Minimax arc of kind (0, δ) with weighted points

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Abstract

In this paper we proved that the existence of (k, $s; f$) – arcs of type (r, s) in PG(2, q); when $q = nt + 1$, derived from a minimax arc C of kind (0, δ), that is a $((n-1)q+1,\ n)$ – arc. We find these (k, $s; f$) – arcs when $s - r = q$, as in the table below. We denote by MIN for minimal, MAX for maximal and X for neither minimal nor maximal.

Summary for $C = ((n-1)q+1, n)$ – arc. [$q = nt+1,\ n > 2$]								
Weights of points	C	0	0	1	$t+1$	$t+1$	$n-1$	$n-t-2$
	E	t	$t+1$	0	0	1	n	$n-1$
	I	$t+1$	$t-n+2$	n	$n-1$	0	0	0
W		MIN	X	MAX	X	MAX	MIN	X

Keywords: projective plane of order q, minimax arcs, $(k, n; f)$–arc of type (m, n), arcs with weighted points.

Introduction

In 1991, the author B. J. Wilson [1] discussed the existence of the minimax arcs in PG(2, q), when q is an odd prime power. We use the results of [1] and combining them with the results of [2, 3, 4, 5, 6, 7, 8] to obtain the minimax arc with weighted points.

Definition 1.[2]

Let π be a projective plane of order q, and denote by $\mathcal{P}$ and $\mathcal{R}$ respectively the set of points and lines of π. Let f be a function from $\mathcal{P}$ into the set N of non – negative integers, and call the weight of p $\in \mathcal{P}$, the value f(p), and the support of f the set of points of the plane having non – zero weight. Using f we can define the function $F: \mathcal{R} \longrightarrow N$ such that for any $r \in \mathcal{R}$ we have $F(r) = \sum_{p \in r} f(p)$. We call $F(r)$ the weight of the line r.

Definition 2.[3]

A $(k,n;f)$– arc K in $PG(2,q)$ is a function $f:\mathcal{P} \longrightarrow N$ such that $k = |$ The support of f $|$ and $n = \max F$.

Observe that a $(k,n) -$ arc$K$ is a $(k,n;f) -$ arc if f is chosen by:

$$f(\text{p}) = \begin{cases} 0, & \text{if} \text{p} \notin K \\ 1, & \text{if} \text{p} \in K \end{cases}.$$

Definition3. (3)

The type of a $(k,n;f)$–arc$K$ is the set of Im F. To write explicitly the type of K we can use the sequence $(n_1, \dots, n_\rho)$ where $n_\lambda \in Im\,F$, $\lambda = 1, \dots, \rho$ and $n_1 < n_2 < \cdots < n_\rho = n$.

Notation

(i) $\omega = \max f$.

(ii) [p] denote the set of lines passing through p.

Lemma 1.(4)

Let K be a $(k,n;f) -$ arc in $PG(2,q)$. Then for every point p we have

$$\sum_{r\in[\text{p}]} F(r) = W + qf(\text{p}).$$

Lemma 2.(4)

The weight W of a $(k,n;f) -$ arc of type (m,n) satisfies:

$$m(q+1) \le W \le (n-\omega)q + n.$$

Definition 4.(4)

We call arcs for which the values in lemma (2) are attained, maximal and minimal$(k,n;f) -$ arcs of type (m,n) respectively.

Theorem 1.(4)

A necessary condition for the existence of a $(k,n;f) -$ arc$K$ of type (m,n), $m > 0$ is that:

(a) $q \equiv 0 \ mod(n-m)$;

(b) $\omega \le n - m$;

(c) $m \le n - 2$.

Minimax arcs

In PG(2, q), where $q = nt + 1$ is an odd prime power and $n > 2$, let C denote a $((n-1)q+1,\ n)$ – arc. Let $P \in C$ and suppose that through P there pass $\rho_i i$ – secant of C for $i = 1, 2, \dots, n$. Then counting lines through P gives

$$\rho_1 + \rho_2 + \cdots + \rho_n = q + 1 \quad (1)$$

And counting points of C on lines through P gives

$$\rho_2 + 2\rho_3 + \cdots + (n-1)\rho_n = (n-1)q. (2)$$

Eliminating ρ_n gives

$(n-1)\rho_1 + (n-2)\rho_2 + \cdots + \rho_{n-1} = n-1$.(3)

The maximum value of ρ_n occurs when

$$\rho_1 = 1, \quad \rho_2 = \cdots = \rho_{n-1} = 0, \quad \rho_n = q \,.$$

And the minimum value of ρ_n occurs when

$$\rho_1 = \rho_2 = \cdots = \rho_{n-2} = 0, \ \rho_{n-1} = n-1, \quad \rho_n = q-n+2 \,.$$

Definition 5.[1]

If every point of C is of one of the above two types then C is called a minmax arc. The points of C are called maximum or minimum points accordingly.

Suppose now that C is a minmax arc having μ maximum points and δ minimum points. We shall refer to C as a minimax arc of kind (μ, δ) when appropriate. Then, using the notation τ_i to denote the number of i – secants of C we have

$\mu + \delta = (n-1)q + 1$(4)

$\tau_1 = \mu$(5)

$\tau_{n-1} = \delta$(6)

$\tau_n = \frac{q\mu + (q-n+2)\delta}{n} = q^2 + 3q - 1 - nq + \mu - \frac{q^2+q+2\mu-2}{n}$(7)

$\tau_0 = (q^2+q+1) - \tau_1 - \tau_{n-1} - \tau_n = (1-q-\mu) + \frac{q^2+q+2\mu-2}{n}$(8)

Let Q be a point not on C and suppose that through Q there pass $\sigma_i i$ – secants for $i = 0,\ 1,\ n-1,\ n$. Then

$\sigma_0 + \sigma_1 + \sigma_{n-1} + \sigma_n = q+1$.(9)and

$\sigma_1 + (n-1)\sigma_{n-1} + n\sigma_n = nq - q + 1$.(10)

From these we may obtain

$n\sigma_0 + (n-1)\sigma_1 + \sigma_{n-1} = n + q - 1$. (11)

Now, suppose that C be of kind($0,\delta$) in PG(2, q) with $q = nt+1$. Through each point of $C$are $q-n+2$, n –secantand $n-1$,$(n-1)$ – secants. Hence the equations (4) – (8), become:

$\delta = (n-1)q + 1 = (n-1)nt + n$.(12)

$\tau_1 = 0$. (13) $\tau_{n-1} = \delta = (n-1)nt + n$. (14)

$$\tau_n = \frac{(q-n+2)\delta}{n} = \frac{(q-n+2)\big((n-1)nt+n\big)}{n}$$

$= (q-n+2)\big((n-1)t+1\big) = (nt-n+3)\big((n-1)t+1\big)$,(15)

$\tau_0 = (1-q) + \frac{q^2+q-2}{n} = \frac{(nt+1)^2 + nt-1}{n} - nt = nt^2 + 3t - nt$. (16)

Since C be of kind (0, δ), then the equations (9), (10) and (11) becomes:

$\sigma_0 + \sigma_{n-1} + \sigma_n = nt + 2$. (17)

$(n-1)\sigma_{n-1} + n\sigma_n = (n-1)nt + n$. (18)

$n\sigma_0 + \sigma_{n-1} = n + nt$. (19)

From equation (18) and (19) we must have $n|\sigma_{n-1}$.Let l be a $(n-1)$ – secant. l has to meet $(\tau_{n-1} - 1 = (n-1)q)$ other, $(n-1)$ – secant. Since l contain $(n-1)$ points of C and through each point there pass $(n-2)$ lines differ from l, then it meets $(n-1)(n-2)$ lines on C and $(n-1)(q-n+2)$ in $(n-1)$ ones in its remaining $(q-n+2)$ points. Hence $\sigma_{n-1} = n$ on $l \setminus C$. Thus the points not on C may be classified as:

Table (1)

Types	σ_0	σ_{n-1}	σ_n
I	$t+1$	0	$(n-1)t+1$
E	t	n	$(n-1)t-n+2$

$$|C| + |E| + |I| = q^2 + q + 1 = n^2t^2 + 3nt + 3.$$

Since $|C| = (n-1)q + 1 = n^2t - nt + n$.

$|E| + |I| = q^2 + 2q - nq = n^2t^2 - n^2 t + 4nt - n + 3$. From equation (12.9) of Lemma (12.1)of [9], we have:

$$(q+1-i)\tau_i = \sum_{Q \in PG(2,q)\setminus C} \sigma_i$$

$$\Longrightarrow (q-n+2)\tau_{n-1} = \sum_{Q \in PG(2,q)\setminus C} \sigma_{n-1} .$$

$$(q-n+2)\big((n-1)nt + n\big) = \sum_{Q \in E} \sigma_{n-1} = n|E| .$$

$n|E| = (nt - n + 3)\big((n-1)nt + n\big)$.

$|E| = (nt - n + 3)\big((n-1)t + 1\big)$

$= n^2t^2 - nt^2 - n^2t + 5nt - 3t - n + 3$. Then

$|I| = nt^2 - nt + 3t$.

A n – secant meets $n(n-1)$,$(n-1)$ – secants on C and hence has to meet $\big(\tau_{n-1} - n(n-1)\big) = n\big((n-1)t - n + 2\big)$, elsewhere in n's. Then on a n – secant are $\big((n-1)t - n + 2\big)$ E points and hence $q + 1 - n - \big((n-1)t - n + 2\big) = t$, I points. On a $(n-1)$ – secant are $q - n + 2 = nt - n + 3$, E points. A 0 – secant has to meet $(n-1)q + 1$, $(n-1)$ – secants in n's and hence on it are

$\frac{(n-1)q+1}{n} = \frac{(n-1)nt+n}{n} = (n-1)t + 1$,E points and hence $q + 1 - \frac{(n-1)q+1}{n} = \frac{q+n-1}{n} = t + 1,$ I points.

Hence we obtain the following table:

Table (2)

Lines	Points on C	Points on E	Points on I
n – secant	n	$((n-1)t - n + 2)$	t
$(n-1)$ – secant	$n-1$	$nt - n + 3$	0
0 – secant	0	$(n-1)t + 1$	$t + 1$

Minimax arc with weighted points

In this section, we discuss (k, n; f) – arc of type (r, s), Im(f) ={0, α, β} where α and β are determined according to the weights of the lines and Table (2). We will divide this section into three cases and each case partition into three subcases:

<u>Case (1):</u>

Now, in this case assign weights α to I points, 0 to points of C and β to E points. Weights of lines are

$$F(n - \text{secant}) = ((n-1)t - n + 2)\,\beta + t\,\alpha.$$

$$F((n-1) - \text{secant}) = (nt - n + 3)\,\beta.$$

$$F(0 - \text{secant}) = ((n-1)t + 1)\,\beta + (t+1)\alpha.$$

(i) Let $F(n - \text{secant}) = F((n-1) - \text{secant})$. Then

$$((n-1)t - n + 2)\,\beta + t\,\alpha = (nt - n + 3)\,\beta.$$

$$t\alpha = (t+1)\beta.$$

Set $\alpha = t + 1$, $\beta = t$.

$$F(n - \text{secant}) = ((n-1)t - n + 2)\,t + t\,(t+1) = nt^2 - n\,t + 3t\,.$$

$$F((n-1) - \text{secant}) = (nt - n + 3)\,t = nt^2 - n\,t + 3t\,.$$

$$F(0 - \text{secant}) = (t+1)^2 + ((n-1)t + 1)\,t = n\,t^2 + 3t + 1\,.$$

$$r = nt^2 - n\,t + 3t \;,$$

$$s = n\,t^2 + 3t + 1\,.$$

Clearly that $(s-r)|q$, because $s-r=nt+1=q$.

Only r – weighting lines pass through points of C= points of weight 0. Then we expect this arc to be minimal with $\omega=\alpha=t+1$ and

$$W=r(q+1)=(nt^2-nt+3t)(q+1)=(nt^2-nt+3t)(n\,t+2)=n^2t^3+5nt^2-n^2t^2-2nt+6t\,.$$

Result 1. There exists $(q^2+2q-nq,\ n\,t^2+3t+1;f)$ – arc of type $(nt^2-n\,t+3t,\ n\,t^2+3t+1)$, when W is minimal and the points of weight 0 are the points of C.

(ii) Let $F(n-\text{secant})=F(0-\text{secant})$. Then

$$\big((n-1)t-n+2\big)\,\beta+t\,\alpha=\big((n-1)t+1\big)\,\beta+(t+1)\alpha\,.$$

$(n-1)\beta+\alpha=0,$ which is impossible. Because $\alpha>0,\ \beta>0$ and $n>2$.

(iii) Let $F\big((n-1)-\text{secant}\big)=F(\ 0-\text{secant})$.

$$(nt-n+3)\,\beta=\big((n-1)t+1\big)\,\beta+(t+1)\alpha\,.$$

$$(t-n+2)\,\beta=(t+1)\alpha$$

Set $\alpha=t-n+2$, $\beta=t+1$, provides that $n<t+2 \implies t>n-2$.

$$F(n-\text{secant})=\big((n-1)t-n+2\big)(t+1)+t\,(t-n+2)$$

$$=nt^2-nt+3t-n+2\,.$$

$$F\big((n-1)-\text{secant}\big)=(nt-n+3)(t+1)=nt^2+3t-n+3\,.$$

$$F(\ 0-\text{secant})=\big((n-1)t+1\big)(t+1)+(t+1)(t-n+2)$$

$$=nt^2+3t-n+3\,.$$

$$r=nt^2-nt+3t-n+2\ ,\qquad s=nt^2+3t-n+3\,.$$

Also, we have $s-r=nt+1=q$.

This arc is neither maximal nor minimal. Since

$$\sum_{l\in[P]}F(l)=W+q\,f(P).$$

If $P \in C \Longrightarrow f(P) = 0$.

$$W = \sum_{l \in [P]} F(l)$$

$$W = (q-n+2)F(n-\text{secant}) + (n-1)F\big((n-1)-\text{secant}\big)$$

$$= n^2t^3 - n^2t^2 + 5nt^2 - nt + 6t - n + 3 .$$

Result 2. There exists $(q^2+2q-nq,\ nt^2+3t-n+3\ ;\ f\ )$ – arc of type $(\ nt^2-nt+3t-n+2\ ,\ nt^2+3t-n+3)$, when W is neither minimal nor maximal and the points of weight 0 are the points of C.

<u>Case (2):</u>

Assign weights α to I points β to points of C and 0 to E points. Weights of lines are

$$F(n-\text{secant}) = t\alpha + n\beta$$

$$F\big((n-1)-\text{secant}\big) = (n-1)\beta$$

$$F(0-\text{secant}) = (t+1)\alpha .$$

(i) Let $F(n-\text{secant}) = F\big((n-1)-\text{secant}\big)$

$t\alpha + n\beta = (n-1)\beta \Longrightarrow t\alpha = -\beta$, which is impossible.

(ii) Let $F(n-\text{secant}) = F(0-\text{secant})$. Then

$$t\alpha + n\,\beta = (t+1)\alpha \Longrightarrow n\,\beta = \alpha .$$

Set $\alpha = n\ ,\ \ \beta = 1$.

$$F(n-\text{secant}) = nt + n = n(t+1).$$

$$F\big((n-1)-\text{secant}\big) = n-1.$$

$$F(0-\text{secant}) = n(t+1).$$

$$r = n-1 \quad , \quad s = n(t+1).$$

Since through every point of maximum weight $(\ \alpha = n\ )$ there pass only s – weighting lines, then this arc is maximal. Then

$$W = (s-\alpha)q + s = ntq + n(t+1) = nt(nt+1) + n(t+1)$$

$$= n^2t^2 + 2nt + n .$$

Result 3. There exists $(nt^2+n^2t-2nt+3t+n\ ,\ n(t+1)\ ;\ f\ )$ – arc of type $(n-1\ ,\ n(t+1))$, when W is maximal and the points of weight 0 are the points of E.

(iii) Let $F\big((n-1)-\text{secant}\big) = F(0-\text{secant})$. Then

$$(n-1)\beta = (t+1)\alpha$$

Set $\alpha = n-1\ ,\ \ \beta = t+1$, provides that $n \neq t+2$.

$$F(n-\text{secant}) = t(n-1) + n(t+1) = 2nt - t + n = s .$$

$$F\big((n-1)-\text{secant}\big) = (n-1)(t+1) = nt-t+n-1 = r\,.$$

$$F(0-\text{secant}) = (n-1)(t+1) = nt-t+n-1 = r\,.$$

$$s = 2nt-t+n\ , \quad r = nt-t+n-1\,.$$

This arc is neither maximal nor minimal. Then

$$\sum_{l\in[P]} F(l) = W + q\,f(P).$$

If $P \in E \Longrightarrow f(P)=0 \quad \Longrightarrow W = \sum_{l\in[P]} F(l)$

$$W = \big((n-1)t-n+2\big)F(n-\text{secant}) + n\,F\big((n-1)-\text{secant}\big) + t\,F(0-\text{secant})$$

$$= 2n^2t^2 - 2nt^2 + 4nt - 3t + n\,.$$

Result 4. There exists $(nt^2+n^2t-2nt+3t+n\,,\,2nt-t+n;f)$ – arc of type $(\,nt-t+n-1\,,\ 2nt-t+n\,)$, when W is neither maximal nor minimal and the points of weight 0 are the points of E.

<u>Case (3):</u>

Assign weights 0 to I points, α to points of C and β to E points. Weights of lines are

$$F(n-\text{secant}) = n\alpha + \big((n-1)t-n+2\big)\beta\,.$$

$$F\big((n-1)-\text{secant}\big) = (n-1)\alpha + (nt-n+3)\,\beta\,.$$

$$F(0-\text{secant}) = \big((n-1)t+1\big)\beta.$$

(i) Let $F(n-\text{secant}) = F\big((n-1)-\text{secant}\big)$. Then

$$n\alpha + \big((n-1)t-n+2\big)\beta = (n-1)\alpha + (nt-n+3)\,\beta\,.$$

$$\alpha = (t+1)\beta\,.$$

Set $\alpha = t+1\ ,\quad \beta = 1.$

$$F(n-\text{secant}) = n(t+1) + \big((n-1)t-n+2\big) = (2n-1)t+2\,.$$

$$F\big((n-1)-\text{secant}\big) = (n-1)(t+1)+nt-n+3 = (2n-1)t+2\,.$$

$$F(0-\text{secant}) = (n-1)t+1\,.$$

$s = (2n-1)t+2\,, \quad r = (n-1)t+1 \quad , \quad \omega = t+1\,.$

This arc is maximal.

$$W = (q+1)(s-\omega)+\omega = (q+1)\big((2n-2)t+1\big)+(t+1)$$

$$= (nt+2)\big((2n-2)t+1\big) + (t+1)$$

$$= n(2n-2)t^2+(5n-3)t+3\,.$$

Result 5. There exists $(n^2t^2-nt^2+4nt-3t+3,\,(2n-1)t+2;\,f\,)$ – arc of type $((n-1)t+1,\,(2n-1)t+2)$, when Im(f) = $\{0,\,1,\,t+1\}$, W is maximal and the points of weight 0 are the points of I.

(ii) Let $F(n-\text{secant}) = F(0-\text{secant})$. Then

$$n\alpha+\big((n-1)t-n+2\big)\beta = \big((n-1)t+1\big)\beta\,.$$

$$n\alpha = (n-1)\beta\,.$$

Set $\alpha = n-1\,, \quad \beta = n\,.$

$$F(n-\text{secant}) = n(n-1)+n\big((n-1)t-n+2\big) = n(n-1)t+n\,.$$

$$F\big((n-1)-\text{secant}\big) = (n-1)^2+n(nt-n+3) = n^2t+n+1\,.$$

$$F(0-\text{secant}) = n\big((n-1)t+1\big) = n(n-1)t+n\,.$$

$s = n^2t+n+1 \quad , \quad r = n(n-1)t+n \;, \quad \omega = n\,.$

This arc is minimal.

$$W = r\,(q+1) = (n(n-1)t+n)(nt+2) = n^2(n-1)t^2+(3n^2-2n)t+2n\,.$$

Result 6. There exists $(n^2t^2-nt^2+4nt-3t+3,\,n^2t+n+1;\,f\,)$ – arc of type ($n(n-1)t+n$, n^2t+n+1), when Im(f) = $\{0,\,n-1,\,n\}$, W is minimal and the points of weight 0 are the points of I.

(iii) Let $F\big((n-1)-\text{secant}\big) = F(0-\text{secant})$. Then

$$(n-1)\alpha+(nt-n+3)\,\beta = \big((n-1)t+1\big)\beta\,.$$

$(n-1)\alpha = \big(n-(t+2)\big)\beta$, which is impossible if $n \le (t+2)$.

Now, we discuss the case in which $n > (t+2)$. Then we suppose that $n = m + (t+2),\ m \in Z^{+}$.

Set $\alpha = n - (t+2) = m$, $\beta = n - 1 = m + t + 1$.

$$F(n-\text{secant}) = nm + \big((n-1)t - n + 2\big)(m+t+1)$$

$$= (n-1)^2\, t - nt + n - 2\,.$$

$$F\big((n-1)-\text{secant}\big) = (n-1)m + (nt - n + 3)(m + t + 1)$$

$$= (n-1)^2\, t + n - 1\,.$$

$$F(0-\text{secant}) = \big((n-1)t+1\big)(n-1) = (n-1)^2\, t + n - 1\,.$$

Since $(n-1)^2\, t + n - 1 - (n-1)^2\, t + nt - n + 2 = nt + 1 = q$, then $s = (n-1)^2\, t + n - 1$, $r = (n-1)^2\, t - nt + n - 2$. Also this arc is neither minimal nor maximal. Then we have

$$\sum_{l\in[P]} F(l) = W + q\, f(P).$$

If $P \in \mathrm{I} \Rightarrow f(P) = 0 \quad \Rightarrow W = \sum_{l\in[P]} F(l)$

$$W = \big((n-1)t+1\big)F(n-\text{secant}) + (t+1)\,F(0-\text{secant})$$

$$W = \big((n-1)t+1\big)\big((n-1)^2 t - nt + n - 2\big) + (t+1)\big((n-1)^2\, t + n - 1\big)$$

$$= (n^3 - 3n^2 + 2n)t^2 + (3n^2 - 7n + 3)t + 2n - 3\,.$$

Result 7. There exists $(n^2t^2 - nt^2 + 4nt - 3t + 3,\ (n-1)^2\, t + n - 1; f)$ – arc of type $((n-1)^2\, t - nt + n - 2,\ (n-1)^2\, t + n - 1)$, when $\mathrm{Im}(f) = \{0, m, n-1\}$, where $m = n - (t+2) \in Z^{+}$. Also W is neither minimal nor maximal and the points of weight 0 are the points of I.

References

(1) B. J. Wilson, (1991), "Minimax arcs", Discrete mathematics 92, 441–450.

(2) M. Y. Abass, (2011), "Existence and non – existence of (k, n; f) – arcs of type $(n-3, n)$ in PG(2, 9)",

M. Sc. Thesis, University of Basrah, Iraq.

(3) E. D'Agostini, (1994), "Onweightedarcswiththreenon–zerocharacters", Journal of Geometry, Vol. 50, 52 – 62.

(4) F. K. Hameed, (1989), "Weighted (k, n) – arcs in the projective plane of order nine", Ph.D. Thesis, University of London Royal Holloway and Bedford New College, England .

(5) F. K. Hameed,M. Hussein and Mohammed Y. Abass, (2011), "On (k, n; f)– arcs of type (n – 5, n) in PG(2, 5)", Journal of Basrah Researches ((Sciences)) Volume 37. Number 4. C, 122 – 131.

(6) R. D. Mahmood, (1990), "(k, n; f) – arcs of type (n – 5, n) in $PG(2,5)$ ", M.Sc. Thesis, University of Mosul, Iraq.

(7) G. Raguso and L. Rella, (1983), " Sui (k, n; f) – architipo (1, n) di un piano proiettivofinito ", Note di Matematica Vol. III, 307 – 320.

(8) B. J. Wilson, (1986), " (k, n; f) – arcs and caps in finite projective spaces", Annals of Discrete mathematics 30, 355 – 362.

(9) J. W. P. Hirschfeld, (1998), "projective geometries over finite fields", Second Edition, Clarendon Press, Oxford, 555+xiv pp.

القوس الأعظم الأصغر من النوع ($0, \delta$)مع النقاط الموزونة

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الخلاصة

في هذا البحث برهنا وجود الأقواس - $(k, s\,; f)$ من النوع (r, s) في PG(2, q) ، عندما $q = nt + 1$، مشتقاً من القوس الأعظم الأصغر Cمن النوع($0, \delta$) أي انه القوس - $((n-1)q+1,\ n)$. نحن وجدنا الأقواس - $(k, s\,; f)$ عندما $q = r -$ و كما مبين في الجدول أدناه . رمزنا بالرمز MAX للقيمة العظمى و MIN للقيمة الصغرى و X للتي ليست عظمى ولا صغرى.

Summary for $C = ((n-1)q+1, n) -$ arc. [$q = nt+1,\ n > 2$]								
Weights of points	C	0	0	1	$t+1$	$t+1$	$n-1$	$n-t-2$
	E	t	$t+1$	0	0	1	n	$n-1$
	I	$t+1$	$t-n+2$	n	$n-1$	0	0	0
W		MIN	X	MAX	X	MAX	MIN	X