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New Category of Soft Topological Spaces

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Abstract In this work, we introduce new category of soft topological space is called soft $L-$closed topological space, also we study in details the properties of soft $L-$closed space and its relation with soft second-countable space, we state that every soft second-countable space is soft $L-$closed but the converse is not true in general, also we describe its relation with soft Lindelof space, soft compact space, and soft absolutely closed space.

Keywords Soft sets, Soft Lindelof space, Soft compact space, Soft $L-$closed space, Soft absolutely closed, Soft second-countable space

1. Introduction

The topological structures of set theories dealing with uncertainties were first discussed by Chang [2]. E. F. Lashin et al. [6] generalized rough set theory in the framework of topological spaces. Shabir and Naz [20] are the first persons who introduce the concept of soft topological spaces which are defined over an initial universe with a fixed parameters. Zorlutuna, et al. [21] introduced some new concepts in soft topological spaces and give some new properties about soft topological spaces. Molodtsov [17] introduced the concept of a soft set as a mathematical tool for dealing with uncertainties. Soft set theory has rich potential for practical applications in several domains. In recently years, soft set theory has been researched in many fields see ([8]-[16]). Compact spaces are one of the most important classes in general topological spaces [3]. They have many well-known properties which can be used in many disciplines. Next, the notion of compact soft spaces around a soft topology is introduced see [21]. Later some other properties of soft topological spaces were also examined: soft countability axioms [19]. In 2017, the notion of soft sequentially absolutely closed space is introduced by S. Mahmood. We introduce in this work the new notation is called soft $L-$closed space and we prove in this work the soft $L-$closed space is a soft topological property, also we study its relation with soft Lindelof space, we show that every soft Lindelof spaces is a soft $L-$closed space but the converse is not true in general, also we show that every soft space satisfies the soft second axiom of countability that is soft $L-$closed space and every soft compact space or soft absolutely closed space is soft $L-$closed space.

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2. Preliminaries

We now begin by recalling some definitions and some of the basic prosperities of the soft sets.

Definition 2.1: ([17])

Let X be an initial universe set and let E be a set of parameters. A pair (F,A) is called a soft set (over X) where $A \subseteq E$ and F is a multivalued function $F : A \to P(X)$. In other words, the soft set is a parameterized family of subsets of the set X. Every set $F(e)$, $e \in E$, from this family may be considered as the set of e-elements of the soft set (F,A), or as the set of $e-$approximate elements of the soft set. Clearly, a soft set is not a set. For two soft sets (F,A) and (G,B) over the common universe X, we say that (F,A) is a soft subset of (G,B) if $A \subseteq B$ and $F(e) \subseteq G(e)$, for all $e \in A$. We write $(F,A) \tilde{\subseteq} (G,B)$. (F,A) is said to be a soft superset of (G,B), if (G,B) is a soft subset of (F,A). Two soft sets (F,A) and (G,B) over a common universe X are said to be soft equal if (F,A) is a soft subset of (G,B) and (G,B) is a soft subset of (F,A). A soft set (F,A) over X is called a null soft set, denoted by $\Phi =$ (ϕ , ϕ), if for each $F(e) = \phi, \forall e \in A$. Similarly, it is called universal soft set, denoted by (X,E), if for each $F(e) = X, \forall e \in A$. The collection of all soft sets over a universe X and the parameter set E is a family of soft sets denoted by $SS(X_E)$. Also, the collection of all soft sets over a universe X and the parameter set $A \subseteq E$ is a family of soft sets denoted by $SS(X_A)$.

Definition 2.2: ([7]) The union of two soft sets (F, A) and (G,B) over X is the soft set (H,C), where $C = A \bigcup B$ and for all $e \in C$, $H(e) = F(e)$ if $e \in A - B$, $G(e)$ if $e \in B - A$, $F(e) \bigcup G(e)$ if $e \in A \bigcap B$. We write $(F, A) \coprod (G,B) = (H,C)$. The intersection (H,C) of (F, A) and (G,B) over X, denoted $(F, A) \prod (G,B)$, is defined as $C = A \bigcap B$, and $H(e) = F(e) \bigcap G(e)$ for all $e \in C$.

Definition 2.3: ([21]) The soft set $(F, A) \in SS(X_E)$ is called a soft point in (X,E), denoted by e_x, if there exist $x \in X$ and $e \in A$, $F(e) = \{x\}$ and $F(e') = \phi$ for all $e' \in A - \{e\}$. The soft point e_x is said to be in the soft set (G, A), denoted by $e_x \tilde{\in} (G, A)$, if for the element $e \in A$ and $F(e) \subseteq G(e)$.

Definition 2.4: ([20]) The difference (H,E) of two soft sets (F,E) and (G,E) over X, denoted by $(F,E) \setminus (G,E)$, is defined as $H(e) = F(e) \setminus G(e)$ for all $e \in E$.

Definition 2.5: ([20]) Let (F, A) be a soft set over X. The complement of (F, A) with respect to the universal soft set (X,E), denoted by $(F, A)^c$, is defined as (F^c, D), where $D = E \setminus \{e \in A \,|\, F(e) = X\} = \{e \in A \,|\, F(e) = X\}^c$, and for all $e \in D$,

$$F^c(e) = \begin{cases} X \setminus F(e)\ ,\ if\ \ e \in A \\ X\ ,\ \ Otherwise\ . \end{cases}$$

Proposition 2.6: ([20]) Let (F,E) and (G,E) be the soft sets over X. Then

(1) $((F,E) \coprod (G,E))^C = (F,E)^C \prod\ (G,E)^C$

(2) $((F,E) \prod (G,E))^C = (F,E)^C \coprod\ (G,E)^C$.

Definition 2.7: ([20])

Let τ be the collection of soft sets over X. Then τ is called a soft topology on X if τ satisfies the following axioms:

(i) Φ, (X,E) belong to τ. (ii) The union of any number of soft sets in τ belongs to τ.

(iii) The intersection of any two soft sets in τ belongs to τ.

The triplet (X,τ,E) is called a soft topological space over X. The members of τ are called soft open sets in X and complements of them are called soft closed sets in X.

Definition 2.8: ([4])

The soft closure of (F, A) is the intersection of all soft closed sets containing (F, A). (i.e) The smallest soft closed set containing (F, A) and is denoted by $cl^S(F, A)$. The soft interior of (F, A) is the union of all soft open set is contained in (F, A) and is denoted by $int^S(F, A)$.

Definition 2.9: ([1]) Let (X,τ,E) be a soft topological space. A sub-collection η of τ is said to be a *base* for τ if every member of τ can be expressed as a union of members of η.

Definition 2.10: ([1]) Let (X,τ,E) be a soft topological space, and let V be a family a soft neighborhood of some soft point e_x. If, for each soft neighborhood (F, A) of e_x, there exists a $(H,B) \in V$ such that $e_x \tilde{\in} (H,B) \tilde{\subseteq} (F, A)$, then we say that V is a soft neighborhoods base at e_x.

Definition 2.11: ([18]) A family $\Psi = \{W_\alpha / \alpha \in \Omega\}$ of soft sets is a cover of a soft set (F, A) if $(F, A) \tilde{\subseteq} \coprod_{\alpha \in \Omega} W_\alpha$. If each member of Ψ is a soft open set, then Ψ is called a soft open cover.

Definition 2.12: ([18]) A soft topological space (X,τ,E) is called soft compact space if each soft open cover of (X,E) has a finite subcover.

Definition 2.14 ([1]) Let (X,τ,E) be a soft topological space. Then (X,τ,E) is called soft absolutely closed iff for each soft open cover $\Psi = \{w_\alpha \,|\, \alpha \in \mathrm{I}\}$ of (X,E), there exist α_1, α_2 ,…, α_n such that $(X,E) = \coprod_{i=1}^{n} cl(w_{\alpha i})$.

Definition 2.15 ([1]) Let (X,τ,E) be a soft topological space. If (X,E) has a countable soft base, then we say that (X,E) is *soft second-countable*.

Definition 2.16: ([1]) A soft space (X,τ,E) is soft Lindelof if each soft open covering Ψ of (X,E) has a countable subcover.

Definition 2.17 ([5]) Let (X,E) and (Y,K) be soft classes and let $u: X \to Y$ and $p: E \to K$ be mappings. Then a mapping $f: (X,E) \to (Y,K)$ is defined as: for a soft set (F, A) in (X,E), $(f(F, A), B)$, $B = p(A) \subseteq K$ is a soft set in (Y,K) given by $f(F, A)(\beta) = u\left(\bigcup_{\alpha \in p^{-1}(\beta) \bigcap A} F(\alpha)\right)$ for $\beta \in K$. $(f(F, A), B)$ is called a soft image of a soft set (F, A). If $B = K$, then we shall write $(f(F, A), K)$

as $f(F,A)$.

Definition 2.18 ([5]) Let $f: (X,E)\to(Y,K)$ be a mapping from a soft class (X,E) to another soft class (Y,K) and (G,C) a soft set in soft class (Y,K) where $C\subseteq K$. Let $u: X\to Y$ and $p: E\to K$ be mappings. Then $(f^{-1}(G,C),D)$, $D=p^{-1}(C)$, is a soft set in the soft classes (X,E) defined as: $f^{-1}(G,C)(\alpha)=u^{-1}(G(p(\alpha)))$ for $\alpha\in D\subseteq E$. $(f^{-1}(G,C),D)$, is called a soft inverse image of (G,C). Hereafter, we shall write $(f^{-1}(G,C),E)$ as $f^{-1}(G,C)$.

Theorem 2.19 ([5]) Let $f:(X,E)\to(Y,K)$, $u: X\to Y$ and $p: E\to K$ be mappings. Then for soft sets (F,A), (G,B) and a family of soft sets (F_i,A_i) in the soft class (X,E) we have:

(1) $f((X,E))=(Y,K)$,

(2) $f((F,A)\coprod(G,B))=f((F,A))\coprod f((G,B))$ in general $f(\coprod_{i\in I}(F_i,A_i))=\coprod_{i\in I} f((F_i,A_i))$,

(3) $f((F,A)\prod(G,B))\subseteq f((F,A))\prod f((G,B))$ in general $f(\prod_{i\in I}(F_i,A_i))\subseteq\prod_{i\in I} f((F_i,A_i))$,

(4) If $(F,A)\subseteq(G,B)$ then $f((F,A))\subseteq f((G,B))$,

(5) $f^{-1}((Y,K))=(X,E)$,

(6) $f^{-1}((F,A)\coprod(G,B))=f^{-1}((F,A))\coprod f^{-1}((G,B))$ in general $f^{-1}(\coprod_{i\in I}(F_i,A_i))=\coprod_{i\in I} f^{-1}((F_i,A_i))$,

(7) $f^{-1}((F,A)\prod(G,B))=f^{-1}((F,A))\prod f^{-1}((G,B))$ in general

$$f^{-1}(\prod_{i\in I}(F_i,A_i))=\prod_{i\in I} f^{-1}((F_i,A_i)),$$

(8) If $(F,A)\subseteq(G,B)$ then $f^{-1}((F,A))\subseteq f^{-1}((G,B))$.

Definition 2.20 ([5]) A soft mapping $f:(X,E)\to(Y,K)$ is said to be soft continuous (briefly s-continuous) if the soft inverse image of each soft open set of (Y,K) is a soft open set in (X,E).

Definition 2.21 ([5]) A soft mapping $f:(X,E)\to(Y,K)$ is said to be soft open (briefly s-open) if soft image of each soft open set of (X,E) is a soft open set in (Y,K).

Definition 2.22 ([5]) A soft mapping *f*: $(X,E)\to(Y,K)$ is said to be soft homeomorphism if *f* is onto and one to one.

3. Soft Lindelof Closed Space

Definition 3.1 Let (X,τ,E) be a topological space we say that (X,E) is a soft Lindelof closed space (soft $L-$closed space) iff each soft open cover $F=\{W_\alpha/\alpha\in\Omega\}$ of (X,E) has a soft countable subfamily whose soft closure covers (X,E) [i.e. $(X,E)=\coprod_{i\in I} cl^s(W_{\alpha_i})$].

Example 3.2 Let $X=R$ be real line, and let $E=\{e\}$. Then R be an uncountable set. Let $\Gamma=\{(F,E)\,|\,F(e)=[a,b);a<b\}$, and let τ be the soft topology generated by Γ as a base. Hence (X,τ,E) is soft $L-$closed space. Indeed, for each soft point e_t, it is easy to see that $\{(F_k^t,E)\,|\,F_k^t(e)=[t,(1/k)+t);k\in N\}$ is a soft neighborhoods base at e_t. Let $E=\{e_t;t\in Q\}$. Thus (X,τ,E) is $L-$ closed space.

Theorem: 3.3 If (X,τ,E) is a soft Lindelof space, then (X,τ,E) is a soft $L-$closed space.

Proof:

Let (X,τ,E) be a soft Lindelof space. Then for each soft open cover $F=\{W_\alpha/\alpha\in\Omega\}$ of (X,E) there is a countable sub-collection of F which also covers (X,E) [i.e. $(X,E)\tilde{\subseteq}\coprod_{i\in I}W_{\alpha_i}$, where $I\subseteq N$].

However, $\coprod_{i\in I}W_{\alpha_i}\tilde{\subseteq}\coprod_{i\in I}cl^s(W_{\alpha_i})$. This implies that $(X,E)=\coprod_{i\in I}cl^s(W_{\alpha_i})$. Hence (X,E) is a soft $L-$closed space.

Remark 3.4: The converse of theorem (3.3) is not true in general. So we will show in following theorem when will be hold.

Theorem: 3.5

Let (X,τ,E) be a soft discrete space. Then (X,τ,E) is soft Lindelof space if and only if (X,τ,E) is $L-$closed.

Proof:

Assume that (X,τ,E) is soft Lindelof space. Then by [theorem (3.3)] we consider that (X,τ,E) is soft $L-$closed space.

Conversely, suppose that (X,E,τ) is a soft $L-$closed space and let $F=\{W_\alpha/\alpha\in\Omega\}$ be a soft open cover of (X,E). Then there is a countable sub family whose closure covers (X,E) [i.e. $(X,E)=\coprod_{i\in I} cl^s(W_{\alpha_i})$, where $I\subseteq N$]. However, for each soft open set in (X,E) we have $W_\alpha=\overline{W_\alpha}$ [since (X,τ,E) is soft discrete space]. Therefore [$(X,E)=\coprod_{i\in I} W_{\alpha_i}$ where $I\subseteq N$], Then (X,T) is a soft Lindelof space.

Theorem 3.6: Every soft compact space is a soft $L-$closed space.

Proof:

Assume that (X,τ,E) is a soft compact space, then for each soft open cover $F=\{W_\alpha/\alpha\in\Omega\}$ of (X,E) there exists a finite sub-collection of F which also covers (X,E), However, each finite family is countable family. Therefore (X,E) is a soft Lindelof space. Hence (X,E) is a soft $L-$closed space by [Theorem, (3.3)].

Theorem 3.7:

Every soft absolutely closed space is a soft $L-$closed space.

Proof:

Assume that (X,τ,E) is a soft topological space where (X,E) is absolutely closed, let $F=\{W_\alpha/\alpha\in\Omega\}$ be a soft open cover of (X,E). Then there is a finite sub-collection of F whose soft closure covers (X,E) [since (X,E) is soft absolutely closed space], thus there is a countable sub-collection of F whose soft closure covers (X,E) [since each finite family is a countable family]. Then (X,E) is a soft $L-$closed space.

Theorem 3.8:

If (X,E,τ) is soft second-countable, then (X,E) is a soft $L-$closed space.

Proof:

Assume that (X,τ,E) is a soft second-countable, then (X,E) has a countable soft base. Therefore for each soft open cover there is a countable sub-cover of (X,E). Then (X,E) is a soft Lindelof space but by [Theorem, (3,3)] we have (X,E) is a soft $L-$closed space.

Remark 3.9: The converse of the theorem (3, 8) is not true in general.

Example 3.10

Let (X,τ,E) be a soft $L-$closed topological space in example (3.2) we shall prove that (X,τ,E) is not soft second-countable. Let V be an arbitrary soft base for (X,τ,E). For arbitrary soft point $e_t\tilde{\in}(X,E)$ and soft open neighborhood (F_1^t,E), there exists a soft open set $(G_t,E)\in V$ such that $e_t\tilde{\in}(G_t,E)\tilde{\subseteq}(F_1^t,E)$, and hence $t\in\inf\{G_t(e)\}$. If $e_t\neq e_r$, then $(G_t,E)\neq(G_r,E)$. Therefore, the set $\{(G_t,E)\mid e_t\tilde{\in}(X,E)\}$ is an uncountable subfamily of V, and thus V is an uncountable family.

Proposition 3.11:

A soft $L-$closeness is a topological property.

Proof:

Assume that $f:(X,E)\to(Y,K)$ is a soft homeomorphism and (X,E) is a soft $L-$closed space, we went to show that (Y,K) is a soft $L-$closed space. Suppose that $G=\{G_\alpha/\alpha\in H\}$ is a soft open cover of (Y,K) [i.e. $(Y,K)=\coprod_{\alpha\in H} cl^s(G_\alpha)\Rightarrow$
$f^{-1}((Y,K))=f^{-1}(\coprod_{\alpha\in H} cl^s(G_\alpha))=\coprod_{\alpha\in H} f^{-1}(cl^s(G_\alpha))$].
However, $(X,E)=f^{-1}((Y,K))$ [since f is on to] and since f is soft continuous function we have $G^\#=\{f^{-1}(G_\alpha)/\alpha\in H\}$ is a soft open cover of (X,E) but (X,E) is a soft $L-$closed space, then there is a countable sub-collection of $G^\#$ whose soft closure covers (X,E) [i.e. $(X,E)=\coprod_{i\in I} cl^s(f^{-1}(G_{\alpha_i}))$, where $I\subseteq N$], therefore $(Y,K)=f(X,K)=$
$f(\coprod_{i\in I} cl^s(f^{-1}(G_{\alpha_i})))=\coprod_{i\in I} f(cl^s(f^{-1}(G_{\alpha_i})))$
$\tilde{\subseteq}\coprod_{i\in I} ff^{-1}(cl^s(G_{\alpha_i}))=\coprod_{i\in I} cl^s(G_{\alpha_i})\tilde{\subseteq}(Y,K).\quad (Y,K).$
Thus $(Y,K)=\coprod_{i\in I} cl^s(G_{\alpha_i})$, where $I\subseteq N$. Then (Y,K) is soft $L-$closed space.

Theorem 3.12: The cartesian product of countably many soft $L-$closed spaces is soft $L-$closed.

Proof: Let $\{(X_\alpha,\tau_\alpha,E_\alpha)\mid\alpha\in N\}$ be a family of countable many soft $L-$closed spaces. For each $\alpha\in N$, let $\Gamma_\alpha=\{B_{\alpha_i}\mid i\in I\}$ be asoft open cover of (X_α,E_α). Then there is a soft countable subfamily of Γ_α whose soft closure covers (X_α,E_α) [i.e. $(X_\alpha,E_\alpha)=\coprod_{i\in I} cl^s(B_{\alpha_i})$]. Put $\Gamma=\{\pi_{\alpha\in N} B_{\alpha_i}\mid (X_\alpha,E_\alpha)=\coprod_{i\in I} cl^s(B_{\alpha_i})$, for countable many values $n\}$. Hence Γ is a countable. Further, $\coprod_{i\in I}(cl^s(\pi_{\alpha\in N} B_{\alpha_i}) = \pi_{\alpha\in N}(\coprod_{i\in I} cl^s(B_{\alpha_i}))$. Then for any soft open cover ψ for $(\pi_{\alpha\in N} X_\alpha, \pi_{\alpha\in N} \tau_\alpha, \pi_{\alpha\in N} E_\alpha)$, there is a soft countable subfamily of ψ whose soft closure covers $(\pi_{\alpha\in N} X_\alpha, \pi_{\alpha\in N} \tau_\alpha, \pi_{\alpha\in N} E_\alpha)$.

Remark 3.13: By the above results we have the following diagram:

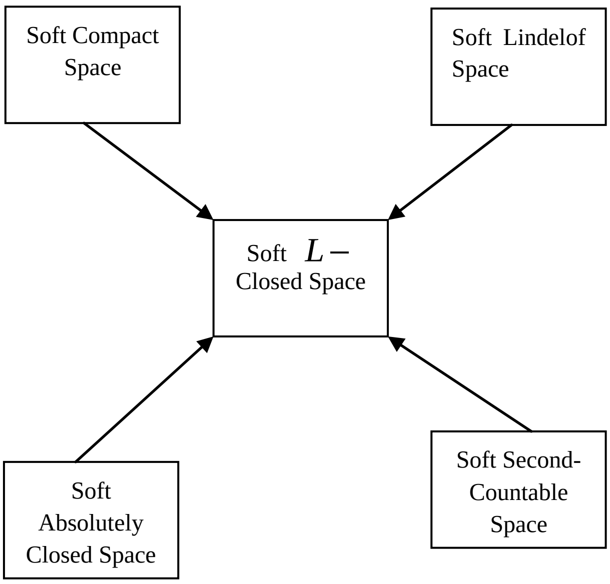

Figure 1. Diagram showing relationships among some of the soft spaces

4. Conclusions

In this paper, the concept of soft $L-$closed spaces is introduced. Further, in this work we describe its relation with soft Lindelof space, soft compact space, and soft absolutely closed space. Assume $f:(X,E)\to(Y,K)$ is a soft strongly open map (soft generality open) from (X,τ,E) soft $L-$closed space into (Y,γ,K) soft countable $H-$closed space (soft sequentially $H-$ closed space). The question we are concerned with is: what is the possible property of soft map f need to provide that $f((X,E))$ is soft countable $H-$closed (soft sequentially $H-$closed) subspace of (Y,γ,K).?

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