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Soft *BCH*-Algebras of the Power Sets

Shuker Mahmood Khalil*, Mayadah Abd Ulrazaq

Department of Mathematics, College of Science, University of Basrah, Basrah, Iraq

Abstract In this paper, in the first we introduce the concept of $BCH-$algebra of the power set and new notions connected to it are investigated and discussed like $BCH-$subalgebra of the power set, soft $BCH-$algebra of the power set and soft $BCH-$subalgebra of the power set. Then some binary operations between two soft $BCH-$algebras of the power set are studied. Further, we state the relations between soft $BCH-$algebra of the power set and soft $BCK/BCL/d/d^*/\rho-$ algebra of the power set. Moreover, several examples are given to illustrate the notations introduced in this work.

Keywords Soft sets theory, Proper $BCH-$algebra, $BCL/BCK/d/d^*/\rho-$algebras

1. Introduction

$BCK/BCI-$algebras two classes of abstract algebras are introduced by Imai and Iseki ([22], [23]). The class of *BCK*-algebras is a proper subclass of the class of $BCI-$algebras. Also, Hu and Li introduced a wider class of abstract algebras, it is said to be BCH-algebras ([5], [6]). Next, the concept of $d-$algebras, which is another useful generalization of *BCK* –algebras are introduced (see [2], [3], [20]). After then, the concept of $\rho-$algebra is introduced and studied [9]. The basic notions of soft sets theory are introduced by Molodtsov ([18]) to deal with uncertainties when solving problems in practice as in engineering, social science, environment, and economics. This notion is convenient and easy to apply as it is free from the difficulties that appear when using other mathematical tools as theory of theory of fuzzy sets, rough sets and theory of vague sets etc. Moreover, many researches on soft sets theory and some of their applications are studied (see [10]-[15]). On other word, many authors applied the notion of soft set on several classes of algebras like soft $BCK/BCL-$algebras [7] and soft $\rho-$algebras [16]. In recent years, for any $|X|<\infty$ (finite set X), the notations of $d-$algebra of the power set, $BCK-$ algebra of the power set, d^*-algebra of the power set, soft $d-$algebra of the power set, soft $BCK-$algebra of the power set, soft $BCL-$algebra of the power set, soft d^*-algebra of the power set, soft edge $d-$algebra of the power set, soft edge $BCK-$algebra of the power set, soft edge d^*-algebra of the power set are introduced (see [16], [17]). The aim of this paper is to introduce new branch of the pure algebra it's called $BCH-$ algebra of the power set. Then some binary operations between two soft $BCH-$algebras of the power set are stated. Further, we study the relations between soft $BCH-$algebra of the power set and soft $BCK/BCL/d/d^*/\rho-$ algebra of the power set. Also, several examples are given to illustrate the notations introduced in this work.

* Corresponding author:
shuker.alsalem@gmail.com (Shuker Mahmood Khalil)
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2. Preliminaries

In this section we recall the basic background needed in our present work.

Definition 2.1: ([20]) A $d-$algebra is a non-empty set X with a constant 0 and a binary operation* satisfying the following axioms:

(*i*)- $x*x=0$

(*ii*)- $0*x=0$

(*iii*)- $x*y=0$ and $y*x=0$ imply that $y=x$ for all *x*, *y* in *X*.

Definition 2.2: ([19]) A $d-$algebra $(X,*,0)$ is said to be $BCK-$ algebra if X satisfies the following additional axioms:

(1). $((x*y)*(x*z))*(z*y)=0$,

(2). $(x*(x*y))*y=0$, for all $x,y\in X$.

Definition 2.3 ([8]) A $\rho-$ algebra $(X,*,f)$ is a non-empty set X with a constant $f\in X$ and a binary operation* satisfying the following axioms:

(i)- $x * x = f$,

(ii)- $f * x = f$,

(iii)- $x * y = f = y * x$ imply that $y = x$,

(iv)- For all $y \neq x \in X - \{f\}$ imply that $x * y = y * x \neq f$.

Definition 2.4: ([21]) A $BCL-$algebra is a non-empty set X with a constant 0 and a binary operation* satisfying the following axioms:

(i)- $x * x = 0$,

(ii)- $x * y = 0$ and $y * x = 0$ imply that $y = x$,

(iii)- $((x * y) * z) * ((x * z) * y) * ((z * y) * x) = 0$, for all *x, y ,z* in *X*.

Definition 2.4: ([5]) A $BCH-$algebra is a non-empty set X with a constant 0 and a binary operation* satisfying the following axioms:

(i)- $x * x = 0$,

(ii)- $x * y = 0$ and $y * x = 0$ imply that $y = x$,

(iii)- $(x * y) * z = (x * z) * y$, for all *x, y ,z* in *X*.

Definition 2.5: ([18]) Let X be an initial universe set and let E be a set of parameters. The power set of X is denoted by $P(X)$. Let K be a subset of E. A pair (F, K) is said to be a soft set over X if F is a set-valued functions of K into the set of all subsets of the set X.

Definition 2.6: ([11]) Let (F, A) and (G, B) be two soft sets over X, then their union is the soft set (H, C), where $C = A \bigcup B$ and for all $e \in C$, $H(e) = F(e)$ if $e \in A - B$, $G(e)$ if $e \in B - A$, $F(e) \bigcup G(e)$ if $e \in A \bigcap B$. We write $(F, A) \coprod (G, B) = (H, C)$. Further, [4] for any two soft sets (F, A) and (G, B) over X their intersection is the soft set (H, C) over X, and we write $(H, C) = (F, A) \prod (G, B)$, where $C = A \bigcap B$, and $H(e) = F(e) \bigcap G(e)$ for all $e \in C$.

Definition 2.7: ([8]) Let (F, K) be a soft set over X. Then (F, K) is said to be a soft $d-$algebra over X if $(F(x), *, 0)$ is a $d-$algebra for all $x \in K$.

Definition 2.8: ([7]) Let (F, K) be a soft set over X. Then (F, K) is said to be a soft $BCK-$algebra over X if $(F(x), *, 0)$ is a $BCK-$algebra for all $x \in K$.

Definition 2.9: ([7]) Let (F, K) be a soft set over X. Then (F, K) is said to be a soft $BCL-$algebra over X if $(F(x), *, 0)$ is a $BCL-$algebra for all $x \in K$.

Definition 2.10: ([16]) Let (F, K) be a soft set over X. Then (F, K) is said to be a soft $\rho-$algebra over X if $(F(x), *, 0)$ is a $\rho-$algebra for all $x \in K$.

Definition 2.10: ([24]) Let (F, K) be a soft set over X. Then (F, K) is said to be a soft $BCH-$algebra over X if $(F(x), *, 0)$ is a $BCH-$algebra for all $x \in K$.

Definition 2.11: ([16]) Let X be non-empty set and $P(X)$ be a power set of X. Then $(P(X), *, A)$ with a constant A and a binary operation* is said to be $d-$algebra of the power set of X if $P(X)$ satisfying the following axioms:

(i)- $B * B = A$

(ii)- $A * B = A$

(iii)- $B * C = A$ and $C * B = A$ imply that $B = C$ for all $B, C \in P(X)$.

Definition 2.12: ([16]) Let $(P(X), *, A)$ be a $d-$algebra of the power set of X. Then $P(X)$ is said to be $BCK-$algebra of the power set of X if it satisfies the following additional axioms:

(1). $((B * C) * (B * D)) * (D * C) = A$,

(2). $(B * (B * C)) * C = A$, for all $B, C \in P(X)$.

Definition: 2.13: ([16]) Let $(P(X), *, A)$ be a $d-$algebra of the power set of X. Then $P(X)$ is said to be a ρ −algebra of the power set of X if it satisfies the identity $(B * C) = (C * B) \neq A$, for all $B \neq C \in P(X) - A$.

Definition 2.14: ([16]) Let $(P(X), *, A)$ be a $d-$algebra of the power set of X. Then $P(X)$ is said to be a d^*-algebra of the power set of X if it satisfies the identity $(B * C) * B = A$, for all $B, C \in P(X)$.

Definition 3.1: ([17]) Let X be non-empty set and $P(X)$ be a power set of X. Then $(P(X), *, A)$ with a constant A and a binary operation ($*$) is said to be $BCL-$ algebra of the power set of X if $P(X)$ satisfying the following axioms:

(i)- $B * B = A$,

(ii)- $B * C = A$ and $C * B = A$ imply that $B = C$ for all $B, C \in P(X)$,

(iii)- $((B * C) * D) * ((B * D) * C) * ((D * C) * B) = A$, for all $B, C, D \in P(X)$.

Definition 2.15: ([16], [17]) Let $(P(X), *, A)$ be a $d-$algebra (ρ −algebra, $BCK-$algebra, d^*-algebra, $BCL-$ algebra) of the power set of X and let $H = \{h_i\}_{i \in I} \subseteq P(X)$ be a collection of some random subsets of X. Then H is said to be $d-$subalgebra (resp. ρ − subalgebra, $BCK-$ subalgebra, d^*- subalgebra, $BCL-$ subalgebra) of the power set of X, if $h_m * h_k \in H$, for any $h_m, h_k \in H$.

Definition 2.16: ([16], [17]) Let $(P(X), *, A)$ be a $d-$ algebra (resp. $\rho-$algebra, $BCK-$ algebra, d^*- algebra, $BCL-$algebra) of the power set of X and let

$F: H \to P(X)$, be a set valued function , where H is a collection of some random subsets of X defined by $F(h)=\{q \in P(X) \mid h \approx q\}$ for all $h \in H$ where $\approx$ is an arbitrary binary operation from H to $P(X)$. Then the pair (F,H) is a soft set over X. Further, (F,H) is said to be a soft $d-$algebra (resp. soft ρ-algebra, soft $BCK-$algebra, soft d^*-algebra, soft $BCL-$algebra) of the power set of X, if $(F(h),*,A)$ is a $d-$subalgebra (resp. $\rho-$ subalgebra, $BCK-$ subalgebra, d^*-subalgebra, $BCL-$subalgebra) of the power set of X for all $h \in H$.

3. Soft *BCH*-algebra of the Power Sets

In this section we introduce the notion of soft $BCH-$algebra of the power set and soft $BCH-$subalgebra of the power set. We will illustrate the definitions with examples.

Definition 3.1 Let X be non-empty set and $P(X)$ be a power set of X. Then $(P(X),*,A)$ with a constant A and a binary operation ($*$) is said to be $BCH-$algebra of the power set of X if $P(X)$ satisfying the following axioms:

(i)- $B*B=A$,

(ii)- $B*C=A$ and $C*B=A$ imply that $B=C$ for all $B,C \in P(X)$,

(iii)- $(B*C)*D=(B*D)*C$, for all $B,C,D \in P(X)$.

Definition 3.2 A $BCH-$algebra of power set is said to be proper $BCH-$algebra of power set if it satisfies $((B*C)*(B*D))*(D*C) \neq A$, for some $B,C,D \in P(X)$.

Example 3.3 Let $X=\{1,2\}$ and let $\otimes: P(X)\times P(X) \to P(X)$ be a binary operation defined by the following table:

Table (1)

$\otimes$	ϕ	$\{1\}$	$\{2\}$	x
ϕ	ϕ	ϕ	ϕ	ϕ
$\{1\}$	$\{1\}$	ϕ	ϕ	$\{1\}$
$\{2\}$	$\{2\}$	$\{2\}$	ϕ	ϕ
X	X	X	X	ϕ

Then $(P(X),\otimes,\phi)$ is a proper $BCH-$algebra of the power set of X, since there are $B=\{1\}, C=X$, $D=\{2\} \in P(X)$ such that

$((\{1\} \otimes X) \otimes (\{1\} \otimes \{2\})) \otimes (\{2\} \otimes X) = (\{1\} \otimes \phi) \otimes \phi = \{1\} \neq \phi$

Definition 3.4 Let $(P(X),*,A)$ be a $BCH-$algebra of the power set of X and let $H=\{h_i\}_{i\in I} \subseteq P(X)$ be a collection of some random subsets of X. Then H is said to be $BCH-$ subalgebra of the power set of X, if $h_m * h_k \in H$, for any $h_m, h_k \in H$.

Example 3.5 Let $(P(X),\otimes,\phi)$ be $BCH-$algebra of the power set of X in example (3.3). Then $H_1=\{\varphi\}$, $H_2=\{\varphi,\{1\}\}, H_3=\{\varphi,\{2\}\}, H_4=\{\varphi,\{1\},\{2\}\}, H_5 = P(X)$ are $BCH-$subalgebra of the power set of X.

Remark 3.6 we will show that not necessary every $BCK/BCL/d/d^*-$algebra of the power set is $BCH-$algebra of the power set.

Example 3.7 Let $X=\{1,2,3\}$ and let $/: P(X) \times P(X) \to P(X)$ be a binary operation defined by $/(B,C)=B/C=\{x \in X \mid x \in B \,\&\, x \notin C\} = B \cap C^c$, for all $B,C \in P(X)$. Then $(P(X),/,\phi)$ is a $BCK/BCL/d/d^*-$algebra of the power set of X with the following table (2).

In other word, $(X/\{1\})/\{2\}=\{2,3\}/\{2\} =\{3\} \neq \{2,3\}=\{1,3\}/\{1\}=(X/\{2\})/\{1\}$, for some $\{1\},\{2\},X \in P(X)$. Then $(P(X),/,\phi)$ is not $BCH-$ algebra. Also, for example, $q_1=\{\phi\}$, $q_2=\{\phi,\{1\}\}$, $q_3=\{\phi,\{2\}\}$, $q_4=\{\phi,\{3\}\}$, $q_5=\{\phi,\{1\},\{2\}\}$, $q_6=\{\phi,\{1\},\{3\}\}$, and $q_7=\{\phi,\{2\},\{3\}\}$ are $BCK/BCL/d/d^*-$subalgebra of the power set of X, but not $BCH-$subalgebra.

Example 3.8: Let $X=\{1,2,3\}$ and let $\oplus: P(X)\times P(X) \to P(X)$ be a binary operation defined by

$$\oplus(A,B)=A\oplus B=\begin{cases} A\cup B, & if \quad B\neq A\neq \phi, \\ \phi, & Otherwise. \end{cases}$$

, for all $A,B \in P(X)$. Then $(P(X),\oplus,\phi)$ is a $BCH/BCL/d/\rho-$algebra of the power set of X with the following table (3).

Table (2)

/	ϕ	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	X
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
{1}	{1}	ϕ	{1}	{1}	ϕ	ϕ	{1}	ϕ
{2}	{2}	{2}	ϕ	{2}	ϕ	{2}	ϕ	ϕ
{3}	{3}	{3}	{3}	ϕ	{3}	ϕ	ϕ	ϕ
{1,2}	{1,2}	{2}	{1}	{1,2}	ϕ	{2}	{1}	ϕ
{1,3}	{1,3}	{3}	{1,3}	{1}	{3}	ϕ	{1}	ϕ
{2,3}	{2,3}	{2,3}	{3}	{2}	{3}	{2}	ϕ	ϕ
X	X	{2,3}	{1,3}	{1,2}	{3}	{2}	{1}	ϕ

Table (3)

$\oplus$	ϕ	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	X
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
{1}	{1}	ϕ	{1,2}	{1,3}	{1,2}	{1,3}	X	X
{2}	{2}	{1,2}	ϕ	{2,3}	{1,2}	X	{2,3}	X
{3}	{3}	{1,3}	{2,3}	ϕ	X	{1,3}	{2,3}	X
{1,2}	{1,2}	{1,2}	{1,2}	X	ϕ	X	X	X
{1,3}	{1,3}	{1,3}	X	{1,3}	X	ϕ	X	X
{2,3}	{2,3}	X	{2,3}	{2,3}	X	X	ϕ	X
X	X	X	X	X	X	X	X	ϕ

Remarks 3.9:

(1) It is not necessary every $BCH-$ algebra of the power set of X is $BCK/d^*/-$ algebra of the power set. In example (3.8), let $A=\{1\}, B=\{2\}$. Then $(P(X),\oplus,\phi)$ is not d^*-algebra of the power set of X , since $(A\oplus B)\oplus A=\{1,2\}\neq\phi.$ Also, let $A=\{1\}$, $B=\{2\}, C=\{3\}$. Then $(P(X),\oplus,\phi)$ is not $BCK-$algebra of the power set of X, since $((A\oplus B)\oplus(A\oplus C))\oplus(C\oplus B)=X\neq\phi.$ Also, $(P(X),\oplus,\phi)$ is a $d/\rho-$algebra of the power set of X. On the other hand, $q_1=\{\phi\}$, $q_2=\{\phi,\{1\}\}$, $q_3=\{\phi,\{2\}\}$ and $q_4=\{\phi,\{3\}\}$ are $\rho-$subalgebras. However, $K_1=\{\phi,\{1\},\{2\}\}$, $K_2=\{\phi,\{1\},\{3\}\}$, and $K_3=\{\phi,\{2\},\{3\}\}$ are not $\rho-$subalgebras. Further, see example (3.7) $(P(X),/,\phi)$ is not ρ-algebra.

(2) Further, if $(P(X),*,f)$ is a $BCH-$algebra of the power set of X satisfying $f*h=f$ for any $h\in P(X)$. Then $(P(X),*,f)$ is a d-algebra of the power set of *X*.

Definition 3.10 Let $(P(X),*,A)$ be a $BCH-$ algebra of the power set of X and let $F:H\to P(X)$, be a set valued function , where H is a collection of some random subsets of X defined by $F(h)=\{q\in P(X)\,|\,h\approx q\}$ for all $h\in H$ where $\approx$ is an arbitrary binary operation from H to $P(X)$. Then the pair (F,H) is a soft set over X . Further, (F,H) is said to be a soft $BCH-$algebra of the power set of X , if $(F(h),*,A)$ is a $BCH-$subalgebra of the power set of X for all $h\in H$.

Example: 3.11 Let $(P(X),\oplus,\phi)$ be a proper $BCH-$algebra of the power set of X with the following table:

Table (4)

$\oplus$	ϕ	$\{1\}$	$\{2\}$	X
ϕ	ϕ	ϕ	ϕ	ϕ
$\{1\}$	$\{1\}$	ϕ	ϕ	$\{1\}$
$\{2\}$	$\{2\}$	X	ϕ	X
X	X	ϕ	ϕ	ϕ

Let (F,H) be a soft set over $X=\{1,2\}$, where $H=\{\phi,\{1\},\{2\},X\}$ and $F:H\to P(X)$ is a set valued function defined by $F(h)=\{k\in P(X)|: h\approx k \leftrightarrow k\oplus(k\oplus h)=\phi\}$ for all $h\in H$. Then $F(\phi)=X$, $F(\{1\})=F(X)=\{\varphi,\{1\}\}$, $F(\{2\})=\{\phi\}$ which are soft $BCH-$ subalgebras of the power set of X. Hence (F,H) is a soft $BCH-$ algebra of the power set of X. The next example shows that there exist set-valued functions $G:B\to P(X)$, where (G,B) the soft set is not a soft $BCH-$ algebra of the power set of X.

Example 3.12: Consider the $BCH-$ algebra in example (3.11) with a set valued function defined by $G(h)=\{k\in P(X)|: h\approx k \quad \leftrightarrow k\oplus h\in\{\{1\},X\}\}$ for all $h\in B=\{\phi,\{1\},\{2\}\}$. We have (G,B) is not a soft $BCH-$algebra of the power set of X, since there exists $\{1\}\in B$, but $G(\{1\})=\{\{2\}\}$ is not soft $BCH-$ subalgebras of the power set of X.

Definition: 3.13 Let (F,H) be a soft $BCH-$algebra of the power set of X, and let $D\subseteq H$, where $F:H\to P(X)$ is defined by $F(h)=\{q\in P(X)\,|\,h\approx q\}$ for all $h\in H$. Then $F|_D: D\to P(X)$ is defined by $F|_D\ (h)=\{q\in P(X)\,|\,h\approx q\}$ for all $h\in D$.

Lemma 3.14: If (F,H) is a soft $BCH-$algebra of the power set of X, then $(F|_D,H)$ is a soft $BCH-$ algebra of the power set of X, for any $D\subseteq H$.

Proof: Let $(P(X),*,A)$ be a $BCH-$algebra of the power set of X and let (F,H) be a soft $BCH-$ algebra of the power set of X, then $(F(h),*,A)$ is a $BCH-$subalgebra of the power set of X, for all $h\in H$. Moreover, for all $h\in D\bigcap H$ we have $F|_D\ (h)=F(h)$, but $D=D\bigcap H$ (since $D\subseteq H$). Hence $(F|_D\ (h),*,A)$ is a $BCH-$subalgebra of the power set of X, for all $h\in D$. Then $(F|_D,H)$ is a soft $BCH-$ algebra of the power set of X.

Definition 3.15: Let (F,H) be a soft $BCH-$algebra of the power set of X. Then (F,H) is said to be a null soft $BCH-$ algebra of the power set of X if $F(h)=\{\phi\}$ for all $h\in H$. Also, (F,H) is said to be an absolutely soft $BCH-$algebra of the power set of X if $F(h)=P(X)$ for all $h\in H$.

Example 3.16 Let $(P(X),\oplus,\phi)$ be the $BCH-$ algebra of the power set of X in example (3.11) and let $F_1: H_1\to P(X)$, $F_2: H_2\to P(X)$, where $H_1=\{\phi,X\}$, $H_2=\{\phi,\{1\},\{2\}\}$ are defined by $F_1(h)=\{k\in P(X)\,|\,h\approx k \Leftrightarrow h\oplus k\in H_1\}$, $\forall h\in H_1$, and $F_2(h)=\{k\in P(X)\,|\quad h\approx k \Leftrightarrow k=h\oplus\{2\}\},\forall h\in H_2$. Thus, $F_1(\phi)=F_1(X)=P(X)$, and hence (F_1,H_1) is an absolutely soft $BCH-$ algebra of the power set X. Moreover, $F_2(\{\phi\})=\quad F_2(\{1\})=F_3(\{2\})=\{\phi\}$ and hence (F_2,H_2) is a null soft $BCH-$algebra of the power set X.

Lemma: 3.17 Let $(P(X),*,L)$ be a $BCK/d/d^*/\rho-$algebra of the power set of X. Then $(P(X),*,L)$ is a $BCL-$algebra of the power set of X, if $(P(X),*,L)$ is a $BCH-$algebra of the power set of X.

Proof: Since $(P(X),*,L)$ is a $BCK/d/d^*/\rho-$ algebra of the power set of X. Then $P(X)$ satisfying the following axioms:

(i)- $B*B=L$,
(ii)- $L*B=L$,
(iii)- $B*C=L$ and $C*B=L$ imply that $B=C$ for all $B,C\in P(X)$.

Also, let $(P(X),*,L)$ be a $BCH-$ algebra of the power set of X, then $(A*B)*C=(A*C)*B$, for all. Hence, from (i) we consider that $((A*B)*C)*((A*C)*B)=L$. Further, from (ii) we have $L*((C*B)*A)=L$ and this implies that $((A*B)*C)*((A*C)*B)\quad *((C*B)*A)=L$. Therefore, the following are hold:

(1)- $B*B=L$,
(2)- $B*C=L$ and $C*B=L$ imply that $B=C$,
(3)- $((A*B)*C)*((A*C)*B)*((C*B)*A)=L$, for all $A,B,C\in P(X)$. The $(P(X),*,L)$ is a $BCL-$ algebra of the power set of X.

Corollary 3.18 If $(P(X),*,L)$ is a $\rho-$algebra of the power set of X. Then $(P(X),*,L)$ is a $BCL-$algebra of the power set of X, if $C*(A*B)=(A*C)*B$ and $A*B=A*C$ for all $A,B,C\in P(X)$.

Proof: Since $(P(X),*,L)$ is a $\rho-$ algebra of the power set of X. Then $P(X)$ satisfying the following axioms:

(i)- $B*B=L$,

(ii)- $L*B=L$,

(iii)- $B*C=L$ and $C*B=L$ imply that $B=C$, for all $B,C\in P(X)$.

(iv)- $(B*C)=(C*B)\neq L$, for all $B\neq C\in P(X)-L$,

Therefore, we consider only the following are hold:

1)- $B*B=L$,

2)- $B*C=L$ and $C*B=L$ imply that $B=C$,

Then, to prove that $(P(X),*,L)$ is a $BCL-$algebra of the power set of X, we need also to prove that $((A*B)*C)*((A*C)*B)*((C*B)*A)=L$, for any $A,B,C\in P(X)$. Thus we have to show that $((A*B)*C)=L$ and hence we have $((A*B)*C)*((A*C)*B)*$ $((C*B)*A)$ $=L*((A*C)*B)*((C*B)*A)=$ $L*((C*B)*A)=L$. For any $A,B,C\in$ $P(X)$ we have the all cases that are cover all probabilities as following:

(1) If $A=B,C\in P(X)\Rightarrow((A*B)*C)=L$.

(2) If $L=A\neq B,C\in P(X)\Rightarrow((A*B)*C)=L$.

(3) If $L=B\neq A,C\in P(X)\Rightarrow$ $B*(A*C)=L$, but $B*(A*C)=(A*B)*C\Rightarrow(A*B)*C=L$.

(4) If $B\neq A=C,B\neq L\neq A\in P(X)$, then $B*(A*C)=(A*B)*C\Rightarrow$ $B*L=(A*B)*C\Rightarrow$ $(A*B)*B=(A*B)*(A*B)*C\Rightarrow(A*B)*B=L*C$ $=L\Rightarrow(A*B)*B*B=L*C=L\Rightarrow(A*B)=L\Rightarrow$ $((A*B)*C)=L$. Hence, for all (1),(2),(3), and (4) we consider that $(P(X),*,L)$ is a $BCL-$ algebra of the power set of X, since $((A*B)*C)*((A*C)*B)*((C*B)*A)=L$, for any $A,B,C\in P(X)$.

(5) If $B\neq A\neq C,B\neq L\neq A\in P(X)$, thus since $A\neq C\in P(X)-L$ and $(P(X),*,L)$ is a $\rho-$ algebra of the power set of X. Then from (iv) we have $(A*C)\neq L$ and this implies that $B*(A*C)=(A*C)*B$ and hence $(A*C)*B=(A*B)*C$. Then $(P(X),*,L)$ is a $BCH-$algebra of the power set of X and by[Lemma (3.17)] we have $(P(X),*,L)$ is a $BCL-$algebra of the power set of X.

Remark 3.19 From [Lemma (3.17) and Corollary (3.18)] we consider that, if $(P(X),*,L)$ is a $\rho-$algebra of the power set of X and $C*(A*B)=(A*C)*B$, for all $A,B,C\in P(X)$. Then $(P(X),*,L)$ is a $BCL-$ algebra of the power set of X.

Theorem 3.20: Let (F,A) and (G,B) be two soft $BCH-$algebras over X. If $A\bigcap B=\phi$, then the union $(H,C)=(F,A)\coprod(G,B)$ is a soft $BCH-$algebra of the power set of X.

Proof: Since $A\bigcap B=\phi$ and by definition (2.6), we have for all $k\in C$,

$$H(k)=\begin{cases}F(k),\ if\ \ k\in A\setminus B,\\ G(k),\ if\ \ k\in B\setminus A.\end{cases}$$

If $k\in A\setminus B$ then $H(k)=F(k)$ is a $BCH-$subalgebra of the power set of X. Similarly, if $k\in B\setminus A$, then $H(k)=G(k)$ is a $BCH-$subalgebra of the power set of X. Hence $(H,C)=(F,A)\coprod(G,B)$ is a soft $BCH-$algebra of the power set of X. Thus the union of two soft $BCH-$algebras of the power set of X is a soft $BCH-$algebra of the power set of X.

Example 3.21: In example (3.4), let (F,A) and (G,B) be two soft sets over $X=\{1,2,3\}$ where $A=\{\phi,\{1\}\{2,3\}\}$ and $B=\{\{2\},\{1,3\}\}$. Define $F:A\to P(X)$ by $F(h)=\{k\in P(X)\mid\ k\approx h\leftrightarrow k\oplus h\in\{\phi,\{1,2\}\}$ for all $h\in A$ and $G:B\to P(X)$ by $G(h)=\{k\in P(X)\mid k\approx h$ $\leftrightarrow k\oplus h\in\{\phi,\{1,3\}\}$ for all $h\in B$. Note that $A\bigcap B=\phi$. Thus, we have $F(\phi)=\{\phi,\{1,2\}\}$, $F(\{1\})=\{\phi,\{1\},\{2\},\{1,2\}\}$, $F(\{2,3\})=\{\phi,\{2,3\}\}$, $G(\{2\})=\{\phi,\{2\}\}$ and $G(\{1,3\})=\{\phi,\{1\},\{3\},\{1,3\}\}$. Then $H(\phi)=F(\phi)=\{\phi,\{1,2\}\}$, $H(\{1\})=F(\{1\})=$ $\{\phi,\{1\},\{2\},\{1,2\}\}$, $H(\{2,3\})=F(\{2,3\})=\{\phi,\{2,3\}\}$, $H(\{2\})=G(\{2\})=\{\phi,\{2\}\}$ and $H(\{1,3\})=$ $G(\{1,3\})=\{\phi,\{1\},\{3\},\{1,3\}\}$ which are $BCH-$subalgebras of the power set of X. Hence, (H,C) is a soft $BCH-$algebra of the power set of X.

Remark 3.22: The condition $A\bigcap B=\phi$ is important as if $A\bigcap B\neq\phi$, then the theorem does not apply. In above example, if $A=\{\phi,\{2\},\{2,3\}\}$ and $B=\{\phi,\{1\}\}$. Then $H(\phi)=F(\phi)\bigcup G(\phi)=\{\phi,\{1,2\},\{1,3\}\}$ which is not a $BCL-$subalgebra of the power set of X. Therefore, is not a soft $BCL-$algebra of the power set of X.

Theorem 3.23: Let (F,A) and (G,B) be two soft $BCH-$algebras over X. If $A\bigcap B=\phi$, then the union $(H,C)=(F,A)\prod(G,B)$ is a soft $BCH-$algebra of the power set of X.

Proof: Since $(H,C)=(F,A)\prod(G,B)$, where $C=A\bigcap B$, and $H(k)=F(k)\bigcap G(k)$ for all $k\in C$ [by definition (2.6)], Note that $H:C\to P(X)$ is a mapping and so (H,C) is a soft set over X. We have,

$H(k) = F(k)$ or $H(k) = G(k)$ is a $BCH-$ subalgebra of the power set of X . Hence, $(H,C) = (F,A) \prod (G,B)$ is a soft $BCH-$algebra of the power set of X . Therefore, the intersection of two soft $BCH-$ algebras is a soft $BCH-$algebra.

Example 3.24: Consider the algebra in example (3.21) with $A = \{\phi,\{2\},\{2,3\}\}$ and $B = \{\phi,\{1\}\}$. Then $H(\phi) = F(\phi) = \{\phi,\{1,2\}\}$ or $H(\phi) = G(\phi) = \{\phi,\{1,3\}\}$. Note that both are $BCH-$subalgebras of the power set of X . Hence, (H,C) is a soft $BCL-$algebras of the power set of X .

Theorem 3.25: Let $(P(X),*,f)$ be a $BCH-$algebra of the power set of X with the condition $f*h = f$ for any $h \in P(X)$. If (F,A) is a soft $BCH-$algebra of the power set of X , then (F,A) is a soft $d-$algebra of the power set of X .

Proof: Straightforward from Definitions [(3.10), (2.7)] and remark [(2)-(3.9)].

Theorem 3.26: Let $(P(X),*,f)$ be a $d-$algebra of the power set of X and (F,A) is a soft $d-$algebra of the power set of X . Then (F,A) is a soft $BCL-$ algebra of the power set of X , if $(P(X),*,f)$ is $BCH-$ algebra of the power set.

Proof: Straightforward from Definitions [(3.10), (2.7)] and [Lemma (3.17)].

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