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# The Orlik-Solomon Algebra and the Supersolvable Class of Arrangements

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**Abstract**

According to the powerful geometric properties of the hypersolvable order on the hyperplanes of a supersolvable arrangement, we introduced a sufficient condition on the Orlik-Solomon algebra for any central arrangement to have supersolvable analogue and we showed this condition as a necessary condition (not sufficient) on the Orlik-Solomon algebra for any central arrangement to be $K(\pi;1)$. Finally as an illustration of our result, we produce to the Orlik-Solomon algebra for the complexification of the Coxeter arrangement of type $A_r$ and $B_r$, for $r \geq 3$, a structure by their supersolvable partitions analogues.

**Keywords**: Hypersolvable arrangement, supersolvable arrangement, Orlik-Solomon algebra, quadratic Orlik-Solomon algebra, $K(\pi;1)$ arrangement, Coxeter arrangements and complex reflection arrangements

## *Introduction*

Let $\mathcal{A} = \{H_1, ..., H_n\}$ be a complex hyperplane $r$-arrangement with complement $M(\mathcal{A}) = \mathbb{C}^r \setminus \bigcup_{i=1}^{n} H_i$. The problem of expressing the cohomological ring for the complement $M(\mathcal{A})$ with arbitrary constant coefficient in terms of generators and relations was firstly studied by Arnold ([3], 1969) in case $\mathcal{A}$ was the Braid arrangement, i.e. $\mathcal{A}(A_r) = \{x_i - x_j | 1 \leq i < j \leq rk(\mathcal{A})\}$. This problem was later studied by Brieskorn ([6], 1971) for general case. Orlik and Solomon ([14], 1980) generalized Brieskorn result by constructing a graded algebra $\mathrm{A}^*_K(\mathcal{A})$ associated to a complex $r$-arrangement $\mathcal{A}$ and their description involves the geometric lattice of intersections, $L(\mathcal{A}) = \{X \subseteq \mathbb{C}^r \mid X = \bigcap_{H\in\mathcal{B}} H$ and $\mathcal{B} \subseteq \mathcal{A}\}$ of $\mathcal{A}$ which is partially ordered by the inclusions and ranked by $\mathrm{rk}(X) = \mathrm{codim}(X) = r - \dim(X)$. For any commutative ring $K$ and an arbitrary total order $\trianglelefteq$ on the hyperplanes of $\mathcal{A}$, they defined $\mathrm{A}^*_K(\mathcal{A})$ to be the quotient of the exterior $K$-algebra $E^*_K = \bigwedge^{j\geq 0}(\bigoplus_{H\in\mathcal{A}} Ke_H)$ by the homogeneous ideal $I(\mathcal{A})$ that generated by the relations $\sum_{k=1}^{p}(-1)^{k-1}e_{H_{i_1}}...\widehat{e_{H_{i_k}}}...e_{H_{i_p}}$, for all $1 \leq i_1 < ... < i_p \leq n$ such that $\{H_{i_1}, ..., H_{i_p}\}$ is dependent. They proved that $\mathrm{A}^*_K(\mathcal{A})$ (which named by their name), is isomorphic to the cohomological ring of the complement $H^*_K(M(\mathcal{A}))$.

For a given total order $\trianglelefteq$ on the hyperplanes of $\mathcal{A}$, by a circuit $\mathcal{C} \subseteq \mathcal{A}$, we mean a minimal (with respect to inclusion) dependent set. We call $\overline{\mathcal{C}} = \mathcal{C}\backslash\{H\}$ a broken circuit of a circuit $\mathcal{C}$, if $H$ is the smallest hyperplane in $\mathcal{C}$ via $\trianglelefteq$, where an NBC base $\mathcal{B}$ of $\mathcal{A}$ is defined to be a subarrangement of $\mathcal{A}$ that contains no broken circuit. The important point to note here is the collection of all monomials that related to the NBC bases of $\mathcal{A}$ forms a basis of the Orlik-Solomon algebra as free graded module. We refer the reader into [13] as a general reference.

Jambu and Papadima in ([9], 1998) and ([10], 2002) introduced the hypersolvable class of arrangements as a generalization of the supersolvable Stanley class ([15], 1972) by using the collinear relations that encoded in the lattice intersection pattern up to codimension two $\mathcal{L}_2(\mathcal{A}) = \{\mathcal{B} \subseteq \mathcal{A} \mid |B| \leq 3\}$.

The aim of this paper is to study the property that encoded in the structure of $\mathrm{A}^*_K(\mathcal{A})$ and inherits to an arrangement $\mathcal{A}$ a fashion as a supersolvable arrangement. We served our goal as follows:

- Firstly, to control the intersections lattice $L(\mathcal{A})$ of a supersolvable arrangement $\mathcal{A}$, we derived a factorization $\Pi$ on $\mathcal{A}$ from the hypersolvable analogue (definition (1.2)), We called it a hypersolvable partition and denoted by SP. The hypersovable partition that we denoted by HP, introduced firstly by Ali in (2007, [1]) and the existences of such partition forms a necessary and sufficient condition to any central arrangement to be hypersolvable arrangement [2]. Consequently, the hypersolvable order on the hyperplanes of $\mathcal{A}$ via a fixed SP was defined (definition(1.3)).

- Björner and Ziegler in ([5], 1991), gave the impression to reconstruct the supersolvable lattice from the incidences of its $\mathcal{L}_2$ by using a suitable order. We used their technique to derived an SP on a supersolvable arrangement, only by using the existence of a suitable order that respects the supersolvable structure as shown in following result:

  **Theorem 0.1.** *Let $\mathcal{A}$ be a central arrangement. $\mathcal{A}$ is supersolvable if, and only if, every subarrangement of $\mathcal{A}$ which contains no 2-broken circuit forms an NBC base of $\mathcal{A}$ under an order that preserves the supersolvable structure.*

It is worth pointing out that the hypersolvable ordering is the best since it is induced from the supersolvable structure.

- Our main result gave a link between the structure of the supersolvable arrangement and the structure of the Orlik-Solomon algebra. The advantage of using the quadratic Orlik-Solomon algebra $\overline{\mathrm{A}}^{*}_{K}(\mathcal{A})$, lies in the fact that it is constructed just from the structure of $\mathcal{L}_2$, (definition (1.6)).

**Theorem 0.2.** *A central $r$-arrangement $\mathcal{A}$ is supersolvable if, and only if, $A^{*}_{K}(\mathcal{A}) = \overline{A}^{*}_{K}(\mathcal{A})$ under an order that preserves the supersolvable structure.*

- For a supersolvable arrangement $\mathcal{A}$ all the higher homotopy groups of the complement are vanished and such arrangements are called $K(\pi, 1)$ arrangements, where $\pi = \pi_1(M(\mathcal{A}))$ is the fundamental group of the complement of $\mathcal{A}$. In ([7], 1962), Fadell and Neuwirth proved that the supersolvable Braid arrangement $\mathcal{A}(A_r)$ is $K(\pi; 1)$. In ([9], 1973), Brieskorn extended this result to a large class of Coxeter groups and conjectured that this is the case for every Coxeter group. All reflection arrangements have been known to be $K(\pi; 1)$ since the late of 1980. These outstanding cases were settled only recently by Bessis [4]. In ([18], 2013), T. Hoge and G. Röhrle classified all supersolvable reflection arrangements. They proved that all the reflection arrangements of type $D_4, G_{24}, G_{25}, G_{26}, G_{27}, G_{29}, G_{31}, G_{32}, G_{33}$ and $G_{34}$ are not supersolvable. That is, they are $K(\pi; 1)$, but they are not supersolvable arrangements. On the Other hand, Papadima and Suciu in ([14], 1998) proved that, a hypersolvable arrangement is $K(\pi; 1)$ if, and only if, it is supersolvable. The assertion above leads us to give an answer to a part of a question given in [11]: Is the quadratic property of Orlik-Solomon algebra $\mathrm{A}^{*}_{K}(\mathcal{A})$ can produce to the complement $M(\mathcal{A})$ of an arrangement $\mathcal{A}$, a structure as $K(\pi; 1)$ space? our answer is in following results as direct application to theorem (0.2) above:

**Corollary 0.1.** *If $A^{*}_{K}(\mathcal{A}) = \overline{A}^{*}_{K}(\mathcal{A})$ of an $r$-arrangement $\mathcal{A}$, then $\mathcal{A}$ is $K(\pi; 1)$. But the converse need not to be true in general.*

**Corollary 0.2.** *The Orlik-Solomon algebra $A^{*}_{K}(\mathcal{A}) = \overline{A}^{*}_{K}(\mathcal{A})$ of a hypersolvable $r$-arrangement $\mathcal{A}$ if, and only if $\mathcal{A}$ is $K(\pi; 1)$.*

**Corollary 0.3.** *A complex reflection arrangement $\mathcal{A}$ is hypersolvable if, and only if the Orlik-Solomon algebra $A^{*}_{K}(\mathcal{A}) = \overline{A}^{*}_{K}(\mathcal{A})$.*

- Finally, as illustrations of our result, we computed the Orlik-Solomon algebra of the complexification Coxeter arrangements of type $A_r$ and $B_r$, $r \geq 3$.

# 1 *Preliminary Notes*

In this section we review some basic definitions and throughout this paper there is no loss of generality in assuming that $\mathcal{A}$ is an essential, central $r$-arrangement of hyperplanes over $\mathbb{C}$ with $|\mathcal{A}| = n$.

**Definition 1.1.** [12] *A partition $\Pi = (\Pi_1, ..., \Pi_\ell)$ of $\mathcal{A}$ is said to be independent if any resulting $\ell$ hyperplanes $H_j \in \Pi_j$, $1 \leq j \leq \ell$ are independent. A sub-arrangement $S = \{H_{i_1}, ..., H_{i_k}\}$ of $\mathcal{A}$ is called a $k$-section of $\Pi$, if for each $1 \leq j \leq k$, $H_{i_j} \in \Pi_{m_j}$ for some $1 \leq m_1 < \cdots < m_k \leq \ell$. Notice that, if $\Pi$ independent, then all it's $k$-sections are independent. By $\boldsymbol{S}_k(A)$ we denote the set of all $k$-sections of $\Pi$ and $\boldsymbol{S}(\mathcal{A}) = \bigcup_{k=1}^{\ell} \boldsymbol{S}_k(\mathcal{A})$. We call $\Pi$ a factorization of $\mathcal{A}$ if it is independent and for each flat $X \in L_k(\mathcal{A})$, the induced partition $\Pi_X = (\Pi_X^1, ..., \Pi_X^k)$ of $\mathcal{A}_X = \{H \in \mathcal{A} | X \subseteq H\}$ contains a singleton block, where for $1 \leq j \leq k$, $\Pi_X^j = \Pi_m \bigcap \mathcal{A}_X \neq \phi$ for some $1 \leq m \leq \ell$.*

**Definition 1.2.** [1] *A partition $\Pi = (\Pi_1, ..., \Pi_\ell)$ of $\mathcal{A}$ is said to be hypersolvable with length $\ell(\mathcal{A}) = \ell$ and denoted by HP, if $|\Pi_1| = 1$ and for a fixed $2 \leq j \leq \ell$, the block $\Pi_j$ of $\Pi$, satisfies the following properties:*

$j$**-closed property of** $\Pi$**:** *For every two distinct hyperplanes $H_1, H_2$ of $\Pi_1 \bigcup \cdots \Pi_j$, there is no $H \in \Pi_{j+1} \bigcup \cdots \Pi_\ell$ such that $rk(H_1, H_2, H) = 2$.*

$j$**-complete property of** $\Pi$**:** *For every two distinct hyperplanes $H_1, H_2$ of $\Pi_j$, there is $H \in \Pi_1 \bigcup \cdots \Pi_{j-1}$ such that $rk(H_1, H_2, H) = 2$. From $(j-1)$-closed property of $\Pi$, the hyperplane $H$ must be unique and it will be denoted by $H_{1,2}$.*

$j$**-solvable property of** $\Pi$**:** *For every three distinct hyperplanes $H_1, H_2, H_3$ of $\Pi_j$, the resulting hyperplanes from the $j$-complete property of $\Pi$, $H_{1,2}, H_{1,3}, H_{2,3} \in \Pi_1 \bigcup \cdots \Pi_{j-1}$ are either equal or $rk(H_{1,2}, H_{1,3}, H_{2,3}) = 2$.*

*A hypersolvable partition $\Pi$ is said to be supersolvable and denoted by SP, if $\ell(\mathcal{A}) = r$. For $1 \leq j \leq \ell$, let $d_j = |\Pi_j|$. The vector of integers $d = (d_1, ..., d_\ell)$ is called the $d$-vector of $\Pi$ and we define the rank of the blocks of $\Pi$ as $rk(\Pi_j) = rk(\bigcap_{H \in \Pi_1 \bigcup \cdots \bigcup \Pi_j} H)$. We call $\Pi_j$ singular if $rk(\Pi_j) = rk(\Pi_{j-1})$ and we call it non singular otherwise.*

Observe that, $\text{rk}(\Pi_{j-1}) \leq \text{rk}(\Pi_j)$ in general and if $\ell \geq 3$, then every three distinct blocks $\Pi_{i_1}, \Pi_{i_2}, \Pi_{i_3} \in \Pi$ are independent.

**Proposition 1.1.** [2] *$\mathcal{A}$ is hypersolvable if, and only if, $\mathcal{A}$ has a hypersolvable partition. $\mathcal{A}$ is supersolvable if, and only if, $\mathcal{A}$ has a supersolvable partition.*

**Definition 1.3.** *Let $\mathcal{A}$ be a hypersolvable arrangement with a fixed HP, $\Pi = (\Pi_1, ..., \Pi_\ell)$. Via the order that given on the blocks of $\Pi$, a hypersolvable order on $\mathcal{A}$ is defined to be a total order $\trianglelefteq$ on the hyperplanes of $\mathcal{A}$ as for any two distinct hyperplanes $H, H' \in \mathcal{A}$, if $H \in \Pi_i$ and $H' \in \Pi_j$ such that $i < j$, then $H \trianglelefteq H'$. Since $\Pi_1$ is a singleton, hence its hyperplane will be the minimal hyperplane of $\mathcal{A}$ via $\trianglelefteq$.*

From now on, if $\mathcal{A}$ is a hypersolvable $r$-arrangement, then the hyperplanes of $\mathcal{A}$ will ordered by a hypersolvable order related to a fixed Hp $\Pi$ and we will use the following notation:

1. We will denote the set of all $k$-broken circuits of $\mathcal{A}$ by $\mathbf{BC}_k(\mathcal{A})$ and $\mathbf{BC}(\mathcal{A}) = \bigcup_{k=1}^{r} \mathbf{BC}_k(\mathcal{A})$.
2. By $\mathbf{NBC}_k(\mathcal{A})$ we denote the set of all $k$-NBC bases of $\mathcal{A}$ and $\mathbf{NBC}(\mathcal{A}) = \bigcup_{k=1}^{r} \mathbf{NBC}_k(\mathcal{A})$.

**Definition 1.4.** ***The Orlik-Solomon algebra.*** *For any commutative ring $K$ and an arbitrary total order $\trianglelefteq$ of $(\mathcal{A})$, defined the Orlik-Solomon algebra $A_K^*(\mathcal{A})$ to be the quotient of the exterior $K$-algebra $E_K^* = \bigwedge^{j\geq 0}(\bigoplus_{H\in\mathcal{A}} Ke_H)$ by a homogeneous ideal $I(\mathcal{A})$ generated by the relations*

$$\sum_{j=1}^{k}(-1)^{j-1} e_{H_{i_1}} ... \widehat{e_{H_{i_j}}} ... e_{H_{i_k}},$$

*for all $1 \leq i_1 < ... < i_k \leq n$ such that $rk(H_{i_1}, ..., H_{i_k}) < k$.*

*Observe that if $\mathcal{I}_k(\mathcal{A}) = \{e_{\mathcal{B}} |\ \mathcal{B} \subseteq \mathcal{A},\ |\mathcal{B}| = k+1$ and $rk(\mathcal{B}) < k+1\}$ be the set of all those monomials that spanned by the dependent subarrangements of $\mathcal{A}$ with cardinality $k+1$, then $\partial_E^{k+1}\mathcal{I}_k(\mathcal{A})$ generates $I_k(\mathcal{A})$, where $I_k(\mathcal{A}) = I(A) \bigcap E_K^k$ and $\partial_E^* : E_K^* \to E_K^*$ is a differentiation defined on $E_K^*$ as; $\partial_E^0(e_{\{\ \}}) = 0$, $\partial_E^1(e_H) = 1$ and for $2 \leq k \leq n$, $\partial_E^k(e_S) = \sum_{j=1}^{k}(-1)^{j-1} e_{H_{i_1}} ... \widehat{e_{H_{i_j}}} ... e_{H_{i_k}}$, for each $S = \{H_{i_1}, ..., H_{i_k}\} \subseteq \mathcal{A}$. Notice that the differentiation $\partial_{\mathrm{A}}^* : A_K^*(\mathcal{A}) \to A_K^*(\mathcal{A})$ which is defined by $\partial_{\mathrm{A}}^* = \psi^* \circ \partial_E^*$ inherits to $(A_K^*(\mathcal{A}), \partial_{\mathrm{A}}^*)$ a structure of an acyclic chain complex, where $\psi : E_K^* \longrightarrow A_K^*(\mathcal{A})$ is the canonical projection.*

*We call the $r$-arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$,* A*-equivalent if they have isomorphic Orlik-Solomon algebras. We mention that L-equivalent arrangements are* A*-equivalent, but the converse need not to be true in general.*

**Definition 1.5.** ***The quadratic Orlik-Solomon algebra.*** *For any commutative ring $K$ and an arbitrary total order $\trianglelefteq$ of $\mathcal{A}$, the quadratic Orlik-Solomon algebra $\overline{A}_K^*(\mathcal{A})$ is defined to be the quotient of the exterior algebra $\bigwedge^{j\geq 0}(\bigoplus_{H\in\mathcal{A}} Ke_H)$ by a homogeneous ideal $J(\mathcal{A})$ generated by the quadratic relations $e_{H_{i_2}}e_{H_{i_3}} - e_{H_{i_1}}e_{H_{i_3}} + e_{H_{i_1}}e_{H_{i_2}}$, for all $1 \leq i_1 < i_2 < i_3 \leq n$ such that $rk\{H_{i_1}, H_{i_2}, H_{i_3}\} = 2$.*

*Observe that the differentiation* $\partial^*_{\overline{A}} : \overline{A}^*_K(\mathcal{A}) \to \overline{A}^*_K(\mathcal{A})$ *which is defined by* $\partial^*_{\overline{A}} = \overline{\psi}^* \circ \partial^*_E$ *inherits to* $(\overline{A}^*_K(\mathcal{A}), \partial^*_{\overline{A}})$ *a structure of an acyclic chain complex, where* $\overline{\psi} : E^*_K \longrightarrow \overline{A}^*_K(\mathcal{A})$ *is the canonical projection.*

**Remark 1.1.** *Observe that, if* $\mathcal{A}$ *be a hypersolvable* $r$*-arrangement with HP* $\Pi = (\Pi_1, ..., \Pi_\ell)$ *and* $d$*-vector* $d = (d_1, ..., d_\ell)$*, then as a* $K$*- module the quadratic Orlik-Solomon algebra can be represented as;*

$$\overline{A}^*_K(\mathcal{A}) \simeq \bigotimes_{k=1}^{\ell} H^*(\bigvee_{|\Pi_k|} S^1; K);$$

*where* $H^*(\bigvee_{|\Pi_k|} S^1; K)$ *is the cohomological ring (*$K$*-module) of the space* $\bigvee_{|\Pi_k|} S^1$ *of wedge of* $|\Pi_k| = d_k$ *of unit circles, (see [9]).*

# 2 *Main Results*

In the follwing theorem we will used the existence of a suitable ordering that classify the set of independent subarrangements of an arrangement $\mathcal{A}$ as a sufficient condition on $\mathcal{A}$ to be supersolvable:

**Theorem 2.1.** *Let* $\mathcal{A}$ *be a central* $r$*-arrangement.* $\mathcal{A}$ *is supersolvable if, and only if, there exists an ordering such that every subarrangement of* $\mathcal{A}$ *which contains no* 2*-broken circuit forms an NBC base of* $\mathcal{A}$*.*

Proof: Let us first assume $\mathcal{A}$ is supersolvable. Then $\mathcal{A}$ has an SP $\Pi = (\Pi_1, ..., \Pi_r)$. Via the order that given on the blocks of $\Pi$, define a total order $\trianglelefteq$ on the hyperplanes of $\mathcal{A}$ such that for any two distinct hyperplanes $H, H' \in \mathcal{A}$, if $H \in \Pi_i$ and $H' \in \Pi_j$ with $i < j$, then $H \trianglelefteq H'$. By using $\trianglelefteq$ that induced from the supersolvable structure of $\Pi$, one can easily check that every section $\mathcal{B}$ of $\Pi$ has no 2-broken circuit via $\trianglelefteq$. It remains to prove that any subarrangement $\mathcal{B}$ of $\mathcal{A}$ forms an NBC-base via $\trianglelefteq$ if, and only if, $\mathcal{B}$ is a section of $\Pi$. We first assume $\mathcal{B}$ is an NBC-base and by contrary it is not a section of $\Pi$, i.e. there are $H, H' \in \mathcal{B} \cap \Pi_i$, for some $1 < i \leq r$. From the $i$-complete property of $\Pi$, there exists a unique hyperplane say $H" \in \Pi_1 \cup ... \cup \Pi_{i-1}$ such that $rk(H", H, H') = 2$. Hence, $H"$ is minimal than $H$ and $H'$ via $\trianglelefteq$. That is $\{H, H'\}$ form a broken circuit and that contradicts our assumption that $\mathcal{B}$ contains no broken circuit. On the other hand, if we assume that $\mathcal{B}$ is a section of $\Pi$, then $\mathcal{B}$ is independent subarrangement of $\mathcal{A}$, i.e. either $\mathcal{B}$ forms an NBC-base or a broken circuit. To obtain a contradiction, suppose $\mathcal{B}$ is a broken circuit. Thus, there exists a hyperplane say $H$ satisfies that $H$ is the minimal hyperplane of $\mathcal{A}$ with $\{H\} \cup \mathcal{B}$ is a circuit. Let $H'$ be the minimal hyperplane of $\mathcal{B}$. Since $\Pi$ is independent partition, $H \trianglelefteq H'$ and $rk(\mathcal{B}) = rk(\{H\} \cup \mathcal{B})$, hence $H$ and $H'$ must be in the

same block say $\Pi_i$, for some $1 < i < r$. By applying $i$-complete property of $\Pi$, there exists a unique hyperplane say $H" \in \Pi_1 \cup ... \cup \Pi_{i-1}$ such that $rk(H", H, H') = 2$ and $H"$ is minimal than $H$ and $H'$ via $\trianglelefteq$. Consequently, $rk(\{H"\} \cup \mathcal{B}) = rk(\{H\} \cup \mathcal{B}) = k$ and this is a contradiction sine $\{H"\} \cup \mathcal{B}$ is $k$-dependent subarrangement of $\mathcal{A}$ and their hyperplanes distributed among $k+1$ independent blocks of $\Pi$. That is, we introduce the order that induced from any SP $\Pi$ on a supersolvable arrangement $\mathcal{A}$ as our best choice to emphasize the property that every subarrangement of $\mathcal{A}$ that contains no 2-broken circuit forms an NBC base of $\mathcal{A}$.
Conversely, assume that there exists an ordering $\trianglelefteq$ on the hyperplanes of $\mathcal{A}$ such that every subarrangement of $\mathcal{A}$ that contains no 2-broken circuit forms an NBC base of $\mathcal{A}$. We will prove that the order $\trianglelefteq$ construct an SP $\Pi$ on $\mathcal{A}$. For this, we will define a relation on $\mathcal{A}$ as; $H_1 \sim H_2$ for $H_1, H_2 \in \mathcal{A}$ if, and only if, either $\{H_1, H_2\}$ is 2-broken circuit or $H_1 = H_2$. We shall prove that this relation is an equivalence relation:

- **For reflexivity:** It clear that if $H \in \mathcal{A}$, we have $H \sim H$.

- **For symmetry:** If $H_1 \sim H_2$ and $H_1 \neq H_2$, then $\{H_1, H_2\}$ is 2-broken circuit. Suppose $H_{1,2}$ be the minimal hyperplane of $\mathcal{A}$ via $\trianglelefteq$ such that $\{H_{1,2}, H_1, H_2\}$ is a circuit. Thus $\{H_{1,2}, H_2, H_1\}$ is a circuit with broken circuit is $\{H_2, H_1\}$. That is $H_2 \sim H_1$.

- **For transitivity:** Suppose $H_1 \neq H_2 \neq H_3$, $H_1 \sim H_2$ and $H_2 \sim H_3$. We need $H_1 \sim H_3$. To this, let $H_{1,2}, H_{2,3} \in \mathcal{A}$ be the minimal hyperplanes via $\trianglelefteq$ such that $\{H_{1,2}, H_1, H_2\}$ and $\{H_{2,3}, H_2, H_3\}$ are circuits. If $H_{1,2} = H_{2,3}$, then $rk\{H_{1,2}, H_1, H_2, H_3\} = 2$ and that means $\{H_{1,2}, H_1, H_3\}$ is a circuit. Therefore, $H_1 \sim H_3$. Now, if $H_{1,2} \trianglelefteq H_{2,3}$, we have $\{H_{1,2}, H_1, H_2, H_3\}$ and $\{H_{1,2}, H_1, H_{2,3}, H_3\}$ are circuits with their broken circuits $\{H_1, H_2, H_3\}$ and $\{H_1, H_{2,3}, H_3\}$ respectively. From our assumption, $\{H_1, H_{2,3}, H_3\}$ contains a 2-broken circuit. We have $H_{2,3} \nsim H_3$, since $H_{2,3}$ be the minimal hyperplane with $\{H_{2,3}, H_2, H_3\}$ is a circuit. Suppose $H_1 \sim H_{2,3}$ and $H$ be the minimal hyperplane via $\trianglelefteq$ such that $\{H, H_1, H_{2,3}\}$ is a circuit. That is, $\{H, H_1, H_{2,3}, H_3\}$ is a circuit and that contradicts our assumption that $H_{1,2}$ is the minimal hyperplane via $\trianglelefteq$ such that $\{H_{1,2}, H_1, H_2, H_3\}$ is a circuit. Therefore, $H_1 \nsim H_{2,3}$ and $\{H_1, H_3\}$ is the unique 2-broken circuit that contained in $\{H_1, H_{2,3}, H_3\}$. That is $H_1 \sim H_3$.

It is suffices now to prove that the partition $\Pi = (\Pi_1, ..., \Pi_\ell)$ that preserves the ordering $\trianglelefteq$ and induces from this equivalence relation is an SP on $\mathcal{A}$.
**For** $|\Pi_1| = 1$**:** If $H$ is the minimal hyperplane of $\mathcal{A}$ via $\trianglelefteq$, then $\Pi_1 = \{H\}$, since $\{H \nsim H'\}$, for any $H' \in \mathcal{A}$.
**For** $2 \leq j \leq \ell$ we have the following:

- **For $j$-closed property $\Pi$:** To show that, suppose $H_1, H_2 \in \Pi_1 \cup \ldots \cup \Pi_j$ and $H \in \Pi_{j+1} \cup \ldots \cup \Pi_\ell$ such that $rk\{H_1, H_2, H\} = 2$. If $H_1$ be the minimal hyperplane of $\mathcal{A}$ such that $\{H_1, H_2, H\}$ is a circuit, then $H_2 \sim H$. But this contradicts the fact that $H$ and $H_2$ are from different equivalence classes. On the other hand, if there is a hyperplane $H'$ minimal than $H_1$ with $rk\{H', H_2, H\} = 2$. Then $H' \in \Pi_1 \cup \ldots \cup \Pi_j$ and $\{H', H_2, H\}$ is a circuit. That is $H_2 \sim H$ which also a contradiction. Therefore, there is no hyperplane $H \in \Pi_{j+1} \cup \ldots \cup \Pi_\ell$ such that $rk\{H_1, H_2, H\} = 2$.

- **For $j$-complete property $\Pi$:** For that, suppose $H_1, H_2 \in \Pi_j$. Thus $H_1 \sim H_2$ and there is a hyperplane $H$ be the minimal via $\trianglelefteq$ such that $\{H, H_1, H_2\}$ is a circuit. Thus, $H \nsim H_1$, $H \nsim H_2$. That means $H \in \Pi_1 \cup \ldots \cup \Pi_{j-1}$ since the structure of $\Pi$ preserves the order $\trianglelefteq$. We remark that the $j-1$-closed property of $\Pi$ implies that the hyperplane $H$ is unique and we will denoted it by $H_{1,2}$.

- **For $j$-solvable property $\Pi$:** To do this, let $H_1, H_2, H_3 \in \Pi_j$ such that $H_1 \trianglelefteq H_2 \trianglelefteq H_3$ and from the complete property for $\Pi_j$ we have $H_{1,2}, H_{1,3}, H_{2,3} \in \Pi_1 \cup \ldots \cup \Pi_{j-1}$. If $rk\{H_1, H_2, H_3\} = 2$, then by applying the closed property for $\Pi_{j-1}$ we have $H_{1,2} = H_{1,3} = H_{2,3}$. On the other hand, if $rk\{H_1, H_2, H_3\} = 3$, then $H_{1,2} \neq H_{1,3} \neq H_{2,3}$. we shall prove that $\{H_{1,3}, H_{2,3}\}$ is a 2-broken circuit and $H_{1,2}$ is the minimal hyperplane of $\mathcal{A}$ such that $\{H_{1,2}, H_{1,3}, H_{2,3}\}$ forms a circuit. If $rk\{H_{1,2}, H_{1,3}, H_{2,3}\} = 3$, then $\{H_{1,2}, H_{1,3}, H_{2,3}\}$ must be a section of $\Pi$, so it is an NBC-base of $\mathcal{A}$ since it contains no 2-broken circuit. Thus, for $1 \leq i \leq 3$, $\{H_{1,2}, H_{1,3}, H_{2,3}, H_i\}$ forms a section of $\Pi$, i.e. it contains no 2-broken circuit. That is, $\{H_{1,2}, H_{1,3}, H_{2,3}, H_i\}$ forms an NBC-base of $\mathcal{A}$ and that contradicts the fact that for $1 \leq k_1 < k-2 \leq 3$, $rk\{H_{k_1,k_2}, H_1, H_2, H_3\} = 3$. Therefore, $rk\{H_{1,2}, H_{1,3}, H_{2,3}\} = 2$.

The above assertion implies that $\Pi$ is an HP on $\mathcal{A}$ and from the fact that every section of $\Pi$ contains no 2-broken circuit, we have $\ell = r$ and $\Pi$ forms an SP on $\mathcal{A}$ and our claim is down. □

The important point to note here is the existences of a hypersolvable order on a supersolvable arrangement $\mathcal{A}$ that satisfied above theorem give rise to an SP $\Pi = (\Pi_1, ..., \Pi_r)$ satisfied; $\mathbf{NBC}_k(\mathcal{A}) = \mathbf{S}_k$, for $1 \leq k \leq r$. Now, We will illustrate our hypersolvable order by the following examples:

**Example 2.1.** *Let $\mathcal{A}(A_r)$ denotes the complexification of the Coxeter arrangements of type ($A_r$, $r \geq 3$) i.e the defining polynomial of $\mathcal{A}(A_r)$ is;*

$$Q(\mathcal{A}(A_r)) = \prod_{1 \leq i < j \leq r+1} (x_i - x_j).$$

*It is known that, $\mathcal{A}(A_r)$ is the Braid arrangement which is non essential r-supersolvable and we will leave to the reader as a simple exercise to show that the partition $\Pi_{A_r} = (\Pi_1^{A_r}, ..., \Pi_r^{A_r})$ forms an SP on $\mathcal{A}(A_r)$, where $\Pi_k^{A_r} = \{x_1 = x_{k+1}, x_2 = x_{k+1}, ..., x_k = x_{k+1}\}$, for $1 \leq k \leq r$. According to the hypersolvable ordering that given in the structures of $\Pi_k^{A_r}$ above, $3 \leq k \leq r$, we mentioned that, every three distinct hyperplanes $H_{i_1}, H_{i_2}, H_{i_3} \in \Pi_k$ has $rk(H_{i_1}, H_{i_2}, H_{i_3}) = 3$, satisfied $H_{i_1} \trianglelefteq H_{i_2} \trianglelefteq H_{i_3}$ if, and only if, $H_{i_1,i_2} \trianglelefteq H_{i_1,i_3} \trianglelefteq H_{i_2,i_3}$, where $H_{i_1,i_2}, H_{i_1,i_3}, H_{i_2,i_3} \in \Pi_1 \cup \cdots \cup \Pi_{k-1}$ are the hyperplanes that arising from the k-complete property of $\Pi$.*
*If $\mathcal{A}$ is an r-arrangement has the defining polynomial*

$$Q(\mathcal{A}) = x_1.x_2...x_r \prod_{1 \leq i < j \leq r} (x_i - x_j);$$

*then $\mathcal{A}$ is r-supersolvable since it has the SP, $\Pi_{\mathcal{A}} = (\Pi_1^{\mathcal{A}}, ..., \Pi_r^{\mathcal{A}})$, where $\Pi_k^{\mathcal{A}} = \{x_k = 0, x_1 = x_k, x_2 = x_k, ..., x_{k-1} = x_k\}$, for $1 \leq k \leq r$. According to the hypersolvable ordering that given in the structures of $\Pi_k^{A_r}$ and $\Pi_k^{\mathcal{A}}$ above, there is a one to one correspondence, $\pi : \Pi_{A_r} \to \Pi_{\mathcal{A}}$ which is define a one to one correspondence $\pi : \mathcal{A}(A_r) \setminus T \to \mathcal{A}$ preserve the hypersolvable order that given above, where $T = \cap_{H \in \mathcal{A}(A_r)} H$. Therefore, $\mathcal{A}(A_r) \setminus T$ and $\mathcal{A}$ are $\mathcal{L}_2$ equivalent and they are L-equivalent as given in theorem (3.2.12) in [1].*

**Example 2.2.** *Let $\mathcal{A}(B_r)$ denotes the complexification of the Coxeter arrangements of type $B_r$ with defining polynomial;*

$$Q(\mathcal{A}(B_r)) = x_1.x_2...x_r \prod_{1 \leq i < j \leq r} (x_i \pm x_j).$$

*$\mathcal{A}(B_r)$ is r-supersolvable has an SP $\Pi_{B_r} = (\Pi_1^{B_r}, ..., \Pi_r^{B_r})$, where $\Pi_k^{B_r} = \{x_k = 0, x_1 = \pm x_k, x_2 = \pm x_k, ..., x_{k-1} = \pm x_k\}$, for $1 \leq k \leq r$. The SP $\Pi_{B_r}$ has a d-vector, $d = (1, 3, 5, ..., 2r - 1)$.*

The following result is the main result of our study. We will give a link between the quadratic property of the supersolvable (fiber-type) arrangement and the quadratic Orlik-Solomon algebra as a sufficient condition on the structure of Orlik-Solomon algebra of any central arrangement to produce a supersolvable (fiber-type) structure.

**Theorem 2.2.** *A central r-arrangement $\mathcal{A}$ is supersolvable (fiber-type) if, and only if, $A_K^*(\mathcal{A}) = \overline{A}_K^*(\mathcal{A})$ under an order that preserves the supersolvable structure.*

Proof: It is known that if $\mathcal{A}$ is supersolvable, then there exists an ordering on the hyperplanes of $\mathcal{A}$ such that $\mathrm{A}_K^*(\mathcal{A}) = \overline{\mathrm{A}}_K^*(\mathcal{A})$. Deduce that under the hypersolvable ordering which preserves the supersolvable structure we have

$\mathrm{A}_K^*(\mathcal{A}) = \overline{\mathrm{A}}_K^*(\mathcal{A})$. Conversely, suppose that there exists an ordering $\trianglelefteq$ on the hyperplanes of $\mathcal{A}$ such that $\mathrm{A}_K^*(\mathcal{A}) = \overline{\mathrm{A}}_K^*(\mathcal{A})$ and by contrary suppose $\mathcal{A}$ is not supersolvable. By applying theorem (2.1), according to $\trianglelefteq$ there exists a subarrangement of $\mathcal{A}$ which contains no rank two broken circuit and it's not an NBC base of $\mathcal{A}$. Thus, for a fixed $3 \leq k \leq r$ we have a $k$-broken circuit say $\mathcal{B}$ of $\mathcal{A}$ which contains no rank two broken circuit. Deduce that if $1 \leq i_1 < i_2 \leq k$, $H_{i_1},\ H_{i_2} \in \mathcal{B}$, then $\{H_{i_1},\ H_{i_2}\} \in \mathbf{NBC}_2(\mathcal{A})$. Suppose $H$ be the minimal hyperplane of $\mathcal{A}$ has the property $\mathcal{C} = \{H\} \bigcup \mathcal{B}$ forms a circuit, since $\mathcal{B}$ is a broken circuit. It is clear that $\mathcal{C}$ is a dependent subarrangement of $\mathcal{A}$ has the following properties:

- For each $H_{i_1},\ H_{i_2} \in \mathcal{B}$, $1 \leq i_1 < i_2 \leq k$, $rk\{H,\ H_{i_1},\ H_{i_2}\} = 3$, i.e. there is no collinear relation among any three different hyperplanes of $\mathcal{C}$.

- $\{H, H'\} \in \mathbf{NBC}_2(\mathcal{A})$ for each $H' \in \mathcal{B}$. In fact, if $\{H, H'\}$ is a 2-broken circuit, this contradicts our assumption that $H$ is the minimal hyperplane of $\mathcal{A}$ with $\{H\} \bigcup \mathcal{B}$ is a circuit.

Thus, $\mathcal{C}$ is a dependent subarrangement of $\mathcal{A}$ which contains no collinear relation among any three hyperplanes of it and contains no rank two broken circuit. Therefore, $e_{\mathcal{C}} \in \mathcal{I}_k$ and $\partial_E^{k+1} e_{\mathcal{C}} \in I_k$, i.e. $\partial_{\mathrm{A}}^{k+1} a_{\mathcal{C}} = a_{\mathcal{B}} - a_H \partial_{\mathrm{A}}^k a_{\mathcal{B}} = 0_{\mathrm{A}_K^k(\mathcal{A})}$ and $a_{\mathcal{B}} = a_H \partial_{\mathrm{A}}^k a_{\mathcal{B}}$. On the other hand, $e_{\mathcal{C}} \notin J_{k+1}$ and $\partial_E^{k+1} e_{\mathcal{C}} \notin J_k$. Thus; if $\overline{\psi}(e_{\mathcal{C}}) = \bar{a}_{\mathcal{C}}$ we have $\partial_{\overline{\mathrm{A}}}^{k+1} \bar{a}_{\mathcal{C}} = \bar{a}_{\mathcal{B}} - \bar{a}_H \partial_{\overline{\mathrm{A}}}^k \bar{a}_{\mathcal{B}} \neq 0_{\overline{\mathrm{A}}_K^k(\mathcal{A})}$. That is, $\bar{a}_{\mathcal{B}} \neq \bar{a}_H \partial_{\overline{\mathrm{A}}}^k \bar{a}_{\mathcal{B}}$ which contradicts our assumption that $\mathrm{A}_K^k(\mathcal{A}) = \overline{\mathrm{A}}_K^k(\mathcal{A})$. Hence $\mathcal{A}$ is supersolvable. □

**Corollary 2.1.** *If $A_K^*(\mathcal{A}) = \overline{A}_K^*(\mathcal{A})$ of an $r$-arrangement $\mathcal{A}$, then $\mathcal{A}$ is $K(\pi; 1)$. But the converse need not to be true in general.*

Proof: The first part is a direct result of theorem (2.2) and every complex reflection arrangement not that neither $\mathcal{A}(A_r)$ nor $\mathcal{A}(B_r)$, forms a counter example of a $K(\pi; 1)$-arrangement which is not supersolvable, ( see [6] and [8]). □

**Corollary 2.2.** *The Orlik-Solomon algebra $A_K^*(\mathcal{A}) = \overline{A}_K^*(\mathcal{A})$ of a hypersolvable $r$-arrangement $\mathcal{A}$ if, and only if $\mathcal{A}$ is $K(\pi; 1)$.*

Proof: Papadima and Suciu in [14] showed that a hypersolvable arrangement $\mathcal{A}$ is $K(\pi; 1)$ if, and only if, it is supersolvable and our cliam is a direct result of theorem (2.2). □

The following result is a direct result of the corollaries (2.1) and (2.2):

**Corollary 2.3.** *A complex reflection arrangement $\mathcal{A}$ is hypersolvable if, and only if the Orlik-Solomon algebra $A_K^*(\mathcal{A}) = \overline{A}_K^*(\mathcal{A})$.*

**Example 2.3.** *Recall example (2.1). As an application of theorem (2.2), we have:*

$$A^*_K(\mathcal{A}(A_r)) = \overline{A}^*_K(\mathcal{A}(A_r)) \simeq \bigotimes_{k=1}^{r} H^*(\bigvee_{k} S^1; K).$$

*As well as, recall structure of the SP in example (2.2), the Orlik-solomon for* $\mathcal{A}(B_r)$ *can be given as:*

$$A^*_K(\mathcal{A}(B_r)) = \overline{A}^*_K(\mathcal{A}(B_r)) \simeq \bigotimes_{k=1}^{r} H^*(\bigvee_{2k-1} S^1; K).$$

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