A comparative study between the Adomain decomposition method (ADM) and the variational iteration method (VIM)

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Abstract

In this paper, three cases are discussed. First, the comparison between Adomain decomposition method (ADM) and the variational iteration method (VIM). Second, the failure of ADM to solve some boundary value problems like one space-dimensional spatially homogeneous heat equation for some initial and boundary conditions. Third, using VIM when the ADM fails to be convergent.

Key words: ADM, VIM, Heat equation.

1. Introduction

Recently a great deal of interest has been focused on the application of ADM developed by Adomain (1998) and VIM developed by He (1998), for solution of many different problems. For example, boundary value problems, algebraic equations and partial differential equations. The ADM, which accurately computes the series solution, is of great interest to applied sciences. The method provides the solution in a rapidly convergent series with components that one elegantly computed . The main advantage of the method is that it can be applied directly for all types of differential and integral equations, linear or nonlinear, homogeneous or inhomogeneous, with constant coefficients or with variable coefficients. Another advantage is that the method is capable of greatly reducing the size of computation work while still maintaining high accuracy of the numerical solution Somail (2007).

Nelson (1988) discussed theoretically the failure of ADM to solve some boundary value problems like one space-dimensional spatially homogeneous heat equation for some boundary conditions.

In this paper, firstly, the convergence and divergence cases of those problems are discussed practically. Secondly, the VIM is used to solve the problem of the case where ADM fails to be convergent.

2. The Methods

In what follows, the main points of each of the two methods are briefly highlighted.

2.1. ADM Javidi (2007)

Here we demonstrate the main algorithm of ADM on general nonlinear partial differential equation:

$$L_t u(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \quad (1)$$

with the initial condition, $u(x,0) = f(x)$, (2)

where $L_t = \frac{\partial}{\partial t}$ and R are linear operators, and R has partial derivatives with respect to x, $Nu(x,t)$ is a nonlinear terms and $g(x,t)$ is an inhomogeneous term.

. We are looking for the solution satisfying equations (1)-(2). The decomposition method consist of approximating the solution of (1)-(2) as an infinite series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \quad (3)$$

and decomposing ϕ as

$$Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, ..., u_n), \quad (4)$$

Where A_n's are the Adomain polynomials given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\alpha^n}[N(\sum_{k=0}^{n}\alpha^k u_k)]_{\alpha=0}, \qquad n = 0,1,2,... \quad (5)$$

Assuming the inverse of operator L_t exists it can be take as $L_t^{-1}(.) = \int_0^t (.)dt$

Therefore, applying on both sides of equation (1) with L_t^{-1} yields

$$u(x,t) = u(x,0) + L_t^{-1}(g(x,t)) - L_t^{-1}(Ru) - L_t^{-1}N(u) \quad (6)$$

Using equations (3) and (4) it follows that

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + L_t^{-1}(g(x,t)) - L_t^{-1}(\sum_{n=0}^{\infty} Ru_n) - L_t^{-1}(\sum_{n=0}^{\infty} A_n)$$
(7)

Therefore, one determines the iterates in the following recursive way:

$$u_0(x,t) = u(x,0) + L_t^{-1}(g(x,t)),$$
$$u_{n+1}(x,t) = -L_t^{-1}(Ru_n + A_n), \quad n = 0, 1, 2, ...$$
(8)

The convergence of this series has been established, using fixed point theorem Cherruault (1989) , Cherruault and Adomain (1993) .

However, in practice, all terms of the series $\sum_{n=0}^{\infty} u_n(x,t)$ cannot be determined, so we use an approximation of the solution from the truncated series $S_M(x,t) = \sum_{n=0}^{M} u_n(x,t)$

with $\lim_{M \to \infty} S_M(x,t) = u(x,t)$.

2.2. VIM

To illustrate the basic concepts of VIM, we consider the same general nonlinear partial differential equation (1), according to VIM , we can construct the following correlation iteration formula:

$$U_{n+1}=U_n+\int_0^t \lambda(s)[L_tU_n+R\tilde{U}_n+N\tilde{U}_n-g]ds \qquad (9)$$

where λ is called Lagrange multiplier which can be identified optimally via variational theory, $R\tilde{U}_n$ and $N\tilde{U}_n$ are considered as restricted variations, i.e. $\delta R\tilde{U}_n=0$, $\delta N\tilde{U}_n=0$.

3. Heat equation

The one space-dimensional spatially homogeneous heat conduction

$$L_{xx}u=L_tu+g \qquad (10)$$

Where $L_t=\frac{\partial}{\partial t}$, $L_{xx}=\frac{\partial^2}{\partial x^2}$ and g is the source term , with the initial condition $u(x,0)=f_1(x)$ and boundary conditions $u(a,t)=f_2(t),\ \frac{\partial u}{\partial x}(b,t)=f_3(t)$.

4. The discussion

4.1. ADM

In order to apply ADM to equation (10), it can put as

$$u=L_{xx}{}^{-1}[L_tu+g]+\alpha_1(t)+\alpha_2(t)x \qquad (11)$$

where $L_{xx}{}^{-1}(.)=\int_0^x\int_0^x(.)dxdx$ and α_1,α_2 can be determined from the boundary conditions and,

$$u=L_t{}^{-1}[L_{xx}u-g]+\gamma(x) \qquad (12)$$

where γ can be determined from the initial condition .

Now Adding equations (11) and (12) and solving the result for u , we obtain

$$u=Ku+u_0 \qquad (13)$$

where

$$u_0=\frac{1}{2}\left[\left(L_{xx}{}^{-1}-L_t{}^{-1}\right)g\right]+\alpha_1(t)+\alpha_2(t)x+\gamma(x) \qquad (14)$$

and the operator K is given by

$$K=\frac{1}{2}[L_t{}^{-1}L_{xx}+L_{xx}{}^{-1}L_t] \qquad (15)$$

equation (13) can be termed "fundamental equation of ADM". By iteration of equation (13), we find that

$$\psi_n=\sum_{i=0}^{n-1}u_i \qquad (16)$$

is an approximation to u , where

$$u_{n+1}=Ku_n \qquad (17)$$

4.2. VIM

Applying VIM to equation (10), we get that

$$U_{n+1}(x,t)=U_n(x,t)+\int_0^t\lambda(s)\left((g(x,s)+L_sU_n(x,s)-L_{xx}\tilde{U}_n(x,s)\right)ds \qquad (18)$$

5. Applications

Consider the one -dimensional heat equation

$$\frac{\partial^2u}{\partial x^2}=\frac{\partial u}{\partial t}+(-t^2+4t-2)e^{-t}\sin x,\quad x\in[0,\frac{\pi}{2}]\ ,\ t\geq 0 \qquad (19)$$

subject to the initial condition $u(x,0)=0$ and boundary conditions $u(0,t)=\frac{\partial u}{\partial t}(\frac{\pi}{2},t)=0$.

The exact solution of this problem is

$$u(x,t)=(\frac{1}{3}t^3+-2t^2+2t)e^{-t}\sin x$$

To solve this problem by ADM , we have $g(x,t)=(-t^2+4t-2)e^{-t}\sin x$ and easily can be determined from equations (11, 12 and 14) that $\alpha_1(t)=\alpha_2(t)=\gamma(x)=0$ and u_0 will be $u_0=(1-t)e^{-t}\sin x$, and by using iteration equation (19) we found that

$u_1=(1-t)e^{-t}\sin x$,

$u_2=(1-t)e^{-t}\sin x$,

$u_3 = (1-t)e^{-t}\sin x$,

$\vdots$

$u_n = (1-t)e^{-t}\sin x$,

and $\psi_n(x,t) = \sum_{i=0}^{n-1} u_i = n(1-t)e^{-t}\sin x$.

Note that $\psi_n(x,t)$ diverge (pointwise) , so ADM fails to solve this problem.

To solve this problem by means of VIM, we get the correlation iteration formula:

$$U_{n+1}(x,t) = U_n(x,t) + \int_0^t \lambda(s)\left((-s^2+4s-2)e^{-s}\sin x + (U_n)_s - (U_n)_{xx}\right)ds$$

Now, calculate variation with respect to U_n , we get

$$\delta U_{n+1}(x,t) = \delta U_n(x,t) + \int_0^t \lambda(s)\left(\delta U_n\right)_s\right)ds$$

$$\delta U_{n+1}(x,t) = \delta U_n(x,t) + \lambda(t)\delta U_n(x,t) - \int_0^t \lambda'(s)\delta U_n\, ds$$

Integrating the above equation by part, we find the stationary conditions,

$$\lambda'(s) = 0$$
$$1 + \lambda(t) = 0,$$

This in turn gives

$$\lambda = -1,$$

so we get the following variational iteration formula:

$$U_{n+1}(x,t) = U_n(x,t) - \int_0^t \left((-s^2+4s-2)e^{-s}\sin x + (U_n)_s - (U_n)_{xx}\right)ds \qquad (20)$$

Using the equation above, the approximate solutions $U_n(x,t)$ are obtained iteratively by substituting $U_0(x,t) = xt(x-\frac{\pi}{2})$ which satisfies the initial and boundary conditions.

Some approximate solutions are listed below,

$$U_1(x,t) = -t(-2e^{-t}(\sin x) + te^{-t}(\sin x) - t)$$

$$U_2(x,t) = -2te^{-t}(\sin x)(t-1)$$

$$U_3(x,t) = -(\sin x)(3t^2e^{-t} - 2 + 2e^{-t})$$

$$U_4(x,t) = -2(\sin x)(2te^{-t} + 2t^2e^{-t} - 4 + 4e^{-t} + t)$$

$$U_5(x,t) = -(\sin x)(10te^{-t} + 5t^2e^{-t} - 20 + 20e^{-t} + 8t - t^2)$$

$$U_6(x,t) = -\frac{1}{3}(\sin x)(54te^{-t} + 18t^2e^{-t} - 120 + 120e^{-t} + 60t - 12t^2 + t^3)$$

$$\vdots$$

and so on.

The approximate solution takes the form $u(x,t) \approx U_n(x,t),$ where n is the final iteration step. Now the figure below show the absolute error between the exact solution and U_6 .

From the results, we can say that the U_6 is closed to the exact solution and can get a closer solution by increasing n .

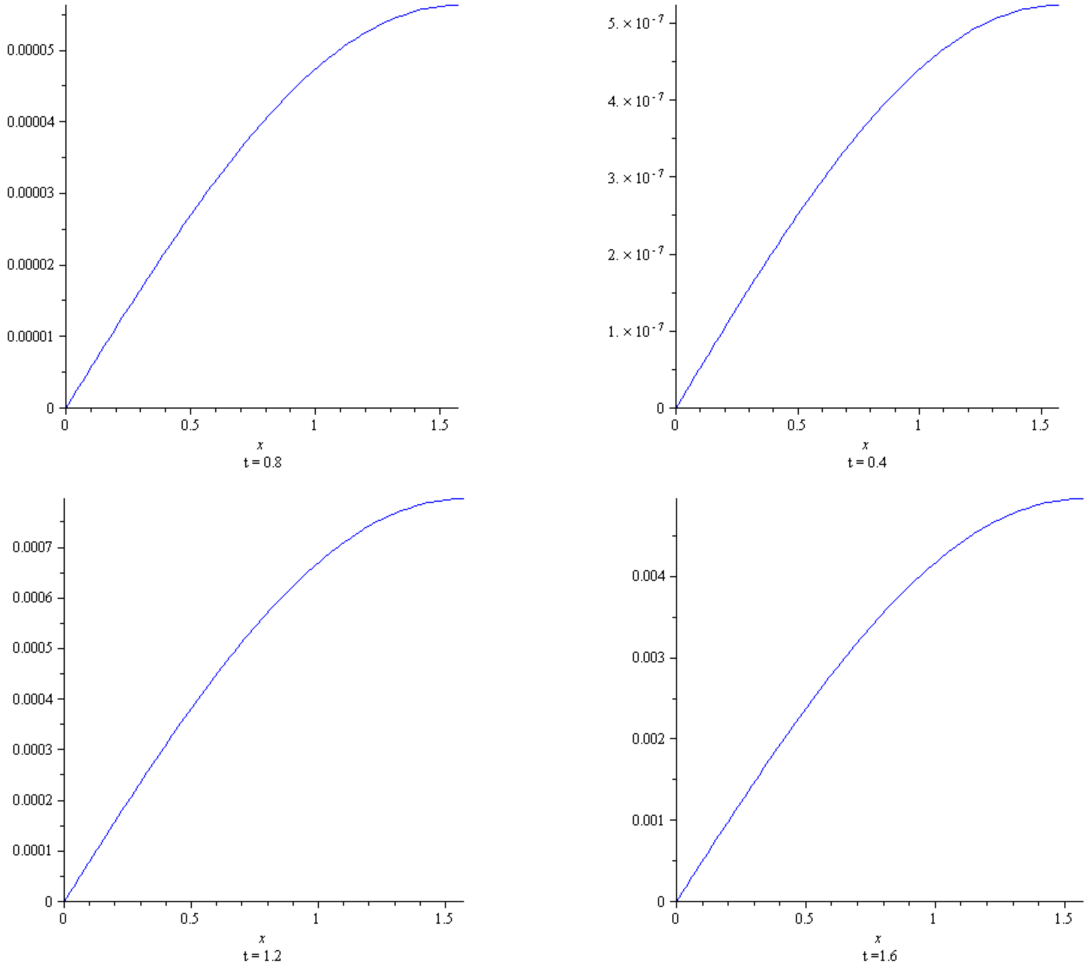

Figure (1):The absolute error between the exact solution

6. Conclusions

The conclusions can be summarized in the following two points:

1. ADM can be applied to the problems with either initial or boundary conditions.
2. ADM fails to be convergent for some boundary value problems that contain initial and boundary conditions like one space-dimensional spatially homogeneous heat equation, because ADM cannot be applied directly for these problems.
3. VIM can be applied directly to solve that boundary value problems mentioned in conclusion (1) by choosing U_0 that satisfy the initial and boundary conditions.

4. VIM reduces the volume of calculations by not requiring the Adomain polynomial like ADM, hence the iterations is direct and straightforward.

7. Future work

As a future work, the modification of ADM to solve the problems that contain initial and boundary conditions can be discussed.

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دراسة مقارنة بين طريقة ADM و طريقة VIM

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المستخلص

في هذا البحث تمت مناقشة ثلاثة حالات، الأولى هي مقارنة بين الطريقتين ADM و VIM ، الثانية هي فشل طريقة ADM لحل بعض مسائل القيم الحدودية مثل معادلة الحرارة ذات البعد الواحد لبعض الشروط الابتدائية والحدودية والثالثة هي استعمال طريقة VIM بديلا عنها في حل تلك المسائل.