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Research Article

$\sigma-$Algebra and $\sigma-$Baire in Fuzzy Soft Setting

Shuker Mahmood Khalil, Mayadah Ulrazaq, Samaher Abdul-Ghani, and Abu Firas Al-Musawi

Department of Mathematics, College of Science, University of Basrah, Basrah 61004, Iraq

Correspondence should be addressed to Shuker Mahmood Khalil; shuker.alsalem@gmail.com

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We first introduce some new notions of Baireness in fuzzy soft topological space (FSTS). Next, their characterizations and basic properties are investigated in this work. The notions of fuzzy soft dense, fuzzy soft nowhere dense, fuzzy soft meager, fuzzy soft second category, fuzzy soft residual, fuzzy soft Baire, fuzzy soft $\delta-$sets, fuzzy soft $\lambda_\sigma-$sets, fuzzy soft $\sigma-$nowhere dense, fuzzy soft $\sigma-$meager, fuzzy soft $\sigma-$residual, fuzzy soft $\sigma-$Baire, fuzzy soft $\sigma-$second category, fuzzy soft $\sigma-$residual, fuzzy, fuzzy soft submaximal space, fuzzy soft $P-$space, fuzzy soft almost resolvable space, fuzzy soft hyperconnected space, fuzzy soft $A-$embedded, fuzzy soft $D-$Baire, fuzzy soft almost $P-$spaces, fuzzy soft Borel, and fuzzy soft $\sigma-$algebra are introduced. Furthermore, several examples are shown as well.

1. Introduction

The concepts of Baire spaces have been studied and discussed extensively in general topology in [1–4]. Thangaraj and Balasubramanian [5] studied the notion of somewhat fuzzy continuous functions. Next, Thangaraj and Anjalmose investigated and discussed the notion of Baire space in fuzzy topology [6]. After that, they introduced the notion of fuzzy Baire space [7].

Soft sets theory was introduced by Molodtsov [8]. It explains new type of mathematical tool of soft sets and it deals with vagueness when solving problems in practice as in engineering, environment, social science, and economics, which cannot be handled as classical mathematical tools. Also, other authors such as Maji et al. [9–19] have further studied the theory of soft sets and used this theory in pure mathematics to solve some decision making problems. Next, the notion of fuzzy soft set is investigated and discussed [20–22]. Since then much attention has been used to generalize the basic notions of fuzzy topology in soft setting. In other words, the modern theories of fuzzy soft topology have been developed.

In recent years, fuzzy soft topology has been found to be very useful in solving many practical problems [23]. Also, rough fuzzy and other applications are studied [24–26]. The main purpose of this work is to introduce new concepts of fuzzy soft Baireness in fuzzy soft topological spaces. In section three, we introduce fuzzy soft dense sets, fuzzy soft nowhere dense sets, fuzzy soft meager sets, fuzzy soft second category sets, fuzzy soft meager spaces, fuzzy soft second category spaces, fuzzy soft residual sets, fuzzy soft Baire spaces, fuzzy soft $\delta-$sets, fuzzy soft $\lambda_\sigma-$sets, fuzzy soft $\sigma-$nowhere dense, fuzzy soft $\sigma-$meager, fuzzy soft $\sigma-$residual, fuzzy soft $\sigma-$Baire, fuzzy soft $\sigma-$second category, fuzzy soft $\sigma-$residual, fuzzy, fuzzy soft submaximal space, fuzzy soft $P-$space, fuzzy soft almost resolvable space, fuzzy soft hyperconnected space, fuzzy soft $A-$embedded, fuzzy soft $D-$Baire, fuzzy soft almost $P-$spaces, fuzzy soft Borel, and fuzzy soft $\sigma-$algebra. Moreover, several examples are given to illustrate the notions introduced in this work.

2. Preliminaries

In this section, we give few definitions and properties regarding fuzzy soft sets.

Definition 1 ([20]). Assume U is an initial universe set and E is a set of parameters. Let I^U refer to the family of all fuzzy soft sets (FSSs) in U and $A \subseteq E$. The multivalued map $F_A : A \longrightarrow I^U$ defined by $F_A(e) = \mu^e_{F_A}$ is said to be a fuzzy soft set (FSS) over (U, E), where $\mu^e_{F_A} \neq \overline{0}$ if $e \in A$ and $\mu^e_{F_A} = \overline{0}$ if $e \in E \setminus A$. We refer to family of all (FSSs) over (U, E) by $FS(U, E)$.

Definition 2 ([20]). We say $F_\phi \in FS(U, E)$ is null (FSS) and we refer to it by Φ, if $\forall e \in E$, $F(e)$ is the null (FSS) $\overline{0}$ of U, where $\overline{0}(x) = 0\ \forall x \in U$.

Definition 3 ([20]). Assume $F_E \in FS(U, E)$ and $F_E(e) = \overline{1}\ \forall e \in E$, where $\overline{1}(x) = 1\ \forall x \in U$. We say F_E is absolute (FSS) and we refer to it by $\overline{E}$.

Definition 4 ([20]). We say F_A is a fuzzy soft subset of a (FSS) G_B over a common universe U if $A \subseteq B$ and $F_A(e) \subseteq G_B(e)\ \forall e \in A$; i.e., if $\mu^e_{F_A}(x) \leq \mu^e_{G_B}(x)\ \forall x \in U$ and $\forall e \in E$ and denoted by $F_A \widetilde{\subseteq} G_B$.

Definition 5 ([20]). Assume F_A and G_B are (FSSs) over a common universe U. We say they are fuzzy soft equal if $F_A \widetilde{\subseteq} G_B$ and $G_B \widetilde{\subseteq} F_A$.

Definition 6 ([20]). Assume F_A and G_B are (FSSs) over a common universe U. Their union is the (FSS) H_C, defined by $H_C(e) = \mu^e_{H_C} = \mu^e_{F_A} \cup \mu^e_{G_B}\ \forall e \in E$, where $C = A \bigcup B$. For this case, we write $H_C = F_A \widetilde{\cup} G_B$.

Definition 7 ([20]). Assume F_A and G_B are (FSSs) over a common universe U. Their intersection is a (FSS) H_C, defined by $H_C(e) = \mu^e_{H_C} = \mu^e_{F_A} \cap \mu^e_{G_B}\ \forall e \in E$, where $C = A \bigcap B$. For this case, we write $H_C = F_A \widetilde{\cap} G_B$.

Definition 8 ([27]). Assume $F_A \in FS(U, E)$ is a (FSS). We refer to its complement by ${F_A}^c$, and it is defined as

$$F^c_A = \begin{cases} \overline{1} - \mu^e_{F_A} & \text{if } e \in A, \\ \overline{1} & \text{if } e \notin A. \end{cases} \tag{1}$$

Remark 9. Let $K = \{F^i_{A_i} \mid i \in I\}$ be a family of fuzzy soft sets. The union (intersection) of any number ($J \subseteq I$) of family K is defined by $H_C(e) = \mu^e_{H_C} = \bigcup_{j \in J} \{\mu^e_{F^i_{A_i}}\} (H_C(e) = \mu^e_{H_C} = \bigcap_{j \in J} \{\mu^e_{F^i_{A_i}}\})$, where $C = \bigcup_{j \in J} \{A_i\}$ $(C = \bigcap_{j \in J} \{A_i\})$.

Definition 10 ([28]). Assume ψ is the family of (FSSs) over U. We say ψ is a fuzzy soft topology on U if ψ the following axioms hold:

(i) $\Phi, \overline{E}$ belong to ψ.

(ii) The union of any number of (FSSs) in ψ belongs to ψ.

(iii) The intersection of any two (FSSs) in ψ belongs to ψ.

We say (U, E, ψ) is a fuzzy soft topological space (FSTS) over U. Each member in ψ is called (FSOS) (FSOS) in U and its complement is called fuzzy soft closed set (FSCS) in U, where $C = \bigcup_{j \in J} \{A_i\}$ $(C = \bigcap_{j \in J} \{A_i\})$.

Definition 11 ([28]). The union of all fuzzy soft open subsets of F_A over (U, E) is called the interior of F_A and is denoted by $int^{fs}(F_A)$.

Proposition 12 ([28]). $int^{fs}(F_A \widetilde{\cap} G_B) = int^{fs}(F_A) \widetilde{\cap} int^{fs}(G_B)$.

Definition 13 ([28]). Let $F_A \in FS(U, E)$ be a (FSS). Then the intersection of all closed sets, each containing F_A, is called the closure of F_A and is denoted by $cl^{fs}(F_A)$.

Remark 14. (1) For any (FSS) F_A in a (FSTS) (U, E, ψ), it is easy to see that $(cl^{fs}(F_A))^c = int^{fs}({F_A}^c)$ and $(int^{fs}(F_A))^c = cl^{fs}({F_A}^c)$.

(2) For any fuzzy soft F_A subset of a (FSTS) (U, E, ψ), we define the fuzzy soft subspace topology ψ_{F_A} on F_A by $K_D \in \psi_{F_A}$ if $K_D = F_A \widetilde{\cap} G_B$ for some $G_B \in \psi$.

(3) For any fuzzy soft H_C in F_A fuzzy soft subspace of a (FSTS), we denote the interior and closure of H_C in F_A by $int^{fs}_{F_A}(H_C)$ and $cl^{fs}_{F_A}(H_C)$, respectively.

3. Fuzzy Soft σ−Baire Spaces

In this section, we introduce new notions of (FSTSs) using new classes of (FSSs) which are introduced in this section and obtained their properties.

Definition 15. A (FSS) F_A in a (FSTS) (U, E, ψ) is called fuzzy soft dense if there exists no (FSCS) G_B in (U, E, ψ) such that $F_A \widetilde{\subset} G_B \widetilde{\subset} \overline{E}$.

Definition 16. A (FSS) F_A in a (FSTS) (U, E, ψ) is called fuzzy soft nowhere dense if there exists no nonempty (FSOS) G_B in (U, E, ψ) such that $G_B \widetilde{\subset} cl^{fs}(F_A)$. That is, $int^{fs}(cl^{fs}(F_A)) = \Phi$.

Definition 17. A (FSS) F_A in a (FSTS) (U, E, ψ) is called fuzzy soft meager if F_A is a countable union of fuzzy soft nowhere dense sets [i.e., if $F_A = \widetilde{\bigcup}_{i \in I} \{F^i_{A_i}\}$, where $F^i_{A_i}$*'s* are fuzzy soft nowhere dense sets in (U, E, ψ), $\forall i \in I \subseteq N$]. Otherwise, F_A will be called a fuzzy soft second category set.

Definition 18. A (FSTS) (U, E, ψ) is called fuzzy soft meager or (fuzzy soft first category) space if the (FSS) $\overline{E}$ is a fuzzy soft meager set in (U, E, ψ). That is, $\overline{E} = \widetilde{\bigcup}_{i \in I} \{F^i_{A_i}\}$, where $({F^i_{A_i}}^c)$*'s* are fuzzy soft nowhere dense sets in (U, E, ψ). Otherwise, (U, E, ψ) will be called a fuzzy soft second category space.

Definition 19. Assume F_A is a fuzzy soft meager set in (U, E, ψ). We say ${F_A}^c$ is a fuzzy soft residual set in (U, E, ψ).

Definition 20. Assume (U, E, ψ) is a (FSTS). We say (U, E, ψ) is a fuzzy soft Baire space if each sequence $\{F^1_{A_1}, F^2_{A_2}, F^3_{A_3}, \ldots\}$ of fuzzy soft nowhere dense sets in (U, E, ψ) such that $int^{fs}(\widetilde{\bigcup}_{i \in I} \{F^i_{A_i}\}) = \Phi$.

Definition 21. A (FSS) F_A in a (FSTS) (U, E, ψ) is called fuzzy soft δ−set in (U, E, ψ) if $F_A = \widetilde{\bigcap}_{i \in I} \{F^i_{A_i}\}$, where $F^i_{A_i} \in \psi\ \forall i \in I$.

Remark 22. Definition 20 can be written in other equivalent forms as follows:

(i) We say (U, E, ψ) is a fuzzy soft Baire space if every countable fuzzy soft closed cover $\{F^i_{A_i}; i \in I\}$ of $\overline{E}$, the set $\widetilde{\bigcup}_{i \in I} \{int^{fs}(F^i_{A_i})\}$ is fuzzy soft dense in $\overline{E}$.

(ii) We say (U, E, ψ) is a fuzzy soft Baire space if every sequence $\{F^1_{A_1}, F^2_{A_2}, F^3_{A_3}, \ldots\}$ of (FSOSs) with the same closure F_A, we have $F_A = cl^{fs}(\widetilde{\bigcap}_{i\in I}\{F^i_{A_i}\})$.

(iii) We say (U, E, ψ) is a fuzzy soft Baire space if every fuzzy soft meager and fuzzy soft $\delta-$set in $\overline{E}$ is fuzzy soft nowhere dense.

Remark 23. We say $SS(U_E)$ is a family of all soft sets over a universe set U and the parameter set E. Moreover, the cardinality of $SS(U_E)$ is given by $n(SS(U_A)) = 2^{n(U)\times n(E)}$. Therefore, in this paper for each (FSS) F_E over (U, E) we can define F_E by using matrix form as follows:

$$F_E = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mk} \end{pmatrix}_{m\times k} \tag{2}$$

The order of this matrix is given by $m \times k$, where $m = n(E)$, $k = n(U)$, and $a_{ij} = \mu^{e_i}_{F_E}(c_j)$, $\forall 1 \le i \le m$ and $1 \le j \le k$.

Definition 24. A (FSS) F_A in a (FSTS) (U, E, ψ) is called fuzzy soft $\lambda_\sigma-$set in (U, E, ψ) if $F_A = \widetilde{\bigcup}_{i\in I}\{F^i_{A_i}\}$, where ${F^i_{A_i}}^c \in \psi$ $\forall i \in I$.

Definition 25. A (FSS) F_A in a (FSTS) (U, E, ψ) is called fuzzy soft $\sigma-$nowhere dense set if F_A is a fuzzy soft $\lambda_\sigma-$set in (U, E, ψ) such that $int^{fs}(F_A) = \phi$.

Example 26. As an illustration, consider the following example. Suppose the (FSSs) $F_E, G_E, H_E, J_E, K_E, L_E, T_E$ describe attractiveness of the cars with respect to the given parameters, which my friends are going to buy. $U = \{x_1, x_2, x_3\}$ which is the set of all cars under consideration. Let I^U be the collection of all fuzzy subsets of U. Also, let $E = \{e_1, e_2, e_3\}$, where e_1, e_2, e_3 stand for the attributes of cheap, qualification, and colorful, respectively. Let $F_E, G_E, H_E, J_E, K_E, L_E, T_E$ be defined as follows:

$$F_E = \begin{pmatrix} .2 & .3 & .6 \\ .15 & .25 & .10 \\ .3 & .12 & .6 \end{pmatrix},$$

$$G_E = \begin{pmatrix} .4 & .7 & .3 \\ .8 & .75 & .7 \\ .8 & .10 & .9 \end{pmatrix},$$

$$H_E = \begin{pmatrix} .1 & .5 & .3 \\ .5 & .75 & .4 \\ .8 & .4 & .1 \end{pmatrix},$$

$$J_E = \begin{pmatrix} .2 & .5 & .6 \\ .5 & .75 & .4 \\ .8 & .4 & .6 \end{pmatrix},$$

$$K_E = \begin{pmatrix} .2 & .3 & .3 \\ .15 & .25 & .10 \\ .3 & .10 & .6 \end{pmatrix},$$

$$L_E = \begin{pmatrix} .1 & .3 & .3 \\ .15 & .25 & .10 \\ .3 & .12 & .1 \end{pmatrix},$$

$$T_E = \begin{pmatrix} .4 & .7 & .6 \\ .8 & .75 & .7 \\ .8 & .12 & .9 \end{pmatrix}. \tag{3}$$

Then, $\psi = \{\Phi, \overline{E}, F_E, G_E, H_E, J_E, K_E, L_E, T_E\}$ is clearly a fuzzy soft topology on $\overline{E}$. Now consider the (FSS)

$$\alpha = (T_E)^c \,\widetilde{\cup}\, (J_E)^c = \begin{pmatrix} .8 & .5 & .4 \\ .5 & .25 & .6 \\ .2 & .88 & .4 \end{pmatrix} \tag{4}$$

in (U, E, ψ). Then α is a fuzzy soft $\lambda_\sigma-$set in (U, E, ψ) and $int^{fs}(\alpha) = \Phi$ and hence α is a fuzzy soft $\sigma-$nowhere dense set in (U, E, ψ). The (FSS)

$$\beta = (F_E)^c \,\widetilde{\cup}\, (H_E)^c = \begin{pmatrix} .9 & .7 & .7 \\ .85 & .75 & .90 \\ .7 & .88 & .9 \end{pmatrix} \tag{5}$$

is a fuzzy soft $\lambda_\sigma-$set in (U, E, ψ) and $int^{fs}(\beta) = F_E \neq \Phi$ and hence β is not a fuzzy soft $\sigma-$nowhere dense set in (U, E, ψ).

Definition 27. A (FSS) F_A in a (FSTS) (U, E, ψ) is called fuzzy soft $\sigma-$meager if $F_A = \widetilde{\bigcup}_{i\in I}\{F^i_{A_i}\}$, where $(F^i_{A_i})$*'s* are fuzzy soft $\sigma-$nowhere dense sets in (U, E, ψ). Any other (FSS) in (U, E, ψ) is said to be of fuzzy soft $\sigma-$second category.

Definition 28. A (FSTS) (U, E, ψ) is called fuzzy soft $\sigma-$meager space if the (FSS) $\overline{E}$ is a fuzzy soft $\sigma-$meager set in (U, E, ψ). That is, $\overline{E} = \widetilde{\bigcup}_{i\in I}\{F^i_{A_i}\}$, where $(F^i_{A_i})$*'s* are fuzzy soft $\sigma-$nowhere dense sets in (U, E, ψ). Otherwise, (U, E, ψ) will be called a fuzzy soft $\sigma-$second category space.

Definition 29. Assume F_A is a fuzzy soft $\sigma-$meager set in a (FSTS) (U, E, ψ). We say ${F_A}^c$ is a fuzzy soft $\sigma-$residual set in (U, E, ψ).

Proposition 30. *Let (U, E, ψ) be a (FSTS). Then the following are equivalent:*

(1) (U, E, ψ) is a fuzzy soft Baire space.

(2) $int^{fs}(F_A) = \Phi$ for every fuzzy soft meager set F_A in (U, E, ψ).

(3) $cl^{fs}(G_B) = \overline{E}$ for every fuzzy soft residual set G_B in (U, E, ψ).

Proof. (1) $\Longrightarrow$ (2). Let F_A be a fuzzy soft meager set in (U, E, ψ). Then $F_A = (\widetilde{\bigcup}_{i\in I}\{F^i_{A_i}\})$, where $(F^i_{A_i})$*'s*, $i \in I$ are fuzzy soft nowhere dense sets in (U, E, ψ). Then, we have $int^{fs}(F_A) = int^{fs}(\widetilde{\bigcup}_{i\in I}\{F^i_{A_i}\})$. Since (U, E, ψ) is a fuzzy soft Baire space, $int^{fs}(\widetilde{\bigcup}_{i\in I}\{F^i_{A_i}\}) = \Phi$. Hence, $int^{fs}(F_A) = \Phi$ for any fuzzy soft meager set F_A in (U, E, ψ).

(2) $\Longrightarrow$ (3). Let G_B be a fuzzy soft residual set in (U, E, ψ). Then ${G_B}^c$ is a fuzzy soft $\sigma-$meager set in (U, E, ψ). By hypothesis, $int^{fs}({G_B}^c) = \Phi$. Then $(cl^{fs}(G_B))^c = \Phi$. Hence, $cl^{fs}(G_B) = \overline{E}$ for any fuzzy soft residual set G_B in (U, E, ψ).

(3) $\Longrightarrow$ (1). Let F_A be a fuzzy soft meager set in (U, E, ψ). Then $F_A = (\widetilde{\bigcup}_{i\in I}\{F^i_{A_i}\})$, where $(F^i_{A_i})$'s are fuzzy soft nowhere dense sets in (U, E, ψ). Now F_A is a fuzzy soft meager set in (U, E, ψ) implying that ${F_A}^c$ is a fuzzy soft residual set in (U, E, ψ). By hypothesis, we have $cl^{fs}(F_A)^c = \overline{E}$. Then $(int^{fs}(F_A))^c = \overline{E}$. Hence, $int^{fs}(F_A) = \Phi$. That is, $int^{fs}(\widetilde{\bigcup}_{i\in I}\{F^i_{A_i}\}) = \Phi$, where $(F^i_{A_i})$'s are fuzzy soft nowhere dense sets in (U, E, ψ). Hence, (U, E, ψ) is a fuzzy soft Baire space. □

Proposition 31. *If F_A is a fuzzy soft dense and fuzzy soft $\delta-$set in a (FSTS) (U, E, ψ), then ${F_A}^c$ is a fuzzy soft meager set in (U, E, ψ).*

Proof. Since F_A is a fuzzy soft $\delta-$set in (U, E, ψ), $F_A = \widetilde{\bigcap}_{i\in I}\{F^i_{A_i}\}$, where $F^i_{A_i} \in \psi$ and since F_A is a fuzzy soft dense set in (U, E, ψ), $cl^{fs}(F_A) = \overline{E}$. Then $cl^{fs}(\widetilde{\bigcap}_{i\in I}\{F^i_{A_i}\}) = \overline{E}$. But $cl^{fs}(\widetilde{\bigcap}_{i\in I}\{F^i_{A_i}\}) \widetilde{\subseteq} \widetilde{\bigcap}_{i\in I}\{cl^{fs}(F^i_{A_i})\}$. Hence, $\overline{E} \widetilde{\subseteq} \widetilde{\bigcap}_{i\in I}\{cl^{fs}(F^i_{A_i})\}$. That is, $\widetilde{\bigcap}_{i\in I}\{cl^{fs}(F^i_{A_i})\} = \overline{E}$. Then we have $cl^{fs}(F^i_{A_i}) = \overline{E}$ for each $F^i_{A_i} \in \psi$ and hence $cl^{fs}(int^{fs}(F^i_{A_i})) = \overline{E}$ which implies that $(cl^{fs}(int^{fs}(F^i_{A_i})))^c = \Phi$ and hence $int^{fs}(cl^{fs}({F^i_{A_i}}^c)) = \Phi$. Therefore, $({F^i_{A_i}}^c)$ is a fuzzy soft nowhere dense set in (U, E, ψ). Now ${F_A}^c = (\widetilde{\bigcap}_{i\in I}\{F^i_{A_i}\})^c = \widetilde{\bigcup}_{i\in I}\{{F^i_{A_i}}^c\}$. Therefore, $F^c_A = \widetilde{\bigcup}_{i\in I}\{{F^i_{A_i}}^c\}$, where $({F^i_{A_i}}^c)$'s are fuzzy soft nowhere dense sets in (U, E, ψ). Hence, ${F_A}^c$ is a fuzzy soft meager set in (U, E, ψ). □

Lemma 32. *If F_A is a fuzzy soft dense and fuzzy soft $\delta-$set in a (FSTS) (U, E, ψ), then F_A is a fuzzy soft residual set in (U, E, ψ).*

Proof. Since F_A is a fuzzy soft dense and fuzzy soft $\delta-$set in (U, E, ψ), by Proposition 31, we have that ${F_A}^c$ is a fuzzy soft meager set in (U, E, ψ) and hence F_A is a fuzzy soft residual set in (U, E, ψ). □

Proposition 33. *In a (FSTS) (U, E, ψ), a (FSS) F_A is a fuzzy soft $\sigma-$nowhere dense set in (U, E, ψ) if and only if ${F_A}^c$ is a fuzzy soft dense and fuzzy soft $\delta-$set in (U, E, ψ).*

Proof. Let F_A be a fuzzy soft $\sigma-$nowhere dense set in (U, E, ψ). Then $F_A = (\widetilde{\bigcup}_{i\in I}\{F^i_{A_i}\})$, where ${F^i_{A_i}}^c \in \psi$, $\forall i \in I$ and $int^{fs}(F_A) = \Phi$. Then $(int^{fs}(F_A))^c = \Phi^c = \overline{E}$ implies that $cl^{fs}({F_A}^c) = \overline{E}$. Also ${F_A}^c = (\widetilde{\bigcup}_{i\in I}\{F^i_{A_i}\})^c = \widetilde{\bigcap}_{i\in I}\{{F^i_{A_i}}^c\}$, where ${F^i_{A_i}}^c \in \psi$, for $i \in I$. Hence, we have ${F^i_{A_i}}^c$ is a fuzzy soft dense and fuzzy soft $\delta-$set in (U, E, ψ).

Conversely, let F_A be a fuzzy soft dense and fuzzy soft $\delta-$set in (U, E, ψ). Then $F_A = \widetilde{\bigcap}_{i\in I}\{F^i_{A_i}\}$, where $F^i_{A_i} \in \psi$ for $i \in I$. Now ${F_A}^c = (\widetilde{\bigcap}_{i\in I}\{F^i_{A_i}\})^c = \widetilde{\bigcup}_{i\in I}\{{F^i_{A_i}}^c\}$. Hence, ${F_A}^c$ is a $\lambda_\sigma-$set in (U, E, ψ) and $int^{fs}({F_A}^c) = (cl^{fs}(F_A))^c = (\overline{E})^c = \Phi$ [since F_A is a fuzzy soft dense]. Therefore, ${F_A}^c$ is a fuzzy soft $\sigma-$nowhere dense set in (U, E, ψ). □

Definition 34. Assume (U, E, ψ) is a (FSTS). We say (U, E, ψ) is a fuzzy soft $\sigma-$Baire space if each sequence $\{F^1_{A_1}, F^2_{A_2}, F^3_{A_3}, \ldots\}$ of fuzzy soft $\sigma-$nowhere dense sets in (U, E, ψ) such that $int^{fs}(\widetilde{\bigcup}_{i\in I}\{F^i_{A_i}\}) = \Phi$.

Example 35. Let $U = \{c_1, c_2, c_3\}$ be the set of three flats and $E = \{\text{costly } (e_1), \text{modern } (e_2), \text{security services } (e_3)\}$ be the set of parameters. Then, we consider that $\psi = \{\Phi, \overline{E}, F_E, G_E, H_E,\} \widetilde{\cup} \{T^i_E \mid i = 1, 2, 3, \ldots, 14\}$ is a fuzzy soft topology on $\overline{E}$ defined as follows:

$$F_E = \begin{pmatrix} .7 & .5 & .6 \\ .6 & .4 & .5 \\ .5 & .3 & .4 \end{pmatrix}, \quad G_E = \begin{pmatrix} .5 & .8 & .7 \\ .4 & .7 & .6 \\ .3 & .6 & .5 \end{pmatrix}, \quad H_E = \begin{pmatrix} .6 & .4 & .8 \\ .5 & .3 & .7 \\ .4 & .2 & .6 \end{pmatrix}. \tag{6}$$

$T^1_E = F_E \widetilde{\cap} G_E$, $T^2_E = F_E \widetilde{\cap} H_E$, $T^3_E = G_E \widetilde{\cap} H_E$, $T^4_E = F_E \widetilde{\cup} G_E$, $T^5_E = F_E \widetilde{\cup} H_E$, $T^6_E = G_E \widetilde{\cup} H_E$, $T^7_E = F_E \widetilde{\cap} (G_E \widetilde{\cup} H_E)$, $T^8_E = F_E \widetilde{\cup} (G_E \widetilde{\cap} H_E)$, $T^9_E = G_E \widetilde{\cap} (F_E \widetilde{\cup} H_E)$, $T^{10}_E = G_E \widetilde{\cup} (F_E \widetilde{\cap} H_E)$, $T^{11}_E = H_E \widetilde{\cap} (F_E \widetilde{\cup} G_E)$, $T^{12}_E = H_E \widetilde{\cup} (F_E \widetilde{\cap} G_E)$, $T^{13}_E = F_E \widetilde{\cup} G_E \widetilde{\cup} H_E$, and $T^{14}_E = F_E \widetilde{\cap} G_E \widetilde{\cap} H_E$. Now consider the (FSSs)

$$\alpha = (F_E)^c \,\widetilde{\cup}\, (T^8_E)^c \,\widetilde{\cup}\, (T^{13}_E)^c = \begin{pmatrix} .3 & .5 & .4 \\ .4 & .6 & .5 \\ .5 & .7 & .6 \end{pmatrix}, \quad \beta = (G_E)^c \,\widetilde{\cup}\, (T^5_E)^c \,\widetilde{\cup}\, (T^{10}_E)^c = \begin{pmatrix} .5 & .5 & .3 \\ .6 & .6 & .4 \\ .7 & .7 & .5 \end{pmatrix}, \tag{7}$$

and

$$\theta = (H_E)^c \,\widetilde{\cup}\, (T^1_E)^c \,\widetilde{\cup}\, (T^4_E)^c \,\widetilde{\cup}\, (T^6_E)^c = \begin{pmatrix} .5 & .6 & .4 \\ .6 & .7 & .5 \\ .7 & .8 & .6 \end{pmatrix}, \tag{8}$$

in (U, E, ψ). Then α, β and θ are fuzzy soft $\lambda_\sigma-$sets in (U, E, ψ) and $int^{fs}(\alpha) = \Phi$, $int^{fs}(\beta) = \Phi$, and $int^{fs}(\theta) = \Phi$.Then α, β and θ are fuzzy soft $\sigma-$nowhere dense sets in (U, E, ψ).

Moreover, $(T_E^2)^c\tilde{\cup}(T_E^3)^c\tilde{\cup}(T_E^7)^c\tilde{\cup}(T_E^9)^c\tilde{\cup}(T_E^{11})^c\tilde{\cup}(T_E^{12})^c\tilde{\cup}(T_E^{14})^c = \theta$ and also $int^{fs}(\alpha\tilde{\cup}\beta\tilde{\cup}\theta) = int^{fs}(\theta) = \Phi$ and therefore (U, E, ψ) is a fuzzy soft $\sigma-$Baire space.

Remark 36. A fuzzy soft $\sigma-$Baire space need not be a fuzzy soft Baire space. For, consider the following example.

Example 37. Assume $X = \{c_1, c_2, c_3\}$ is a set of soldiers under consideration and $E = \{$ courageous (e_1); strong (e_2); smart(e_3) $\}$ is a set of parameters framed to choose the best soldier. Then, we consider that $\psi = \{\Phi, \overline{E}, F_E, G_E, H_E\}\tilde{\cup}\{T_E^i \mid i = 1, 2, 3, \ldots, 9\}$ is a fuzzy soft topology on $\overline{E}$ defined as follows:

$$H_E = \begin{pmatrix} .8 & .5 & .1 \\ .7 & .4 & .1 \\ .9 & .6 & .1 \end{pmatrix},$$

$$F_E = \begin{pmatrix} .1 & .3 & .8 \\ .1 & .2 & .7 \\ .1 & .4 & .9 \end{pmatrix}, \quad (9)$$

$$G_E = \begin{pmatrix} .4 & .1 & .3 \\ .3 & .1 & .2 \\ .5 & .1 & .4 \end{pmatrix}.$$

$T_E^1 = F_E\tilde{\cap}G_E$, $T_E^2 = F_E\tilde{\cap}H_E$, $T_E^3 = G_E\tilde{\cap}H_E$, $T_E^4 = F_E\tilde{\cup}G_E$, $T_E^5 = F_E\tilde{\cup}H_E$, $T_E^6 = G_E\tilde{\cup}H_E$, $T_E^7 = G_E\tilde{\cup}(F_E\tilde{\cap}H_E)$, $T_E^8 = H_E\tilde{\cup}(G_E\tilde{\cap}F_E)$, and $T_E^9 = H_E\tilde{\cap}(F_E\tilde{\cup}G_E)$. Now $\{{F_E}^c, {G_E}^c, {H_E}^c, (T_E^1)^c, (T_E^2)^c, (T_E^3)^c, (T_E^5)^c, (T_E^7)^c, (T_E^8)^c, (T_E^9)^c\}$ are fuzzy soft nowhere dense sets in (U, E, ψ). $(T_E^1)^c = {F_E}^c\tilde{\cup}{G_E}^c\tilde{\cup}{H_E}^c\tilde{\cup}(T_E^2)^c\tilde{\cup}(T_E^3)^c\tilde{\cup}(T_E^5)^c\tilde{\cup}(T_E^7)^c\tilde{\cup}(T_E^8)^c\tilde{\cup}(T_E^9)^c$. Therefore, $(T_E^1)^c$ is a fuzzy soft meager set in (U, E, ψ). $int^{fs}((T_E^1)^c) = T_E^1 \neq \Phi$. Hence, (U, E, ψ) is not a fuzzy soft Baire space. Now consider the (FSSs) $\alpha = (H_E)^c\tilde{\cup}(T_E^5)^c\tilde{\cup}(T_E^6)^c$, and $\beta = (F_E)^c\tilde{\cup}(T_E^2)^c\tilde{\cup}(T_E^4)^c\tilde{\cup}(T_E^7)^c\tilde{\cup}(T_E^8)^c$ in (U, E, ψ) and also $int^{fs}(\alpha) = \Phi$, $int^{fs}(\beta) = \Phi$. Then α and β are fuzzy soft $\sigma-$nowhere dense sets in (U, E, ψ). Now the (FSS) $(\alpha\tilde{\cup}\beta)$ is a fuzzy soft $\sigma-$meager set in (U, E, ψ) and $int^{fs}(\alpha\tilde{\cup}\beta) = \Phi$. Hence, (U, E, ψ) is a fuzzy soft $\sigma-$Baire space.

Proposition 38. *Let (U, E, ψ)be a (FSTS). Then the following are equivalent:*

(1) (U, E, ψ) is a fuzzy soft $\sigma-$Baire space.

(2) $int^{fs}(F_A) = \Phi$ for every fuzzy soft $\sigma-$meager set F_A in (U, E, ψ).

(3) $cl^{fs}(G_B) = \overline{E}$ for every fuzzy soft $\sigma-$residual set G_B in (U, E, ψ).

Proof. (1) $\Longrightarrow$ (2). Let F_A be a fuzzy soft $\sigma-$meager set in (U, E, ψ). Then $F_A = (\tilde{\bigcup}_{i\in I}\{F_{A_i}^i\})$, where $((F_{A_i}^i)$'s, $i \in I$ are fuzzy soft $\sigma-$nowhere dense sets in (U, E, ψ). Then, we have $int^{fs}(F_A) = int^{fs}(\tilde{\bigcup}_{i\in I}\{F_{A_i}^i\})$. Since (U, E, ψ) is a fuzzy soft $\sigma-$Baire space, $int^{fs}(\tilde{\bigcup}_{i\in I}\{F_{A_i}^i\}) = \Phi$. Hence, $int^{fs}(F_A) = \Phi$ for any fuzzy soft $\sigma-$meager set F_A in (U, E, ψ).

(2) $\Longrightarrow$ (3). Let G_B be a fuzzy soft $\sigma-$residual set in (U, E, ψ). Then ${G_B}^c$ is a fuzzy soft $\sigma-$meager set in (U, E, ψ). By hypothesis, $int^{fs}(G_B^c) = \Phi$. Then $(cl^{fs}(G_B^c))^c = \Phi$. Hence, $cl^{fs}(G_B) = \overline{E}$ for any fuzzy soft $\sigma-$residual set G_B in (U, E, ψ).

(3) $\Longrightarrow$ (1). Let F_A be a fuzzy soft $\sigma-$meager set in (U, E, ψ). Then $F_A = (\tilde{\bigcup}_{i\in I}\{F_{A_i}^i\})$, where $(F_{A_i}^i)$*'s* are fuzzy soft $\sigma-$nowhere dense sets in (U, E, ψ). Now F_A is a fuzzy soft $\sigma-$meager set in (U, E, ψ) implying that ${F_A}^c$ is a fuzzy soft $\sigma-$residual set in (U, E, ψ). By hypothesis, we have $cl^{fs}(F_A)^c = \overline{E}$. Then $(int^{fs}(F_A))^c = \overline{E}$. Hence, $int^{fs}(F_A) = \Phi$. That is, $int^{fs}(\tilde{\bigcup}_{i\in I}\{F_{A_i}^i\}) = \Phi$, where $(F_{A_i}^i)$*'s* are fuzzy soft $\sigma-$nowhere dense sets in (U, E, ψ). Hence, (U, E, ψ) is a fuzzy soft $\sigma-$Baire space. □

Proposition 39. *If the (FSTS) (U, E, ψ) is a fuzzy soft $\sigma-$Baire space, then $cl^{fs}(\tilde{\bigcup}_{i\in I}\{F_{A_i}^i\}) = \overline{E}$, where the (FSSs) $(F_{A_i}^i)$'s $\forall i \in I$ are fuzzy soft dense and fuzzy soft $\delta-$sets in (U, E, ψ).*

Proof. Let $(F_{A_i}^i)$*'s*, $i \in I$ be fuzzy soft dense and fuzzy soft $\delta-$sets in (U, E, ψ). By Proposition 33, $(F_{A_i}^i)$*'s* are fuzzy soft $\sigma-$nowhere dense sets in (U, E, ψ). Then the (FSS) $F_A = \tilde{\bigcup}_{i\in I}\{{F_{A_i}^i}^c\}$ is a fuzzy soft $\sigma-$meager set in (U, E, ψ). Now $int^{fs}(F_A) = int^{fs}(\tilde{\bigcup}_{i\in I}\{{F_{A_i}^i}^c\}) = int^{fs}(\tilde{\bigcap}_{i\in I}\{F_{A_i}^i\})^c = (cl^{fs}(\tilde{\bigcap}_{i\in I}\{F_{A_i}^i\}))^c$. Since (U, E, ψ) is a fuzzy soft $\sigma-$Baire space, by Proposition 38, we have $int^{fs}(F_A) = \Phi$. Then $(cl^{fs}(\tilde{\bigcap}_{i\in I}\{F_{A_i}^i\}))^c = \Phi$. This implies that $cl^{fs}(\tilde{\bigcap}_{i\in I}\{F_{A_i}^i\}) = \overline{E}$. □

Proposition 40. *If the (FSTS) (U, E, ψ) is a fuzzy soft $\sigma-$Baire space, then $int^{fs}(\tilde{\bigcup}_{i\in I}\{{F_{A_i}^i}^c\}) = \Phi$, where the (FSSs) sets $(F_{A_i}^i)$'s, $i \in I$ are fuzzy soft meager sets formed from the fuzzy soft dense and fuzzy soft $\delta-$sets $F_{A_i}^i$ in (U, E, ψ).*

Proof. Let the (FSTS) (U, E, ψ) be a fuzzy soft $\sigma-$Baire space and the (FSSs) $(F_{A_i}^i)$*'s*, $i \in I$ be fuzzy soft dense and fuzzy soft $\delta-$sets in (U, E, ψ). By Proposition 39, $cl^{fs}(\tilde{\bigcap}_{i\in I}\{F_{A_i}^i\}) = \overline{E}$. Then $(cl^{fs}(\tilde{\bigcap}_{i\in I}\{F_{A_i}^i\}))^c = \Phi$. This implies that $int^{fs}(\tilde{\bigcup}_{i\in I}\{{F_{A_i}^i}^c\}) = \Phi$. Also by Proposition 31, $(F_{A_i}^i)$*'s* are fuzzy soft meager sets in (U, E, ψ). Hence, $int^{fs}(\tilde{\bigcup}_{i\in I}\{{F_{A_i}^i}^c\}) = \Phi$, where the (FSSs) $(F_{A_i}^i)$*'s*, $i \in I$ are fuzzy soft meager sets formed from the fuzzy soft dense and fuzzy soft $\delta-$sets $F_{A_i}^i$ in (U, E, ψ). □

Proposition 41. *If the fuzzy soft meager sets are formed from the fuzzy soft dense and fuzzy soft $\delta-$sets in a fuzzy soft $\sigma-$Baire space (U, E, ψ), then (U, E, ψ) is a fuzzy soft Baire space.*

Proof. Let the (FSTS) (U, E, ψ) be a fuzzy soft $\sigma-$Baire space. By Proposition 40, $int^{fs}(\tilde{\bigcup}_{i\in I}\{{F_{A_i}^i}^c\}) = \Phi$, where the (FSSs) $(F_{A_i}^i)$*'s*, $i \in I$ are fuzzy soft meager sets formed from the fuzzy soft dense and fuzzy soft $\delta-$sets $F_{A_i}^i$ in (U, E, ψ). Now $\tilde{\bigcup}_{i\in I}(int^{fs}\{{F_{A_i}^i}^c\})\tilde{\subseteq}int^{fs}(\tilde{\bigcup}_{i\in I}\{{F_{A_i}^i}^c\})$. Then we have

$\widetilde{\bigcup}_{i\in I}\{int^{fs}(F^{i}_{A_i}{}^{c})\} = \Phi$. This implies that $int^{fs}(F^{i}_{A_i}{}^{c}) = \Phi$, where $(F^{i}_{A_i}{}^{c})$ is a fuzzy soft meager set in (U,E,ψ). By Proposition 30, (U,E,ψ) is a fuzzy soft Baire space. □

Proposition 42. *If the (FSTS) (U,E,ψ) is a fuzzy soft $\sigma-$meager space, then (U,E,ψ) is not a fuzzy soft $\sigma-$Baire space.*

Proof. Let the (FSTS) (U,E,ψ) be a fuzzy soft $\sigma-$meager space. Then $\widetilde{\bigcup}_{i\in I}\{F^{i}_{A_i}\} = \overline{E}$, where $(F^{i}_{A_i})$'s are fuzzy soft $\sigma-$nowhere dense sets in (U,E,ψ). Now $int^{fs}(\widetilde{\bigcup}_{i\in I}\{F^{i}_{A_i}\}) = int^{fs}(\overline{E}) \neq \Phi$. Hence, by definition, (U,E,ψ) is not a fuzzy soft $\sigma-$Baire space. □

Remark 43. By Proposition 42, we consider that if the (FSTS) (U,E,ψ) is a fuzzy soft $\sigma-$Baire space, then (U,E,ψ) is a fuzzy soft $\sigma-$second category space. Moreover, the converse is not true in general. That means a fuzzy soft $\sigma-$second category space need not be a fuzzy soft $\sigma-$Baire space.

Example 44. Let there be three houses in the universe U given by $U=\{s_1,s_2,s_3\}$ and let $E=\{$stone (e_1); steel (e_2); and brick $(e_3)\}$ be the set of parameters framed to choose one house to rent, where (brick) means the brick built houses, (steel) means the steel built houses, and (stone) means the stone built houses. Then, we consider $\psi = \{\Phi,\overline{E},F_E,G_E,H_E\}\widetilde{\cup}\{T^{i}_{E} \mid i = 1,2,3,\ldots,10\}$ is a fuzzy soft topology defined as follows:

$$F_E = \begin{pmatrix} .7 & .3 & .4 \\ .5 & .1 & .2 \\ .8 & .4 & .5 \end{pmatrix},$$
$$G_E = \begin{pmatrix} .4 & .5 & .6 \\ .2 & .3 & .4 \\ .5 & .6 & .7 \end{pmatrix}, \tag{10}$$
$$H_E = \begin{pmatrix} .6 & .4 & .7 \\ .4 & .2 & .5 \\ .7 & .5 & .8 \end{pmatrix}.$$

$T^{1}_{E} = F_E\widetilde{\cap}G_E$, $T^{2}_{E} = G_E\widetilde{\cap}H_E$, $T^{3}_{E} = F_E\widetilde{\cap}H_E$, $T^{4}_{E} = F_E\widetilde{\cup}G_E$, $T^{5}_{E} = G_E\widetilde{\cup}H_E$, $T^{6}_{E} = F_E\widetilde{\cup}H_E$, $T^{7}_{E} = G_E\widetilde{\cup}(F_E\widetilde{\cap}H_E)$, $T^{8}_{E} = F_E\widetilde{\cup}(G_E\widetilde{\cap}H_E)$, $T^{9}_{E} = H_E\widetilde{\cap}(F_E\widetilde{\cup}G_E)$, $T^{10}_{E} = F_E\widetilde{\cup}G_E\widetilde{\cup}H_E$. Now consider the (FSSs) $\alpha = (G_E)^c\widetilde{\cup}(T^{4}_{E})^c\widetilde{\cup}(T^{5}_{E})^c$, and $\beta = (F_E)^c\widetilde{\cup}(H_E)^c$ in (U,E,ψ). Then α and β are fuzzy soft $\lambda_\sigma-$sets in (U,E,ψ) and $int^{fs}(\alpha)=\Phi$, $int^{fs}(\beta)=\Phi$. Then α and β are fuzzy soft $\sigma-$nowhere dense sets in (U,E,ψ). Now $(\alpha\widetilde{\cup}\beta)\neq\overline{E}$. Therefore, (U,E,ψ) is a fuzzy soft $\sigma-$second category space. But $int^{fs}(\alpha\widetilde{\cup}\beta)\neq\Phi$ and therefore (U,E,ψ) is not a fuzzy soft $\sigma-$Baire space.

Proposition 45. *If $(\widetilde{\bigcap}_{i\in I}\{F^{i}_{A_i}\})\neq\Phi$, where the (FSSs) $(F^{i}_{A_i})$'s are fuzzy soft dense and fuzzy soft $\delta-$sets in a (FSTS) (U,E,ψ), then (U,E,ψ) is a fuzzy soft $\sigma-$second category space.*

Proof. Let $(F^{i}_{A_i})$'s, $i\in I$ be fuzzy soft dense and fuzzy soft $\delta-$sets in (U,E,ψ). By Proposition 33, $(F^{i}_{A_i}{}^{c})$'s are fuzzy soft $\sigma-$nowhere dense sets in (U,E,ψ). Now $(\widetilde{\bigcap}_{i\in I}\{F^{i}_{A_i}\})\neq\Phi$ implies that $(\widetilde{\bigcap}_{i\in I}\{F^{i}_{A_i}\})^c\neq\overline{E}$. Then $\widetilde{\bigcup}_{i\in I}\{(F^{i}_{A_i})^c\}\neq\overline{E}$. Hence, (U,E,ψ) is not a fuzzy soft $\sigma-$meager space and therefore (U,E,ψ) is a fuzzy soft $\sigma-$second category space. □

Proposition 46. *If F_A is a fuzzy soft $\sigma-$meager set in (U,E,ψ), then there is a fuzzy soft $\lambda_\sigma-$set G_B in (U,E,ψ) such that $F_A\widetilde{\subseteq}G_B$.*

Proof. Let F_A be a fuzzy soft $\sigma-$meager set in (U,E,ψ). Thus, $F_A=\widetilde{\bigcup}_{i\in I}\{F^{i}_{A_i}\}$, where $(F^{i}_{A_i})$'s are fuzzy soft $\sigma-$nowhere dense sets in (U,E,ψ). Now $[(cl^{fs}(F^{i}_{A_i}))^c]$'s are (FSOSs) in (U,E,ψ). Then $T=\widetilde{\bigcap}_{i\in I}\{(cl^{fs}(F^{i}_{A_i}))^c\}$is a fuzzy soft $\delta-$set in (U,E,ψ) and $T^c = (\widetilde{\bigcap}_{i\in I}\{(cl^{fs}(F^{i}_{A_i}))^c\})^c = \widetilde{\bigcup}_{i\in I}\{cl^{fs}(F^{i}_{A_i})\}$. Now $F_A = \widetilde{\bigcup}_{i\in I}\{F^{i}_{A_i}\}\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}\{cl^{fs}(F^{i}_{A_i})\} = T^c$. That is, $F_A\widetilde{\subseteq}T^c$ and T^c is a fuzzy soft $\lambda_\sigma-$set in (U,E,ψ). Let $G_B{}^c\widetilde{\subseteq}T$. Hence, if F_A is a fuzzy soft $\sigma-$meager set in (U,E,ψ), then there is a fuzzy soft $\lambda_\sigma-$set G_B in (U,E,ψ) such that $F_A\widetilde{\subseteq}G_B$. □

Proposition 47. *If G_B is a fuzzy soft $\sigma-$residual set in a (FSTS) (U,E,ψ) such that G_B in (U,E,ψ) such that $F_A\widetilde{\subseteq}G_B$, where F_A is a fuzzy soft dense and fuzzy soft $\delta-$set in (U,E,ψ), then (U,E,ψ) is a fuzzy soft $\sigma-$Baire space.*

Proof. Let G_B be a fuzzy soft $\sigma-$residual set in a (FSTS) (U,E,ψ). Thus, $G_B{}^c$ is a fuzzy soft $\sigma-$meager set in (U,E,ψ). Now by Proposition 46, there is a fuzzy soft $\lambda_\sigma-$set T in (U,E,ψ) such that $G_B{}^c\widetilde{\subseteq}T$. This implies that $T^c\widetilde{\subseteq}G_B$. Let $F_A = T^c$. Then F_A is a fuzzy soft $\delta-$set in (U,E,ψ) and $F_A\widetilde{\subseteq}G_B$ implies that $cl^{fs}(F_A)\widetilde{\subseteq}cl^{fs}(G_B)$. If $cl^{fs}(F_A)=\overline{E}$, then we have $cl^{fs}(G_B)=\overline{E}$. Hence, by Proposition 30, (U,E,ψ) is a fuzzy soft $\sigma-$Baire space. □

Proposition 48. *If the (FSTS) (U,E,ψ) is a fuzzy soft $\sigma-$Baire space and if $\widetilde{\bigcup}_{i\in I}\{F^{i}_{A_i}\}=\overline{E}$, then there exists at least one $\lambda_\sigma-$set $F^{i}_{A_i}$ such that $int^{fs}(F^{i}_{A_i})\neq\Phi$.*

Proof. Suppose that $int^{fs}(F_A) = \Phi$, $\forall$ $(i\in I)$, where $(F^{i}_{A_i})$'s are fuzzy soft $\sigma-$nowhere dense sets in (U,E,ψ). Then $\widetilde{\bigcup}_{i\in I}\{F^{i}_{A_i}\}=\overline{E}$, implying that $int^{fs}(\widetilde{\bigcup}_{i\in I}\{F^{i}_{A_i}\}) = int^{fs}(\overline{E}) = \overline{E}\neq\Phi$, a contradiction to (U,E,ψ) being a fuzzy soft $\sigma-$Baire space. Hence, $int(F^{i}_{A_i})\neq\Phi$, for at least one $\lambda_\sigma-$set $F^{i}_{A_i}$ in (U,E,ψ). □

Proposition 49. *If the (FSTS) (U,E,ψ) is a fuzzy soft $\sigma-$Baire space, then no nonempty (FSOS) is a fuzzy soft $\sigma-$meager set in (U,E,ψ).*

Proof. Let F_A be nonempty (FSOS) in a fuzzy soft $\sigma-$Baire space (U,E,ψ). Suppose that $F_A = \widetilde{\bigcup}_{i\in I}\{F^{i}_{A_i}\}$, where the (FSSs) $(F^{i}_{A_i})$'s are fuzzy soft $\sigma-$nowhere dense sets in (U,E,ψ). Then $int^{fs}(F_A)=int^{fs}(\widetilde{\bigcup}_{i\in I}\{F^{i}_{A_i}\})$. Since (U,E,ψ) is a fuzzy soft σ−Baire space, $int^{fs}(\widetilde{\bigcup}_{i\in I}\{F^i_{A_i}\}) = \Phi$. This implies that $int^{fs}(F_A) = \Phi$. Then we will have $F_A = \Phi$, which is a contradiction, since $F_A \in \psi$ implies that $int^{fs}(F_A) = F_A \neq \Phi$. Hence, no nonempty (FSOS) is a fuzzy soft σ−meager set in (U, E, ψ). □

Definition 50. A (FSTS) (U, E, ψ) is called a fuzzy soft submaximal space if for each (FSS) F_A in (U, E, ψ) such that $cl^{fs}(F_A) = \overline{E}$; then $F_A \in \psi$ in (U, E, ψ).

Proposition 51. *If the (FSTS) (U, E, ψ) is a fuzzy soft submaximal space and if F_A is a fuzzy soft σ−meager set in (U, E, ψ), then F_A is a fuzzy soft meager set in (U, E, ψ).*

Proof. Let $F_A = \widetilde{\bigcup}_{i\in I}\{F^i_{A_i}\}$ be a fuzzy soft σ−meager set in (U, E, ψ), where the (FSSs) $(F^i_{A_i})$'s, $i \in I$ are fuzzy soft σ−nowhere dense sets in (U, E, ψ). Then we have $int^{fs}(F^i_{A_i}) = \Phi$ and $(F^i_{A_i})$'s, $i \in I$ are fuzzy soft λ_σ−sets in (U, E, ψ). Now $int^{fs}(F^i_{A_i}) = \Phi$, implying that $(int^{fs}(F^i_{A_i}))^c = \Phi^c = \overline{E}$ and hence $cl^{fs}(F^{i\ c}_{A_i}) = \overline{E}$. Since (U, E, ψ) is a fuzzy soft submaximal space, the fuzzy soft dense sets $(F^{i\ c}_{A_i})$'s are (FSOSs) in (U, E, ψ) and hence $(F^i_{A_i})$'s are (FSCSs) in (U, E, ψ). Then $cl^{fs}(F^i_{A_i}) = (F^i_{A_i})$ and $int^{fs}(F^i_{A_i}) = \Phi$ imply that $int^{fs}(cl^{fs}(F^i_{A_i})) = int^{fs}(F^i_{A_i}) = \Phi$. That is, $(F^i_{A_i})$'s are fuzzy soft nowhere dense sets in (U, E, ψ). Therefore, $F_A = \widetilde{\bigcup}_{i\in I}\{F^i_{A_i}\}$ is a fuzzy soft meager set in (U, E, ψ). □

Proposition 52. *If the (FSTS) (U, E, ψ) is a fuzzy soft σ−Baire space and fuzzy soft submaximal space, then (U, E, ψ) is a fuzzy soft Baire space.*

Proof. Let F_A be a fuzzy soft σ−meager set in (U, E, ψ). Since (U, E, ψ) is a fuzzy soft submaximal space, by Proposition 51, F_A is a fuzzy soft meager set in (U, E, ψ). Since (U, E, ψ) is a fuzzy soft σ−Baire space, by Proposition 38, $int^{fs}(F_A) = \Phi$. Hence, for the fuzzy soft meager set F_A in (U, E, ψ), we have $int^{fs}(F_A) = \Phi$. Therefore, by Proposition 30, (U, E, ψ) is a fuzzy soft Baire space. □

Definition 53. A fuzzy soft P−space is a $(FSTS)(U, E, \psi)$ with the property that states that if countable intersection of fuzzy soft open sets in (U, E, ψ) is fuzzy soft open. That is, every non−empty fuzzy soft δ−set in (U, E, ψ) is fuzzy soft open in (U, E, ψ).

Proposition 54. *If the (FSTS) (U, E, ψ) is a fuzzy soft σ−Baire space and fuzzy soft P−space, then (U, E, ψ) is a fuzzy soft Baire space.*

Proof. Let the (FSTS) (U, E, ψ) be a fuzzy soft σ−Baire space. Then, by Proposition 39, $cl^{fs}(\widetilde{\bigcap}_{i\in I}\{F^i_{A_i}\}) = \overline{E}$, where the (FSSs) $(F^i_{A_i})$'s, $i \in I$ are fuzzy soft dense and fuzzy soft δ−sets in (U, E, ψ). Now from $cl^{fs}(\widetilde{\bigcap}_{i\in I}\{F^i_{A_i}\}) = \overline{E}$, we have $(cl^{fs}(\widetilde{\bigcap}_{i\in I}\{F^i_{A_i}\}))^c = \Phi$. This implies that $int^{fs}(\widetilde{\bigcup}_{i\in I}(F^i_{A_i})^c) = \Phi$. Since the (FSSs) $(F^i_{A_i})$'s are fuzzy soft dense in (U, E, ψ), $cl^{fs}(F^i_{A_i}) = \overline{E}$. Then we have $(cl^{fs}(F^i_{A_i}))^c = \Phi$. This implies that $int^{fs}(F^{i\ c}_{A_i}) = \Phi$. Also, since (U, E, ψ) is a fuzzy soft P−space, the non−empty fuzzy soft δ−sets $(F^i_{A_i})$'s in (U, E, ψ) are fuzzy soft open in (U, E, ψ). Then $(F^{i\ c}_{A_i})$'s are (FSCSs) in (U, E, ψ). Then $cl^{fs}(F^{i\ c}_{A_i}) = F^{i\ c}_{A_i}$ and $int^{fs}(F^{i\ c}_{A_i}) = \Phi$ imply that $int^{fs}(cl^{fs}(F^{i\ c}_{A_i})) = int^{fs}(F^{i\ c}_{A_i}) = \Phi$. That is, $(F^{i\ c}_{A_i})$'s are fuzzy soft nowhere dense sets in (U, E, ψ). Therefore, we have $int^{fs}(\widetilde{\bigcup}_{i\in I}\{F^{i\ c}_{A_i}\}) = \Phi$, where $(F^{i\ c}_{A_i})$'s are fuzzy soft nowhere dense sets in (U, E, ψ). Hence, by Proposition 30, (U, E, ψ) is a fuzzy soft Baire space. □

Definition 55. A (FSTS) (U, E, ψ) is called a fuzzy soft almost resolvable space if $\widetilde{\bigcup}_{i\in I}\{F^i_{A_i}\} = \overline{E}$, where the (FSSs) $(F^i_{A_i})$'s in (U, E, ψ) are such that $int^{fs}(F^i_{A_i}) = \Phi$. Otherwise, (U, E, ψ) is called a fuzzy soft almost irresolvable space.

Proposition 56. *If the (FSTS) (U, E, ψ) is a fuzzy soft almost irresolvable space, then (U, E, ψ) is a fuzzy soft σ−second category space.*

Proof. Let $(F^i_{A_i})$'s, $i \in I$ be the fuzzy soft dense and fuzzy soft δ−sets in (U, E, ψ). Now $cl^{fs}(F^i_{A_i}) = \overline{E}$ implies that $(cl^{fs}(F^i_{A_i}))^c = \Phi$. That is, $int^{fs}(F^{i\ c}_{A_i}) = \Phi$. Since (U, E, ψ) is a fuzzy soft almost irresolvable space, $\widetilde{\bigcup}_{i\in I}\{F^{i\ c}_{A_i}\} \neq \overline{E}$, where the (FSSs) $(F^i_{A_i})$'s in (U, E, ψ) are such that $int^{fs}(F^{i\ c}_{A_i}) = \Phi$. Now $\widetilde{\bigcup}_{i\in I}\{F^{i\ c}_{A_i}\} \neq \overline{E}$ implies that $(\widetilde{\bigcup}_{i\in I}\{(F^{i\ c}_{A_i}\})^c \neq \Phi$. Hence, we have $\widetilde{\bigcap}_{i\in I}\{F^i_{A_i}\} \neq \Phi$, where the (FSSs) $(F^i_{A_i})$'s are fuzzy soft dense and fuzzy soft δ−sets in a (FSTS) (U, E, ψ). Thus, by Proposition 45, (U, E, ψ) is a fuzzy soft σ−second category space. □

Definition 57. A (FSTS) (U, E, ψ) is called a fuzzy soft hyperconnected space if every (FSOS) F_A is fuzzy soft dense in (U, E, ψ). That is, $cl^{fs}(F_A) = \overline{E}\ \forall \Phi \neq F_A \in \psi$.

Proposition 58. *If $cl^{fs}(\widetilde{\bigcap}_{i\in I}\{F^i_{A_i}\}) = \overline{E}$, where $(F^i_{A_i})$'s are fuzzy soft dense and fuzzy soft δ−sets in (U, E, ψ), then (U, E, ψ) is a fuzzy soft σ−Baire space.*

Proof. The proof is obvious. □

Proposition 59. *If $cl^{fs}(\widetilde{\bigcap}_{i\in I}\{F^i_{A_i}\}) = \overline{E}$, where the (FSSs) $(F^i_{A_i})$'s are fuzzy soft δ−sets in a fuzzy soft hyperconnected and fuzzy soft P−space (U, E, ψ), then (U, E, ψ) is a fuzzy soft σ−Baire space.*

Proof. Let $(F^i_{A_i})$'s, $i \in I$ be the fuzzy soft δ−sets in (U, E, ψ) such that $cl^{fs}(\widetilde{\bigcap}_{i\in I}\{F^i_{A_i}\}) = \overline{E}$. Since (U, E, ψ) is a fuzzy soft P−space, the fuzzy soft δ−sets $(F^i_{A_i})$'s in (U, E, ψ) are fuzzy soft open in (U, E, ψ). Also since (U, E, ψ) is a fuzzy soft hyperconnected space, the (FSOSs) $(F^i_{A_i})$'s in (U, E, ψ) are fuzzy soft dense sets in (U, E, ψ). Hence, the (FSSs) $(F^i_{A_i})$'s, $i \in I$ are fuzzy soft dense and fuzzy soft δ−sets in (U, E, ψ) and $cl^{fs}(\widetilde{\bigcap}_{i\in I}\{F^i_{A_i}\}) = \overline{E}$. Hence, by Proposition 58, (U, E, ψ) is a fuzzy soft σ−Baire space. □

Definition 60. Let F_A be a fuzzy soft subset of a fuzzy soft space $\overline{E}$. Then F_A is said to be fuzzy soft A−embedded in $\overline{E}$ if each fuzzy soft δ−subset G_B of $\overline{E}$ which is contained in F_A is fuzzy soft nowhere dense in F_A (i.e., $int^{fs}_{F_A}(cl^{fs}_{F_A}(H_C)) = \Phi$).

Proposition 61. *Let F_A be a fuzzy soft dense subspace of a fuzzy soft Baire space $\overline{E}$. If $\overline{E} \setminus F_A$ is fuzzy soft A−embedded in $\overline{E}$, then F_A is a fuzzy soft Baire space.*

Proof. Observe that if F_A is not a fuzzy soft Baire space; then there is a sequence $G_B \widetilde{\supseteq} F^1_{A_1} \widetilde{\supseteq} F^2_{A_2} \widetilde{\supseteq} F^3_{A_3} \widetilde{\supseteq} \ldots$ of fuzzy soft open subsets of F_A such that each $F^i_{A_i}$ is fuzzy soft dense in G_B and yet $\widetilde{\bigcap}_{i\in I}\{F^i_{A_i}\} = \Phi$. Then there is a sequence $H_C \widetilde{\supseteq} K^1_{D_1} \widetilde{\supseteq} K^2_{D_2} \widetilde{\supseteq} K^3_{D_3} \widetilde{\supseteq} \ldots$ of fuzzy soft open subsets of $\overline{E}$ such that $G_B = H_C \widetilde{\cap} F_A$ and $F^i_{A_i} = K^i_{D_i} \widetilde{\cap} F_A$. Each $K^i_{D_i}$ is fuzzy soft dense in H_C and H_C is a fuzzy soft Baire space. Hence, $\widetilde{\bigcap}_{i\in I}\{K^i_{D_i}\}$ is fuzzy soft dense in H_C and therefore in $H_C \widetilde{\cap} \overline{E} \setminus F_A$. Since $\widetilde{\bigcap}_{i\in I}\{K^i_{D_i}\} \widetilde{\subseteq} \overline{E} \setminus F_A, \overline{E} \setminus F_A$ is not fuzzy soft A−embedded in $\overline{E}$. □

Proposition 62. *Let F_A be a fuzzy soft dense subspace of a fuzzy soft Baire space $\overline{E}$. If $\overline{E} \setminus F_A$ is dense in $\overline{E}$, then F_A is a fuzzy soft Baire space if and only if $\overline{E} \setminus F_A$ is fuzzy soft A−embedded in $\overline{E}$.*

Proof. Assume that $\overline{E} \setminus F_A$ is not fuzzy soft A−embedded in $\overline{E}$. Let G_B be a fuzzy soft δ−subset of $\overline{E}$ which is contained in $\overline{E} \setminus F_A$ and which is fuzzy soft dense in some relatively (FSOS) H_C of $\overline{E} \setminus F_A$. Let K_D be an fuzzy soft open subset of $\overline{E}$ with $K_D \widetilde{\cap} (\overline{E} \setminus F_A) = H_C$. Then $T_M = K_D \widetilde{\cap} G_B$ is a δ−subset of $\overline{E}$ which is fuzzy soft dense in K_D and which is contained in H_C. Let $T_M = \widetilde{\bigcap}_{i\in I}\{T^i_{M_i}\}$, where each $T^i_{M_i}$ is open in $\overline{E}$ and $T^i_{M_i} \widetilde{\subseteq} K_D$. The (FSSs) $\{T^i_{M_i} \widetilde{\cap} F_A\}$ are fuzzy soft open and fuzzy soft dense subsets of $K_D \widetilde{\cap} F_A$ and yet $\widetilde{\bigcap}_{i\in I}\{T^i_{M_i} \widetilde{\cap} F_A\}$. It follows that $K_D \widetilde{\cap} F_A$ is not a fuzzy soft Baire space. Consequently, F_A is not a fuzzy soft Baire space. Conversely, assume that $\overline{E} \setminus F_A$ is fuzzy soft A−embedded in $\overline{E}$ and, by Proposition 61, then F_A is a fuzzy soft Baire space. □

Lemma 63. *Let $\overline{E}$ be a fuzzy soft Baire space. If $G_B = \widetilde{\bigcap}_{i\in I}\{T^i_{M_i}\}$ is a nonempty fuzzy soft nowhere dens δ−set of $\overline{E}$, where $T^i_{M_i}$ is a fuzzy soft open subset of $\overline{E}$ $\forall i \in I$, then for every nonempty fuzzy soft open subset T_M of $\overline{E}$ there is $i \in I$ such that $int^{fs}((\overline{E} \setminus T^i_{M_i}) \widetilde{\cap} (T_M \setminus cl^{fs}(G_B))) \neq \Phi$.*

Proof. Let T_M be a nonempty fuzzy soft open subset of $\overline{E}$. Then, $T_M \setminus cl^{fs}(G_B)$ is a nonempty fuzzy soft open subset of $\overline{E}$; hence, $T_M \setminus cl^{fs}(G_B)$ is also fuzzy soft Baire. Since $T_M \setminus cl^{fs}(G_B) \widetilde{\subseteq} \widetilde{\bigcup}_{i\in I}(\overline{E} \setminus T^i_{M_i})$ and each $\overline{E} \setminus T^i_{M_i}$ is a fuzzy soft closed subset of $\overline{E}$, by Remark 22, then there is $i \in I$ such that $int^{fs}((\overline{E} \setminus T^i_{M_i}) \widetilde{\cap} (T_M \setminus cl^{fs}(G_B))) \neq \Phi$. □

Proposition 64. *Let $\overline{E}$ be a fuzzy soft Baire space and let $F_A \widetilde{\subseteq} \overline{E}$ be fuzzy soft dense. Then F_A is a fuzzy soft Baire space if and only if every fuzzy soft δ−set in $\overline{E}$ contained in $\overline{E} \setminus F_A$ is fuzzy soft nowhere dense.*

Proof.

Necessity. Let $G_B = \widetilde{\bigcap}_{i\in I}\{T^i_{M_i}\}$, where $T^i_{M_i}$ is a fuzzy soft open subset of $\overline{E}$ for each $i \in I$, which is contained in $\overline{E} \setminus F_A$. Then, $F_A \widetilde{\subseteq} \widetilde{\bigcup}_{i\in I}(\overline{E} \setminus T^i_{M_i})$. In virtue of Remark 22, $\widetilde{\bigcup}_{i\in I} int^{fs}_{F_A}(F_A \widetilde{\cap} (\overline{E} \setminus T^i_{M_i}))$ is fuzzy soft dense in F_A. Suppose that $int_{fs}(cl^{fs}(G_B)) \neq \Phi$. Then, there is $n \in I$ such that $\Phi \neq int^{fs}(cl^{fs}(G_B)) \widetilde{\cap} int^{fs}_{F_A}(F_A \widetilde{\cap} (\overline{E} \setminus T^n_{M_n}))$. On the other hand, we know that $cl^{fs}(G_B) \widetilde{\subseteq} cl^{fs}(T^n_{M_n}) = cl^{fs}(T^n_{M_n} \widetilde{\cap} F_A)$. Hence, $\Phi \neq int^{fs}(cl^{fs}(G_B)) \widetilde{\cap} int^{fs}_{F_A}(F_A \widetilde{\cap} (\overline{E} \setminus T^n_{M_n})) \widetilde{\subseteq} cl^{fs}(T^n_{M_n} \widetilde{\cap} F_A) \widetilde{\cap} F_A = cl^{fs}_{F_A}(T^n_{M_n} \widetilde{\cap} F_A)$ which implies that $int^{fs}(cl^{fs}(G_B)) \widetilde{\cap} int^{fs}_{F_A}(F_A \widetilde{\cap} (\overline{E} \setminus T^n_{M_n})) \widetilde{\cap} cl^{fs}(T^n_{M_n} \widetilde{\cap} F_A) \neq \Phi$, but this is impossible.

Sufficiency. Assume that F_A is no fuzzy soft Baire. According to Remark 22, there is a countable fuzzy soft closed cover $\{F^i_{A_i}; i \in I\}$ of F_A such that $\widetilde{\bigcup}_{i\in I}\{int^{fs}(F^i_{A_i})\}$ is not dense in F_A. For each $i \in I$, choose a fuzzy soft closed subset $T^i_{M_i}$ of $\overline{E}$ such that $F^i_{A_i} = T^i_{M_i} \widetilde{\cap} F_A$, for each $i \in I$. Let $T_M = \widetilde{\bigcap}_{i\in I}(\overline{E} \setminus T^i_{M_i})$ which is a δ−set of $\overline{E}$ contained in $\overline{E} \setminus F_A$. If $T_M = \Phi$, then $\{T^i_{M_i}; i \in I\}$ would be a fuzzy soft closed cover of $\overline{E}$ and, by Remark 22, then $\widetilde{\bigcup}_{i\in I}\{int^{fs}(T^i_{M_i})\}$ would be fuzzy soft dense in $\overline{E}$ which is not possible. So, $T_M \neq \Phi$. Choose a nonempty fuzzy soft open subset K_D of $\overline{E}$ such that $K_D \widetilde{\cap} F_A \widetilde{\cap} int^{fs}_{F_A}(F^i_{A_i}) = \Phi$, $\forall i \in I$. By Lemma 63, we can find $r \in I$ such that $int^{fs}(T^r_{M_r}) \widetilde{\cap} (K_D \setminus int^{fs}(T_M)) \neq \Phi$. Hence, $\Phi \neq int^{fs}(T^r_{M_r}) \widetilde{\cap} (K_D \setminus int^{fs}(T_M)) \widetilde{\cap} F_A \widetilde{\subseteq} int^{fs}_{F_A}(T^r_{M_r} \widetilde{\cap} F_A) \widetilde{\cap} K_D \widetilde{\cap} F_A \widetilde{\subseteq} int^{fs}_{F_A}(F^i_{A_i}) \widetilde{\cap} K_D \widetilde{\cap} F_A$, but this is a contradiction. Thus, F_A is fuzzy soft Baire. □

4. Baireness in Fuzzy Soft Setting

In this section, we shall study the new class of fuzzy soft Baire spaces.

Definition 65. We say a space $\overline{E}$ is fuzzy soft D−Baire if every fuzzy soft dense subspace of $\overline{E}$ is fuzzy soft Baire.

An immediate consequence of Proposition 64 is the following.

Corollary 66. *Suppose that $\overline{E}$ is a fuzzy soft Baire space. Then, every fuzzy soft $\delta-$set in $\overline{E}$ with empty interior is fuzzy soft nowhere dense iff $\overline{E}$ is fuzzy soft D−Baire*

Proof. It follows from Proposition 64 and Definition 65. □

Definition 67. We say that a (FSTS) $\overline{E}$ is a fuzzy soft almost $P-$space if every non−empty fuzzy soft $\delta-$set in $\overline{E}$ has a nonempty interior.

Corollary 68. *Every fuzzy soft Baire and fuzzy soft almost P−space is fuzzy soft D−Baire.*

Proof. This is a consequence of Proposition 64 and Definition 67. □

Definition 69. A fuzzy soft Borel set is any (FSS) in a (FSTS) that can be formed from (FSOSs) (or, equivalently, from (FSCSs)) through the operations of countable union, countable intersection, and relative complement.

Definition 70. Let $\overline{E}$ be a (FSTS). Then, the class $FSPB(\overline{E})$ is the fuzzy soft $\sigma-$algebra in $\overline{E}$ generated by all (FSOSs) and all fuzzy soft nowhere dense sets.

Remark 71. (1) For a (FSTS) (U, E, ψ), the collection of all fuzzy soft Borel sets on $\overline{E}$ forms a fuzzy soft $\sigma-$algebra.

(2) The fuzzy soft $\sigma-$algebra of fuzzy soft Borel sets is contained in the class $FSPB(\overline{E})$.

(3) It is clear to show that $F_A\tilde{\subseteq}\overline{E}$ belongs to the class $FSPB(\overline{E})$ if and only if F_A may be expressed in the form $F_A = G_B\tilde{\cup}H_C$, where G_B is a fuzzy soft $\delta-$set and H_C is fuzzy soft meager.

Theorem 72. *The following seven conditions on a space (U, E, ψ) are equivalent.*

(1) (U, E, ψ) is fuzzy soft D−Baire.

(2) (U, E, ψ) is fuzzy soft Baire and every fuzzy soft $\delta-$set with empty interior is fuzzy soft nowhere dense.

(3) Every fuzzy soft meager subset $F_A\tilde{\subseteq}\overline{E}$ is fuzzy soft nowhere dense.

(4) (U, E, ψ) is fuzzy soft Baire and every fuzzy soft dense $\delta-$set has fuzzy soft dense interior.

(5) (U, E, ψ) is fuzzy soft Baire and every set in the class $FSPB(\overline{E})$ with empty interior is fuzzy soft nowhere dense.

(6) (U, E, ψ) is fuzzy soft Baire and every fuzzy soft Borel set with empty interior is fuzzy soft nowhere dense.

(7) (U, E, ψ) is fuzzy soft Baire and the union of a fuzzy soft $\delta-$set with empty interior and a fuzzy soft meager set of $\overline{E}$ is fuzzy soft nowhere dense.

Proof. (1)⟺(2). This is Corollary 66.

(2) ⟹ (3). Let $F_A\tilde{\subseteq}\overline{E}$ be a fuzzy soft meager set. Assume $F_A = \tilde{\bigcup}_{i\in I}\{F^i_{A_i}\}$, where $F^i_{A_i}$ is fuzzy soft nowhere dense $\forall i \in I$. Therefore, $L_M = \overline{E} \setminus \tilde{\bigcup}_{i\in I}\{cl^{fs}(F^i_{A_i})\} = \tilde{\bigcap}_{i\in I}\{\overline{E} \setminus cl^{fs}(F^i_{A_i})\}$ is a $\delta-$set in $\overline{E}$ and L_M is fuzzy soft dense in $\overline{E}$ because its complement is a fuzzy soft meager set and $\overline{E}$ is fuzzy soft Baire. Let $V_D = int^{fs}(L_M)$. The (FSS) $L_M - V_D$ clearly has empty interior. Hence, $L_M - cl^{fs}(V_D)$ is a fuzzy soft $\delta-$set with empty interior; by hypothesis, $L_M - cl^{fs}(V_D)$ is fuzzy soft nowhere dense. Also $L_M\tilde{\cap}Fr^{fs}(V_D)$ is a fuzzy soft nowhere dense set. Therefore, $L_M - V_D = (L_M - cl^{fs}(V_D))\tilde{\cup}(L_M\tilde{\cap}Fr^{fs}(V_D))$ is a fuzzy soft nowhere dense set as well. On the other hand, $\overline{E} \setminus V_D = (L_M \setminus V_D)\tilde{\cup}(\overline{E} \setminus L_M) = (L_M \setminus V_D)\tilde{\cup}(\bigcup_{i\in I}\{F^i_{A_i}\})$ is a fuzzy soft meager set. Since $\overline{E}$ is fuzzy soft Baire, $\Phi = int^{fs}(\overline{E} \setminus V_D) = \overline{E} \setminus cl^{fs}(V_D)$. Therefore, $cl^{fs}(V_D) = \overline{E}$ and $F_A\tilde{\subseteq}\overline{E}\setminus V_D = Fr^{fs}(V_D)$ is fuzzy soft nowhere dense.

(3) ⟹ (4). It follows from Remark 22 that $\overline{E}$ is a fuzzy soft Baire space. Let $L_M\tilde{\subseteq}\overline{E}$ be a fuzzy soft dense $\delta-$set of $\overline{E}$. Since $\overline{E}\setminus L_M$ is a fuzzy soft meager set, the hypothesis implies that $\overline{E} \setminus L_M$ is fuzzy soft nowhere dense; i.e., $cl^{fs}(\overline{E} \setminus L_M)$ has empty interior. Therefore, $V_D = \overline{E} \setminus cl^{fs}(\overline{E} \setminus L_M) = int^{fs}(L_M)$ is a fuzzy soft open dense subspace of $\overline{E}$.

(4) ⟹ (2). Let G_B be a $\delta-$set with empty. First observe that $int^{fs}(cl^{fs}(G_B))\tilde{\subseteq}cl^{fs}(cl^{fs}(G_B) \setminus G_B)$. Since $cl^{fs}(G_B) \setminus G_B$ is an $\lambda_\sigma-$set with empty interior, $\overline{E} \setminus (cl^{fs}(G_B) \setminus G_B)$ is a fuzzy soft dense $\delta-$set of $\overline{E}$ By assumption, $int^{fs}(\overline{E}\setminus (cl^{fs}(G_B)\setminus G_B))$ is also fuzzy soft dense in $\overline{E}$. That is, $\overline{E} \setminus cl^{fs}(cl^{fs}(G_B) \setminus G_B)$ is fuzzy soft dense in $\overline{E}$. Hence, $int^{fs}(cl^{fs}(cl^{fs}(G_B) \setminus G_B)) = \Phi$ and so $int^{fs}(cl^{fs}(G_B)) = \Phi$.

(4) ⟹ (5). We have already established above the equivalence among clauses (1), (2), (3), and (4). The fifth clause follows directly from the properties of the class $FSPB(\overline{E})$ and clauses (2) and (3).

(5) ⟹ (6). This implication is obvious because the fuzzy soft $\delta-$algebra of fuzzy soft Borel sets is contained in the class $FSPB(\overline{E})$.

(6) ⟹ (1). It is enough to observe that (6) ⟹ (2) ⟹ (1).

(1)⟹(7). We know the first six statements are equivalent to each other. Thus, clause (7) follows directly from clauses (2) and (3). (7) ⟹ (1). This is a consequence of Corollary 66. □

5. Conclusion

In the present paper, we have introduced and discussed new notions of Baireness in fuzzy soft topological spaces. Furthermore, there are many problems and applications in algebra that deal with group theory and spaces. So, future work in this regard would be required to study some applications using the properties of ψ in our new fuzzy soft spaces and new operations depend on fuzzy soft operations $\tilde{\cup}$ and $\tilde{\cap}$ to consider new fuzzy soft groups and fuzzy soft commutative rings. Also, let us say (U, E, ψ) is fuzzy soft $N-$Baire if every fuzzy soft set in (U, E, ψ) with empty interior is fuzzy soft nowhere dense. The question we are concerned with is as follows: what are the possible relationships considered between fuzzy soft $N-$Baire and each concept of our notions that are given in this work?

Data Availability

Data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

References

[1] R. C. Haworth and R. A. McCoy, "Baire spaces," *Dissertationes Math*, vol. 141, pp. 1–77, 1977.

[2] Z. e. Frolik, "Baire spaces and some generalizations of complete metric spaces," *Czechoslovak Mathematical Journal*, vol. 11 (86), pp. 237–248, 1961.

[3] G. Gruenhage and D. Lutzer, "Baire and Volterra spaces," *Proceedings of the American Mathematical Society*, vol. 128, no. 10, pp. 3115–3124, 2000.

[4] T. Neubrunn, "A note on mappings of Baire spaces," *Math. Slovaca*, vol. 27, no. 2, pp. 173–176, 1977.

[5] G. Thangaraj and G. Balasubramanian, "On somewhat fuzzy continuous functions," *Journal of Fuzzy Mathematics*, vol. 11, no. 3, pp. 725–736, 2003.

[6] G. Thangaraj and S. Anjalmose, "On fuzzy Baire spaces," *Journal of Fuzzy Mathematics*, vol. 21, no. 3, pp. 667–676, 2013.

[7] G. Thangaraj and S. Anjalmose, "On fuzzy D-Baire spaces," *Annals of Fuzzy Mathematics and Informatics*, vol. 7, no. 1, pp. 99–108, 2014.

[8] D. Molodtsov, "Soft set theory-first results," *Computers & Mathematics with Applications*, vol. 37, no. 4-5, pp. 19–31, 1999.

[9] P. K. Maji, A. R. Roy, and R. Biswas, "An application of soft sets in a decision making problem," *Computers & Mathematics with Applications*, vol. 44, no. 8-9, pp. 1077–1083, 2002.

[10] P. K. Maji, R. Biswas, and A. R. Roy, "Soft set theory," *Computers & Mathematics with Applications*, vol. 45, no. 4-5, pp. 555–562, 2003.

[11] S. Mahmood, "Soft Regular Generalized b-Closed Sets in Soft Topological Spaces," *Journal of Linear and Topological Algebra*, vol. 3, no. 4, pp. 195–204, 2014.

[12] S. Mahmood, "Tychonoff Spaces in Soft Setting and their Basic Properties," *International Journal of Applications of Fuzzy Sets and Artificial Intelligence*, vol. 7, pp. 93–112, 2017.

[13] S. Mahmood, "Soft Sequentially Absolutely Closed Spaces," *International Journal of Applications of Fuzzy Sets and Artificial Intelligence*, vol. 7, pp. 73–92, 2017.

[14] S. Mahmood and M. Alradha, "Soft Edge ρ-Algebras of the power sets," *International Journal of Applications of Fuzzy Sets and Artificial Intelligence*, vol. 7, pp. 231–243, 2017.

[15] S. M. Khalil and A. F. al Musawi, "Soft BCL-algebras of the power sets," *International Journal of Algebra*, vol. 11, pp. 329–341, 2017.

[16] S. Mahmood and M. Abd Ulrazaq, "Soft BCH-Algebras of the Power Sets," *American Journal of Mathematics and Statistics*, vol. 8, no. 1, pp. 1–7, 2018.

[17] S. M. Khalil and F. H. Khadhaer, "An algorithm for generating permutation algebras using soft spaces," *Journal of Taibah University for Science*, vol. 12, no. 3, pp. 299–308, 2018.

[18] S. Mahmood and F. Hameed, "An algorithm for generating permutations in symmetric groups using soft spaces with general study and basic properties of permutation spaces," *Journal of Theoretical and Applied Information Technology*, vol. 96, no. 9, pp. 2445–2457, 2018.

[19] S. Mahmood, "Dissimilarity fuzzy soft points and their applications," *Fuzzy Information and Engineering*, vol. 8, no. 3, pp. 281–294, 2016.

[20] P. K. Maji, R. Biswas, and A. R. Roy, "Fuzzy soft sets," *Journal of Fuzzy Mathematics*, vol. 9, no. 3, pp. 589–602, 2001.

[21] P. K. Maji, R. Biswas, and A. R. Roy, "Intuitionistic fuzzy soft sets," *Journal of Fuzzy Mathematics*, vol. 9, no. 3, pp. 677–692, 2001.

[22] S. Mahmood and Z. Al-Batat, "Intuitionistic Fuzzy Soft LA-Semigroups and Intuitionistic Fuzzy Soft Ideals," *International Journal of Applications of Fuzzy Sets and Artificial Intelligence*, vol. 6, pp. 119–132, 2016.

[23] S. Mahmood Khalil, "On intuitionistic fuzzy soft b-closed sets in intuitionistic fuzzy soft topological spaces," *Annals of Fuzzy Mathematics and Informatics*, vol. 10, no. 2, pp. 221–233, 2015.

[24] K. Tyagi and A. Tripathi, "Rough fuzzy automata and rough fuzzy grammar," *International Journal of Fuzzy System Applications*, vol. 6, no. 1, pp. 36–55, 2017.

[25] A. Taeib and A. Chaari, "Optimal tuning strategy for mimo fuzzy predictive controllers," *International Journal of Fuzzy System Applications*, vol. 4, no. 4, pp. 87–99, 2015.

[26] H. Zoulfaghari, J. Nematian, and A. A. K. Nezhad, "A resource-constrained project scheduling problem with fuzzy activity times," *International Journal of Fuzzy System Applications*, vol. 5, no. 4, pp. 1–15, 2016.

[27] N. Cagman, S. Enginoglu, and F. Citak, "Fuzzy soft set theory and its applications," *Iranian Journal of Fuzzy Systems*, vol. 8, no. 3, pp. 137–147, 170, 2011.

[28] S. Roy and T. K. Samanta, "An introduction to open and closed sets on fuzzy soft topological spaces," *Annals of Fuzzy Mathematics and Informatics*, vol. 6, no. 2, pp. 425–431, 2013.