theta^{-1}(t_1 \bullet t_2)\}, \sup\{\nu_\alpha(a_2) \mid a_2 \in \Theta^{-1}(t_2)\}\} = \max\{\Theta_{\sup}(\nu_\alpha)(t_1 \bullet t_2), \Theta_{\sup}(\nu_\alpha)(t_2)\}$. Thus we consider that $\Theta(\alpha) =\prec b, (\Theta_{\sup}(\mu_\alpha), \Theta_{\inf}(\nu_\alpha)) \succ$ is $(IF\rho i)$ of Y. □

Theorem 28. *Assume $\Theta : \mathfrak{R} \longrightarrow Y$ is $\rho-$homomorphism of $\rho-$algebra $\mathfrak{R}$ into $\rho-$algebra Y and $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ \mid a \in \mathfrak{R}\}$ is $(IF\overline{\rho}i)$ of $\mathfrak{R}$. Then $\Theta(\alpha) =\prec b, (\Theta_{\sup}(\mu_\alpha), \Theta_{\inf}(\nu_\alpha)) \succ$ is $(IF\overline{\rho}i)$ of Y.*

Proof. Since $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ \mid a \in \mathfrak{R}\}$ is $(IF\overline{\rho}i)$ of $\mathfrak{R}$. Then by Theorem 26 and Remark 14 we have $\Theta(\alpha) =\prec b, (\Theta_{\sup}(\mu_\alpha), \Theta_{\inf}(\nu_\alpha)) \succ$ as $(IF\rho s)$ of Y. Hence condition (2) in Definition 12 is held. Assume that $0_{\mathfrak{R}}$ and 0_Y are constants of ρ − algebras $\mathfrak{R}$ and Y, respectively. Since $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ \mid a \in \mathfrak{R}\}$ is $(IF\overline{\rho}i)$ of $\mathfrak{R}$, hence $\mu_\alpha(0_{\mathfrak{R}}) \geq \mu_\alpha(a)$ and $\nu_\alpha(0_{\mathfrak{R}}) \leq \nu_\alpha(a)$, $\forall a \in \mathfrak{R}$. Since Θ is $\rho-$homomorphism of $\rho-$algebras, then $\Theta(0_{\mathfrak{R}}) = 0_Y$, where $0_{\mathfrak{R}}$ and 0_Y are constants for $\rho-$algebras $\mathfrak{R}$ and Y, respectively. Noticing that $0_{\mathfrak{R}} \in \Theta^{-1}(0_Y)$ and $\{a \mid a \in \Theta^{-1}(b)\} \subseteq \{a \mid a \in \mathfrak{R}\}$ for any $b \in Y$, then we have $\Theta_{\sup}(\mu_\alpha)(0_Y) = \sup\{\mu_\alpha(a) \mid a \in \Theta^{-1}(0_Y)\} = \mu_\alpha(0_{\mathfrak{R}}) \geq \sup\{\mu_\alpha(a) \mid a \in \mathfrak{R}\} \geq \sup\{\mu_\alpha(a) \mid a \in \Theta^{-1}(b)\} = \Theta_{\sup}(\mu_\alpha)(b)$. Also, $\Theta_{\sup}(\nu_\alpha)(0_Y) = \inf\{\nu_\alpha(a) \mid a \in \Theta^{-1}(0_Y)\} = \nu_\alpha(0_{\mathfrak{R}}) \leq \inf\{\nu_\alpha(a) \mid a \in \mathfrak{R}\} \leq \inf\{\nu_\alpha(a) \mid a \in \Theta^{-1}(b)\} = \Theta_{\sup}(\nu_\alpha)(b)$. Hence $\Theta(\alpha) =\prec b, (\Theta_{\sup}(\mu_\alpha), \Theta_{\inf}(\nu_\alpha)) \succ$ is $(IF\overline{\rho}i)$ of Y. □

4. Some Applications on $IF\rho_S(\mathfrak{R})$

In this section, some applications on $IF\rho_S(\mathfrak{R})$ are shown like the binary relations $\underset{\mu}{\approx}$, $\underset{\nu}{\approx}$ and Γ_r on $IF\rho_S(\mathfrak{R})$. Also, in this section the equivalence classes for theses binary relations are given, and some of their basic properties are studied.

4.1. Equivalence Classes Modulo ($\underset{\mu}{\approx}/\underset{\nu}{\approx}$). Denote the collection of all $(IF\rho_S)$ of $\mathfrak{R}$ by $IF\rho_S(\mathfrak{R})$ and let $r \in [0, 1]$. Define binary relations $\underset{\mu}{\approx}$ and $\underset{\nu}{\approx}$ on $IF\rho_S(\mathfrak{R})$ as follows.

$\alpha \underset{\mu}{\approx} \beta \Longleftrightarrow W(\mu_\alpha, r) = W(\mu_\beta, r)$ and $\alpha \underset{\nu}{\approx} \beta \Longleftrightarrow L(\nu_\alpha, r) = L(\nu_\beta, r)$, respectively, for $\alpha =\prec a, \mu_\alpha, \nu_\alpha \succ$ and $\beta =\prec a, \mu_\beta, \nu_\beta \succ$ in $IF\rho_S(\mathfrak{R})$. Moreover, it is clear that $\underset{\mu}{\approx}$ and $\underset{\nu}{\approx}$ are equivalence relations on $IF\rho_S(\mathfrak{R})$. If $\alpha =\prec a, \mu_\alpha, \nu_\alpha \succ \in IF\rho_S(\mathfrak{R})$, then we refer to the equivalence class of $\alpha =\prec a, \mu_\alpha, \nu_\alpha \succ$ modulo $\underset{\mu}{\approx}$ (resp., $\underset{\nu}{\approx}$) by $\langle\alpha\rangle_\mu$ (resp., $\langle\alpha\rangle_\nu$), and we refer to the family of all equivalence classes of α modulo $\underset{\mu}{\approx}$ (resp., $\underset{\nu}{\approx}$) by $IF\rho_S(\mathfrak{R})/\underset{\mu}{\approx}$ (resp., $IF\rho_S(X)/\underset{\nu}{\approx}$); i.e., $IF\rho_S(\mathfrak{R})/\underset{\mu}{\approx} = \{\langle\alpha\rangle_\mu \mid \alpha =\prec a, \mu_\alpha, \nu_\alpha \succ \in IF\rho_S(\mathfrak{R})\}$ (resp., $IF\rho_S(\mathfrak{R})/\underset{\nu}{\approx} = \{\langle\alpha\rangle_\mu \mid \alpha =\prec a, \mu_\alpha, \nu_\alpha \succ \in IF\rho_S(\mathfrak{R})\}$). Moreover, denote the collection of all $\rho-$ideals of $\mathfrak{R}$ by $\rho_I(\mathfrak{R})$ and let $r \in [0, 1]$. Let σ_r and η_r be maps from $IF\rho_S(\mathfrak{R})$ to $\rho_I(\mathfrak{R}) \cup \{\phi\}$ by $\sigma_r(\alpha) = W(\mu_\alpha, r)$ and $\eta_r(\alpha) = L(\nu_\alpha, r)$, respectively, $\forall \alpha =\prec a, \mu_\alpha, \nu_\alpha \succ \in IF\rho_S(\mathfrak{R})$. In other words, σ_r and η_r are well-defined.

Theorem 29. *Let σ_r and η_r be the maps from $IF\rho_S(\mathfrak{R})$ to $\rho_I(\mathfrak{R}) \cup \{\phi\}$. Then σ_r and η_r are surjective, for each $r \in (0, 1)$.*

Proof. Let $r \in (0,1)$. Then $\widehat{0} = \prec a, \overline{0}, \overline{1} \succ$ is in $IF\rho_S(\mathfrak{R})$, where each one of $\overline{0}$ and $\overline{1}$ is (FS) in $\mathfrak{R}$ defined by $\overline{0}(a) = 0$ and $\overline{1}(a) = 1$, $\forall a \in \mathfrak{R}$. Furthermore, $\sigma_r(\widehat{0}) = W(\overline{0}, r) = \phi = L(\overline{1}, r) = \eta_r(\widehat{0})$. Let $\phi \neq H \in \rho_I(\mathfrak{R})$. $\forall a \in \mathfrak{R}$, let $\mu_H(a) = \left\{\begin{smallmatrix} 1, & if\ a\in H \\ 0, & if\ a\notin H \end{smallmatrix}\right.$, and $\nu_H(a) = 1 - \mu_H(a)$; thus $\sigma_r(\widehat{H}) = W(\mu_H, r) = H = L(\nu_H, r) = \eta_r(\widehat{H})$. Now, we want to prove that $\widehat{H} = \prec x, \mu_H, \nu_H \succ \in IF\rho_S(\mathfrak{R})$. Since $H \in \rho_I(\mathfrak{R})$, then by condition (1) in Definition 5 we have H as ρ−subalgebra of $\mathfrak{R}$ and this implies that $W(\mu_H, r)$ and $L(\nu_H, r)$ are ρ−subalgebras of $\mathfrak{R}$. By Theorem 22 we consider $\widehat{H} = \prec a, \mu_H, \nu_H \succ \in IF\rho_S(\mathfrak{R})$. Therefore, $\forall H \in \rho_I(\mathfrak{R})$ we consider $\sigma_r(\widehat{H}) = H$ and $\eta_r(\widehat{H}) = H$ for some $\widehat{H} \in IF\rho_S(\mathfrak{R})$. This completes the proof. □

Theorem 30. *Let $IF\rho_S(\mathfrak{R})/\underset{\mu}{\approx}$ and $IF\rho_S(\mathfrak{R})/\underset{\nu}{\approx}$ be quotient sets. Then they are equipotent to $\rho_I(\mathfrak{R}) \cup \{\phi\}$, $\forall r \in (0,1)$.*

Proof. Assume $r \in (0,1)$ and let σ'_r(resp. η'_r) be a map from $IF\rho_S(\mathfrak{R})/\underset{\mu}{\approx}$ (resp., $IF\rho_S(\mathfrak{R})/\underset{\nu}{\approx}$) to $\rho_I(\mathfrak{R}) \cup \{\phi\}$ and they are defined by $\sigma'_r(\langle\alpha\rangle_\mu) = \sigma_r(\alpha)$ (resp. $\eta'_r(\langle\alpha\rangle_\nu) = \eta_r(\alpha)$), $\forall \alpha = \prec a, \mu_\alpha, \nu_\alpha \succ \in IF\rho_S(\mathfrak{R})$. Hence, $\alpha \underset{\mu}{\approx} \beta$ and $\alpha \underset{\nu}{\approx} \beta$, $\forall \alpha = \prec a, \mu_\alpha, \nu_\alpha \succ$ and $\beta = \prec a, \mu_\beta, \nu_\beta \succ$ in $IF\rho_S(\mathfrak{R})$, if $W(\mu_A, r) = W(\mu_B, r)$ and $L(\nu_A, r) = L(\nu_B, r)$. Then $\langle\alpha\rangle_\mu = \langle\beta\rangle_\mu$ and $\langle\alpha\rangle_\nu = \langle\beta\rangle_\nu$. This implies the maps σ'_r and η'_r are injective. Moreover, let $\phi \neq H \in \rho_I(\mathfrak{R})$ and $\forall a \in \mathfrak{R}$, let

$$\mu_H(a) = \begin{cases} 1, & if\ a \in H \\ 0, & if\ a \notin H, \end{cases} \tag{5}$$

$\nu_H(a) = 1 - \mu_H(a)$, and thus $\widehat{H} = \prec a, \mu_H, \nu_H \succ \in IF\rho_S(\mathfrak{R})$. We consider that $\sigma'_r(\langle\widehat{H}\rangle_\mu) = \sigma_r(\widehat{H}) = W(\mu_H, r) = H$, and $\eta'_r(\langle\widehat{H}\rangle_\nu) = \eta_r(\widehat{H}) = L(\nu_H, r) = H$. Finally, for $\widehat{0} = \prec a, \overline{0}, \overline{1} \succ \in IF\rho_S(\mathfrak{R})$ we have $\sigma'_r(\langle\widehat{0}\rangle_\mu) = \sigma_r(\widehat{0}) = W(\overline{0}, r) = \phi$ and $\eta'_r(\langle\widehat{0}\rangle_\nu) = \eta_r(\widehat{0}) = L(\overline{1}, r) = \phi$. Therefore σ'_r and η'_r are surjective, and we are done. □

4.2. Equivalence Class Modulo Γ_r. Another relation Γ_r on $IF\rho_S(\mathfrak{R})$ is defined by $(\alpha, \beta) \in \Gamma_r \Longleftrightarrow W(\mu_\alpha, r) \cap L(\nu_\alpha, r) = W(\mu_\beta, r) \cap L(\nu_\beta, r)$, $\forall r \in [0,1]$ and, $\forall \alpha = \prec a, \mu_\alpha, \nu_\alpha \succ$, $\beta = \prec a, \mu_\beta, \nu_\beta \succ \in IF\rho_S(\mathfrak{R})$. Moreover, the relation Γ_r is also an equivalence relation on $IF\rho_S(\mathfrak{R})$. Let $\langle\alpha\rangle_{\Gamma_r}$ denote the equivalence class of $\alpha = \prec a, \mu_\alpha, \nu_\alpha \succ$ modulo Γ_r, $\forall \alpha = \prec a, \mu_\alpha, \nu_\alpha \succ \in IF\rho_S(\mathfrak{R})$.

Theorem 31. *For any $r \in (0,1)$, the map $\psi_r : IF\rho_S(\mathfrak{R}) \longrightarrow \rho_I(\mathfrak{R}) \cup \{\phi\}$ defined by $\psi_r(\mathfrak{R}) = \sigma_r(\mathfrak{R}) \cap \eta_r(\mathfrak{R})$, $\forall \alpha = \prec a, \mu_\alpha, \nu_\alpha \succ \in IF\rho_S(\mathfrak{R})$ is surjective.*

Proof. Let $r \in (0,1)$. For $\widehat{0} = \prec a, \overline{0}, \overline{1} \succ \in IF\rho_S(\mathfrak{R})$, we get $\psi_r(\widehat{0}) = \sigma_r(\widehat{0}) \cap \eta_r(\widehat{0}) = W(\overline{0}, r) \cap L(\overline{1}, r) = \phi$. For any $H \in IF\rho_S(\mathfrak{R})$, there exists $\widehat{H} = \prec a, \mu_H, \nu_H \succ \in IF\rho_S(\mathfrak{R})$, where

$$\mu_H(a) = \begin{cases} 1, & if\ a \in H \\ 0, & if\ a \notin H \end{cases} \tag{6}$$

and $\nu_H(a) = 1 - \mu_H(a)$ such that $\psi_r(\widehat{H}) = \sigma_r(\widehat{H}) \cap \eta_r(\widehat{H}) = W(\mu_H, r) \cap L(\nu_H, r) = H$. This completes the proof. □

Theorem 32. *For any $r \in (0,1)$, the quotient set $IF\rho_S(\mathfrak{R})/\Gamma_r$ is equipotent to $\rho_I(X) \cup \{\phi\}$.*

Proof. Assume $r \in (0,1)$ and $\psi'_r : IF\rho_S(\mathfrak{R})/\Gamma_r \longrightarrow \rho_I(\mathfrak{R}) \cup \{\phi\}$ is a map defined by $\psi'_r(\langle\alpha\rangle_{\Gamma_r}) = \psi_r(\alpha)$, $\forall \langle\alpha\rangle_{\Gamma_r} \in IF\rho_S(\mathfrak{R})/\Gamma_r$.

Suppose that $\psi'_r(\langle\alpha\rangle_{\Gamma_r}) = \psi'_r(\langle\beta\rangle_{\Gamma_r})$ for any $\langle\alpha\rangle_{\Gamma_r}, \langle\beta\rangle_{\Gamma_r} \in IF\rho_S(\mathfrak{R})/\Gamma_r$. We consider that $\sigma_r(\alpha) \cap \eta_r(\alpha) = \sigma_r(\beta) \cap \eta_r(\beta)$, i.e., $W(\mu_\alpha, r) \cap L(\nu_\alpha, r) = W(\mu_\beta, r) \cap L(\nu_\beta, r)$. Hence $(\alpha, \beta) \in \Gamma_r$, and so $\langle\alpha\rangle_{\Gamma_r} = \langle\beta\rangle_{\Gamma_r}$. Therefore ψ'_r is injective. Furthermore, for $\widehat{0} = \prec a, \overline{0}, \overline{1} \succ \in IF\rho_S(\mathfrak{R})$ we get $\psi'_r(\langle\widehat{0}\rangle_{\Gamma_r}) = \psi_r(\widehat{0}) = \sigma_r(\widehat{0}) \cap \eta_r(\widehat{0}) = W(\overline{0}, r) \cap L(\overline{1}, r) = \phi$. Let $\widehat{H} = \prec a, \mu_H, \nu_H \succ \in IF\rho_S(\mathfrak{R})$, $\forall H \in IF\rho_S(\mathfrak{R})$, be the same $(IF\rho_S)$ of X that is defined in the proof of Theorem 22. Then we have $\psi'_r(\langle\widehat{H}\rangle_{\Gamma_r}) = \psi_r(\widehat{H}) = \sigma_r(\widehat{H}) \cap \eta_r(\widehat{H}) = W(\mu_H, r) \cap L(\nu_H, r) = H$. Hence ψ'_r is surjective. This completes the proof. □

5. Conclusion

In this work, we introduce the notions of $(IF\rho s)$, $(IF\rho i)$, $(IF\overline{\rho}i)$, and others; then we proved that for any $\rho-$subalgebra of X can be considered as both μ−level ρ−subalgebra and ν−level ρ−subalgebra of some $(IF\rho s)$ of $\mathfrak{R}$. At the same time, we proved that intersection of any family of $(IF\rho s)$ of X is $(IF\overline{\rho}i)$ of X. Also, we show that if IFS $\alpha = \{\prec a, (\mu_\alpha(a), \nu_\alpha(a)) \succ \mid a \in \mathfrak{R}\}$ of ρ − algebra X such that the sets $W(\mu_\alpha, r)$ and $L(\nu_\alpha, r)$ are ρ−subalgebras of $\mathfrak{R}$. Then α is $(IF\rho s)$ of $\mathfrak{R}$. Further, some interesting theorems about ρ−homomorphism are given. Finally, some binary relations $\underset{\mu}{\approx}$, $\underset{\nu}{\approx}$ and Γ_r on $IF\rho_S(\mathfrak{R})$ are obtained, and some of their basic properties are discussed. In future work, we will investigate IF in new types of algebras like BCL^+−algebras, BCL^+−subalgebras, BCL^+−ideals and others. Next, we will study their characteristics.

Data Availability

Data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest.

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