On the hypersolvable graphic arrangements

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Abstract:

This paper is devoted to study the hypersolvable graphic arrangements which are originally introduced by Papadima and Suciu in 2002. Motivated by our aim, we defined the hypersolvable partition (which we denoted by Hp), and the hypersolvable ordering on a graph, in order to introduce the existences of them as necessary and sufficient conditions of any graph to be hypersolvable. On the other hand, we studied the hypersolvable graphic matroids and we introduced a comparison between the hypersolvable graphic matroid which is not supersolvable and its deformed supersolvable matroid that obtained from Jambu's-Papadima's deformation method in 1998-2002. Finally, this paper included some of applications and illustrations.

Key words: Hyperplane arrangement, hypersolvable arrangement, connected graph, graphic arrangement, hypersolvable (supersolvable) graph, simplicial complex, matroid, homological group.

(1) Introduction:

By a hyperplane H in a finite dimensional vector space V over a field $K = \mathbb{R}$ or $\mathbb{C}$, we mean an affine subspace of dimension $(\dim V - 1 = r - 1)$ and an arrangement A is a finite collection of hyperplanes H in V. The variety of A is $N(A) = \bigcup_{H\in A} H$ and its complement is $M(A) = V \setminus \bigcup_{H\in A} H$. One of the essential problems in the topological study of hyperplane arrangement is, how we can reflect the combinatorics of A which are encoded in the associated intersection lattice $L(A)$, (which is forms a geometric lattice), into topological information about the topological structure of $M(A)$.

In section (2), we review a brief summary of a notion "hypersolvable arrangement", which was originally introduced by Jambu and Papadima in (1998, [8]) and (2002, [9]) as a generalization of supersolvable (Stanly) class (1972, [21]). And we looked more closely at the construction of a "hypersolvable partition", $\Pi = (\Pi_1, \dots, \Pi_\ell)$ on the hyperplanes of a hypersolvable arrangement A that induced from its hypersolvable analogue, for more details see [2]. Naturally, the properties of the blocks of Π define a hypersolvable ordering on the hyperplanes of A by the collinear relations which is denoted by $\trianglelefteq$ (see definition (2.4)). As well as a lot of various examples and fundamental results were specified throughout this paper. Also, we review the notion of NBC (no broken circuits) of A. Where by a circuit $C \subseteq A$ we mean a minimal (with respect to inclusion) dependent set of hyperplanes and it has a corresponding broken circuit $\bar{C} = C \setminus \{H\}$, where H is the smallest hyperplane in C via a fixed ordering on the hyperplanes of A. We call $B \subseteq A$ is NBC base, if it contains no broken circuit. Note that such a set must be independent, then we denoted B by i-NBC base if $|B| = i$. by $NBC_k(A)$ we denoted the set of all k-NBC bases of A and $NBC(A) = \bigcup_{k=1}^{r} NBC_k(A)$.

The set of all the NBC bases of A forms an explicit bases of the cohomological group of $M(A)$, we refer the reader to [13] as a general reference. Ali in her thesis [2], studied the concept of "NBC bases" of any hypersolvable arrangement.

Jambu and Papadima in ([8] 1998 and [9] 2002) define a vertical deformation method which deformed the hypersolvable arrangement A with s-singular blocks into supersolvable arrangement $\tilde{A} = \tilde{A}_1$ by one-parameter family of arrangements $\{\tilde{A}_t\}_{t\in\mathbb{C}}$ in $\mathbb{C}^r \times \mathbb{C}^s = \mathbb{C}^\ell$, with preserving all the collinear relations of A, i.e. this method preserves the lattice intersection pattern up to codimension two $\ell_2(A) = \{B \subseteq A \big| \; |B| \leq 3\} \approx \ell_2(\tilde{A})$. An algorithm to compute the deformed hypersolvable arrangement $\tilde{A}$ by using the hypersolvable partition analogue was given in [2], with a comparison between the structures of the NBC bases of A and the structures of the NBC bases of $\tilde{A}$, which enables us to see the deformed properties in each block of A.

Graph theory is a fundamental and powerful mathematical tool for a wide range of applications. Many problems are arising in such various fields as chemistry, industrial and electrical engineering, transportation planning, management, marketing, and education can be posed as problems from graph theory [11]. In the network can be modeled by a graph. Conversely, any graph can also be considered as a topological structure of some interconnection network [10]. In section (3) we specialized on the “hypersolvable graphs” which is firstly defined by Papadima and Suciu in (2002, [14]). In general, if G is a finite simple non oriented graph and A_G be the corresponding graphic arrangement, then the correspondence $G \mapsto A_G$ gives a map from the class of finite simple non oriented graphs into the class of arrangements. This map may be used to "pull back" results concerning arrangements to results concerning graphs. Thus we will used this duality between the notions “graphs” and “graphic arrangements” to reflect some known results in the class of hypersolvable arrangements into the class of hypersolvable graphs by using hypersolvable partitions analogue. In section (3) we define the notion a "hypersolvable partition" $\Pi = (\Pi^V, \Pi^\mathcal{E})$ of a graph G which inherits to G a fashion as a hypersolvable graph and we proved that the existences of it as a necessary and sufficient condition of any graph to be hypersolvable.

Moreover, in section (3) we study certain special central arrangements obtained from finite non oriented graphs, they are called graphic arrangements and we specialized on the hypersolvable graphic arrangements in order to introduce applications of the hypersolvable partition on a hypersolvable graph.

Section (4) is devoted to introduce the notion of "Matroids". A matroid is a pair $M = (A, \Delta)$, where A is a finite set and Δ is a non-empty collection of subsets of A called independent sets such that Δ forms a simplicial complex and every induced subcomplex of Δ is a pure, i.e. if $B \subseteq A$, the maximal elements of $\Delta \cap 2^B$ have the same cardinality, where $2^B = \{C \subseteq A \;|C \subseteq B\}$. That is a matroid M is essentially a set with some kind of 'independence structure' defined on it. With a finite matroid M there associated several simplicial complexes that are interrelated in an appealing way. They carry some of significant invariants of M as face numbers and Betti numbers that give rise to useful algebraic structures. Such complexes are: the G-complex Δ, the broken circuit complex $NBC_{\trianglelefteq}(M)$ and the reduced broken circuit complex $\overline{NBC}_{\trianglelefteq}(M)$ via a fixed ordering $\trianglelefteq$ of the underlying set A of M. In particular, the broken circuit complex carries the chromatic properties of M. The homology of geometric lattice complexes was determined in the pioneering work of Folkman (1966), see [6]. On the other hand, Folkman's theorem for homology made the geometric lattices, one of the motivating examples for the theory of Cohen-Macaulay Posets (see Stanley, [19]). Orlik and Solomon (1980), showed that the singular cohomology ring of the complement of a complex arrangement of hyperplanes can be described entirely in terms of the order homology of the geometric lattice of intersections. Hence in these connections, the geometric lattice homology is related to interesting applications of matroids within mathematics. In ([2], 2010), Al-Ta'ai and Ali define the hypersolvable partition complex $S_{\Pi}(A)$ of a hypersolvable matroid $M = (A, \Delta)$ and some applications was investigated.

Let I_{Δ} be the homogenous ideal of the polynomial algebra in n-indeterminate, $\mathrm{A} = K[x_1, \dots, x_n]$ generated by all the "minimal" non-faces of Δ. The ring $\mathrm{A}_{\Delta} = \mathrm{A}/I_{\Delta}$ was first considered by M. Hochster (who suggested it to student G. Reisner, see [7] for further study) and independently by this researcher [19] and [22]). Motivated by the fact that the homological information of M is encoded in the associated minimal free resolution of A_{Δ};

$$0 \longrightarrow M_h \longrightarrow M_{h-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow \mathrm{A}_{\Delta} \longrightarrow 0;$$

section (4) included this concept, (see section (2), [4]). In section (4) we studied the hypersolvable graphic matroid which associates to a hypersolvable graph and some results were given. Finally, section (4) included some of applications and illustrations of hypersolvable graphic matroids.

(2) THE HYPERSOLVABLE PARTITION AND THE NBC-BACES OF A HYPERSOLVABLE ARRANGEMENT

The aim of this section is to review a brief summary of the notion a "hypersolvable partition" of an arrangement that is defined by Ali in ([2], 2007).

Definition (2.1): [3] and [13]

1. A partition $\Pi = (\Pi_1, \dots, \Pi_\ell)$ of an arrangement A is said to be *independent*, if for every choice of hyperplanes $H_i \in \prod_i$ for $1 \leq i \leq \ell$, the resulting ℓ-hyperplanes are independent, i.e. $rk\{H_1 \cap \dots \cap H_\ell\} = \ell$. Let $X \in L(A)$ and $\Pi = (\Pi_1, \dots, \Pi_\ell)$ be a partition of A. Then the *induced partition* Π_X is a partition of A_X with blocks are the non-empty subsets $\Pi_i \cap A_X$, $1 \leq i \leq \ell$.
2. Call $S = \{H_1, \dots, H_k\}$ a k-*section* of Π if, for each $1 \leq i \leq k$, $H_i \in \Pi_{m_i}$, where $1 \leq m_1 < \cdots < m_k \leq \ell$. It has been noticed that if Π is independent, then all it's k-sections are independent. By $S_{\Pi}^k(A)$ we denoted the set of all k-sections of Π and $S_{\Pi}(A) = \cup_{K=1}^{\ell} S_{\Pi}^k(A)$.

Definition (2.2): [3]

Let A be a central r-arrangement. A partition $\Pi = (\Pi_1, \dots, \Pi_\ell)$ of A is said to be *hypersolvape* with length $\ell(A) = \ell$, *exponent vector*, (or d-*vector*), $d = (d_1, \dots, d_\ell)$, (where $d_i = |\Pi_i|$ for, $1 \leq i \leq \ell$) and denoted by Hp, if $|\Pi_1| = 1$ (i.e. Π_1 is a singleton) and for fixed $2 \leq j \leq \ell$, Π_j satisfies the following properties:

(closed property of $\mathbf{\Pi}_j$***):*** For any $H_1, H_2 \in \Pi_1 \cup \dots \cup \Pi_j$, there is no hyperplane $H \in \Pi_{j+1} \cup \dots \cup \Pi_\ell$ such that $rk\{H_1, H_2, H\} = 2$.

(complete property of Π_j): For each $H_1, H_2 \in \Pi_j$, there exists $H \in \Pi_1 \cup \ldots \cup \Pi_{j-1}$ such that $rk\{H_1, H_2, H\} = 2$. It has been noticed that, form closed property of Π_j, the hyperplanene H is uniqe and we will denote it by $H = H_{1,2}$.

(solvable property of Π_j): If $H_1, H_2, H_3 \in \Pi_j$, then the hyperplanes, $H_{1,2}, H_{1,3}, H_{2,3} \in \Pi_1 \cup \ldots \cup \Pi_{j-1}$, either $H_{1,2} = H_{1,3} = H_{2,3}$ or $rk\{H_{1,2}, H_{1,3}, H_{2,3}\} = 2$.

For $1 \leq j \leq \ell$, we define the rank of block Π_j of Π as $rk(\Pi_j) = rk\left(\bigcap_{H \in \Pi_1 \cup \ldots \cup \Pi_j} H\right)$. We call Π_j singular if $rk(\Pi_j) = rk(\Pi_{j-1})$ and we call it non singular otherwise. An Hp Π is said to be supersolvable if it is independent. Observe that $rk(\Pi_{j-1}) \leq rk(\Pi_j)$ in general and if $\ell \geq 3$, then every $\Pi_{i_1}, \Pi_{i_2}, \Pi_{i_3} \in \Pi$ are independent, where $1 \leq i_1 < i_2 < i_3 \leq \ell$.

theorem (2.3): [12]

Let A be an essential central complex r-arrangement. A is a hypersolvable if, and only if, A has an Hp Π and A is supersolvable it has a supersolvable partition.

Definition (2.4): [2]

Let $A = \{H_1, \ldots, H_n\}$ be a hypersolvable r-arrangement with Hp $\Pi = (\Pi_1, \ldots, \Pi_\ell)$ and exponent vector $d = (d_1, \ldots, d_\ell)$. For $1 \leq i \leq \ell$, partitioned Π_i into two blocks as; $\Pi_{i*1} = \{H_{i_1}, \ldots, H_{i_k}\} \subseteq \Pi_i$ such that $rk\left(H_{i_1}, \ldots, H_{i_k}\right) = 2$ and $\Pi_{i*2} = \Pi_i \setminus \Pi_{i*1}$, such that, $rk\left(H_{i_1}, \ldots, H_{i_k}, H\right) = 2$ for each $H \in \Pi_{i*2}$. Define a *hypersolvable order* of A associated to Hp Π and denoted by $\trianglelefteq$, as follows:

1. If $H_i \in \Pi_i$ and $H_j \in \Pi_j$ with $1 \leq i < j \leq \ell$, put $H_i \trianglelefteq H_j$.
2. For fixed $1 \leq i \leq \ell$, we give the hyperplanes of the subblock Π_{i_1} of Π_i an arbitrary total order with preserving the order of Π_{i_2} and Π_j form $1 \leq j \leq i-1$, as follows: if $H_1, H_2, H_3 \in \Pi_i$ with $rk(H_1, H_2, H_3) = 3$, we put $\{H_{i_1}, H_{i_2}, H_{i_3}\} = \{H_1, H_2, H_3\}$, such that $H_{i_1} \trianglelefteq H_{i_2} \trianglelefteq H_{i_3}$ if, and only if, $H_{i_1, i_2} \trianglelefteq H_{i_1, i_3} \trianglelefteq H_{i_2, i_3}$.

Proposition (2.5): [2]

Let A be a hypersolvable arrangement with an Hp $\Pi = (\Pi_1, \ldots, \Pi_\ell)$ and $rk(A) = r$. Then for $2 \leq i \leq r$, every an i-NBC base of A must be an i-section of Π.

Definition (2.6): [3]

Given two r-arrangements $A_1 = \{H_1^1, \ldots, H_n^1\}$ and $A_2 = \{H_1^2, \ldots, H_n^2\}$:

1. We will say A_1 and A_2 have the same lattice or L-equivalent and denoted by $L(A_1) \approx L(A_2)$, if for each $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq k \leq n$ we have $rk(H_{i_1}^1, \ldots, H_{i_k}^1) = rk(H_{i_1}^2, \ldots, H_{i_k}^2)$.
2. For $2 \leq k \leq r-1$, set $\ell_k(A_i) = \{B_i \subseteq A_i \,\big|\, |B_i| \leq k+1\}$ to be the lattice intersection pattern up to codimension k of A_i and $i = 1,2$. We say A_1 and A_2 are ℓ_k-equivalent and denoted by $\ell_k(A_1) \approx \ell_k(A_2)$ if for each $1 \leq i_1 < \cdots < i_j \leq n$ and $j \leq k+1$ we have $rk(H_{i_1}^1, \ldots, H_{i_j}^1) = rk(H_{i_1}^2, \ldots, H_{i_j}^2)$.

Note that, if A_1 and A_2 are L-equivalent, then they are ℓ_k-equivalent for $2 \leq k \leq r-1$. But the converse needs not to be true in general, see [2].

Theorem (2.7): [2]

Let A be a hypersolvable r-arrangement with $rk(A) = r \geq 3$ and Hp, $\Pi = (\Pi_1, \ldots, \Pi_\ell)$ has an exponent vector $d = (1, \ldots, 1)$. Then:-

1. If $|A| = \ell = r$, then A is supersolvable and for $1 \leq j \leq r$;
$$b_j = \left|NBC_j(A)\right| = \binom{r}{j}.$$
2. If $r < m(A) = r + 1 \leq |A| = \ell$, then A is r-generic and for $1 \leq j \leq r$;
$$b_j = \left|NBC_j(A)\right| = \binom{\ell}{j} \text{ and } b_r = |NBC_r| = \binom{\ell}{r} - \binom{\ell-1}{r} = \binom{\ell-1}{r-1}.$$
3. $m(A) \leq r < |A| = \ell$, then for $1 \leq j \leq r$, $b_j = \left|NBC_j(A)\right| \leq \binom{\ell}{j}$.

(3) THE HYPERSOLVABLE PARTITION GRAPHS AND APPLICATIONS OF THE HYPERSOLVABLE PARTITION ON A HYPERSOLVABLE GRAPH

This section is devoted to define a hypersolvable partition of a graph G which inherits to G a fashion as a hypersolvable graph.

Definition (3.1): [13]

A finite simple graph $G = (V, \mathcal{E})$ is an ordered pair consisting of the set $V$of vertices and the set $\mathcal{E}$ of edges with the following two conditions:

1. V is a finite set,
2. $\mathcal{E}$ is a collection of 2-element subsets of V.

A graph $G = (V, \mathcal{E})$ is called *complete* when the set $\mathcal{E}$ is the set of all 2-element subsets of V. From now on we will use the square brackets $[i, j]$ to denote an edge $\{i, j\} \in \mathcal{E}$ in order to distinguish it from the subset of two vertices.

Definition (3.2): [13]

Let $G = (V, \mathcal{E})$ be a graph. The *chromatic function* $\chi(G, t)$ is a function define on the set of nonnegative integers by;

$\chi(G, t) =$ The number of colorings of G with t colors.

Since the chromatic function is a polynomial. Then from now we call it the *chromatic polynomial* of G.

Definition (3.3):

Let $G = (V, \mathcal{E})$ be a connected graph with a finite set of vertices, i.e. $V = \{\upsilon_1, \ldots, \upsilon_m\}$. A pair of partitions, $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$ is said to be a *hypersolvable partition* of G and denoted by Hp Π^G, if $\Pi^V = (\Pi_1^V, \ldots, \Pi_{m-1}^V)$ and $\Pi^{\mathcal{E}} = (\Pi_1^{\mathcal{E}}, \ldots, \Pi_\ell^{\mathcal{E}})$ are partitions of V and $\mathcal{E}$ respectively, such that the following properties are satisfied:

HP$_1$: $\Pi_1^V = \{\upsilon_1, \upsilon_2\}$ and $\Pi_1^{\mathcal{E}} = \{e_1\}$, such that $e_1 = [\upsilon_1, \upsilon_2]$, i.e. $\Pi_1^{\mathcal{E}}$ is a singleton.

HP$_2$: For each $2 \leq j \leq m-1$, the block Π_j^V is a singleton.

HP$_3$: For each $2 \leq k \leq \ell$, the block $\Pi_k^{\mathcal{E}}$ satisfying the following properties:

HP$_3 i$: For each $e_{i_1}, e_{i_2} \in \Pi_1^{\mathcal{E}} \cup \ldots \cup \Pi_k^{\mathcal{E}}$, there is no edge $e \in \Pi_{k+1}^{\mathcal{E}} \cup \ldots \cup \Pi_\ell^{\mathcal{E}}$ such that $\{e_{i_1}, e_{i_2}, e\}$ forms a set of edges of a triangle.

HP$_3 ii$: There exists a positive integer $1 < m_k \leq m-1$, such that $V_k = \Pi_1^V \cup \ldots \cup \Pi_{m_k}^V$ is a subset of V that contains all the end points of the edges in $\Pi_1^{\mathcal{E}} \cup \ldots \cup \Pi_k^{\mathcal{E}}$, i.e. $G_k = (V_k, \Pi_1^{\mathcal{E}} \cup \ldots \cup \Pi_k^{\mathcal{E}})$ forms a subgraph of G. Then, either;

1. $\Pi_k^{\mathcal{E}} = \{e\}$ such that $V_k = V_{k-1}$,

or;

2. $\Pi_k^{\mathcal{E}} = \{e_{i_1}, \dots, e_{i_{d_k}}\}$, such that $V_k \setminus V_{k-1} = \Pi_{m_{k-1}+1}^V = \Pi_{m_k}^V = \{v\}$ and for $1 \le j \le d_k$, $e_{i_j} = [v_{i_j}, v]$, for some $v_{i_j} \in \Pi_1^V \cup \dots \cup \Pi_{m_{k-1}}^V$, where $\{v_{i_1}, \dots, v_{i_{d_k}}\} \subseteq V_{k-1} = \Pi_1 \cup \dots \cup \Pi_{m_{k-1}}$ induces a complete subgraph of G.

The number of the blocks of $\Pi^{\mathcal{E}}$ is called the length of Π and denoted by $\ell(G) = \ell$. For $1 \le k \le \ell$, let $d_k = |\Pi_k^{\mathcal{E}}|$ and $d = (d_1, \dots, d_\ell)$ is said to be the exponent vector (or d-vector) of Π. Define the rank of $\Pi_k^{\mathcal{E}}$ as $rk\, \Pi_k^{\mathcal{E}} = |V_k| - 1$ and $rk(G) = rk\, \Pi_\ell^{\mathcal{E}} = m - 1$. We will call the block $\Pi_k^{\mathcal{E}}$ *singular block*, if $|V_{k-1}| = |V_k|$ and *non-singular* otherwise, i.e. $\Pi_k^{\mathcal{E}}$ is non-singular if $|V_k \setminus V_{k-1}| = 1$.

A hypersolvable partition Π is said to be supersolvable if, and only if, $\Pi^{\mathcal{E}}$ has no singular block.

We will call a hypersolvable partition Π^G, generic if $\ell \ge m$, the exponent vector $d = (1, \dots, 1)$ and every k-eadges of $\mathcal{E}$ cannot be an k-cycle, $3 < k \le m - 1$.

***Remark (3.4)*:**

It has been noticed that;

1. For $1 \le k \le \ell$, the positive integer m_k needs not to be equal to $k - 1$ in general.
2. $\ell \ge m - 1 = rk(G)$.
3. $\ell = m - 1$ if, and only if, Π is supersolvable.
4. $\Pi_2^{\mathcal{E}}$ cannot be a singular block, for $|V_2| = 3$.
5. We call $\mathrm{HP}_3 i$ the closed property of $\Pi_k^{\mathcal{E}}$.

***Corollary (3.5)*:**

For $3 \le k \le \ell$, if $\Pi_k^{\mathcal{E}}$ is a singular block, then $\Pi_k^{\mathcal{E}}$ is a singleton, i.e. $|\Pi_k^{\mathcal{E}}| = 1$.

Proof: By contrary, if $\Pi_k^{\mathcal{E}}$ contains a triangle, then $|V_{m_k} \setminus V_{m_{k-1}}| \ge 3$. Which contradicts our assumption that (G, K) is solvable, and from definition (3.1), $|V_G \setminus V_K| = 0,1$ or 2. □

Note (3.6):

The worth point to note here that every supersolvable graph forms a connected hypersolvable graph. So, there is no loss of generality in assuming that all the graphs that will be used from now on are connected.

Theorem (3.7):

Let G be a connected graph. Then G is hypersolvable if, and only if, G has a hypersolvable partition.

***Proof*:** Firstly, suppose G is a hypersolvable graph, we need to show that G has an Hp. Since G is a hypersolvable graph, hence G has a hypersolvable composition series say,

$$G_1 \subset \cdots \subset G_k \subset G_{k+1} \subset \cdots \subset G_\ell.$$

For $1 \le k \le \ell$, if $G_k = (V^k, \mathcal{E}^k)$, then put;

1. $\Pi_1^V = V^1$ and $\Pi_1^{\mathcal{E}} = \mathcal{E}^1$.
2. For $2 \le j \le \ell$, deduce that, if $V^j \setminus V^{j-1} = \{v, v'\}$, (i.e. $v, v' \notin V^{j-1}$), then for each $v'' \in V^{j-1}$, there exist paths from v'' into v and v' respectivly. But $\{v'', v, v'\}$ can not be a triangle, since there is no edge of $\mathcal{E}^{j-1}$ contains v or v' as end point. Thus, without loss of generality we can rearrange the composition series above such that our choices will be either $V^j \setminus V^{j-1} = \emptyset$ or $V^j \setminus V^{j-1} = \{v\}$. Therefore, there exist

$2 \leq j_2 < j_3 < \cdots < j_{m-1} < \ell$, such that $V^{j_k} \backslash V^{j_{k-1}}$ is non-empty sets, $2 \leq k \leq m - 1$, where $m = |V|$. Put;

$$\Pi_k^V = V^{j_k} \backslash V^{j_{k-1}}.$$

3. For $2 \leq j \leq \ell$, put $\Pi_j^{\mathcal{E}} = \mathcal{E}^j \backslash \mathcal{E}^{j-1}$.

Deduce that $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}}) = ((\Pi_1^V, \dots, \Pi_{m-1}^V), (\Pi_1^{\mathcal{E}}, \dots, \Pi_\ell^{\mathcal{E}}))$ forms a hypersolvable partition of G.

Conversely, suppose that G has a hypersolvable partition, say;

$$\Pi^G = (\Pi^V = (\Pi_1^V, \dots, \Pi_{m-1}^V), \Pi^{\mathcal{E}} = (\Pi_1^{\mathcal{E}}, \dots, \Pi_\ell^{\mathcal{E}})).$$

Put,

$$G_1 = (V^1 = \Pi_1^V, \mathcal{E}^1 = \Pi_1^{\mathcal{E}}).$$

For $2 \leq k \leq \ell$;

$$G_k = (V^k = V_k, \mathcal{E}^k = \Pi_1^{\mathcal{E}} \cup \dots \cup \Pi_k^{\mathcal{E}});$$

where $V_k = \Pi_1^V \cup \dots \cup \Pi_{mk}^V$ as given in definition (3.3). It is clear that $G_1 \subset \cdots \subset G_k \subset G_{k+1} \subset \cdots \subset G_\ell$ forms a hypersolvable composition series of G. □

Theorem (3.8):

A connected hypersolvable graph G is supersolvable if, and only if, G has a supersolvable partition.

Proof: Suppose G is a hypersolvable graph which is supersolvable. Then G has a hypersolvable composition series:

$$G_1 \subset \cdots \subset G_{m-1} = G.$$

$(\ell = m - 1)$ such that:

1. For each $1 \leq k \leq m - 1$, there is a single vertex in $G_k \backslash G_{k-1}$ say v_k,
2. The subgraph of G_k that induced by v_k and its neighbors in G_k is complete.

By applying the proof of theorem (3.7) above the hypersolvable composition series induces a hypersolvable partition say;

$$\Pi^G = (\Pi^V = (\Pi_1^V, \dots, \Pi_{m-1}^V), \Pi^{\mathcal{E}} = (\Pi_1^{\mathcal{E}} \dots, \Pi_{m-1}^{\mathcal{E}})).$$

Thus, $\Pi^{\mathcal{E}}$ has no singular block, thus Π is supersolvable.

Conversely, suppose that G is a hypersolvable graph has a supersolvable partition say;

$$\Pi^G = (\Pi^V = (\Pi_1^V, \dots, \Pi_{m-1}^V), \Pi^{\mathcal{E}} = (\Pi_1^{\mathcal{E}} \dots, \Pi_{m-1}^{\mathcal{E}})).$$

Since $\Pi^{\mathcal{E}}$ has no singular block, hence $\ell = m - 1$. If we apply the proof of theorem (3.7), we have;

$$G_1 \subset \cdots \subset G_k \subset G_{k+1} \subset \cdots \subset G_{m-1},$$

is a hypersolvable composition series. Let $V^k \backslash V^{k-1} = \{v\}$ and let $\{v_{i_1}, \dots, v_{i_{d_k}}\}$ be the set of its neighbors of;

$$G_k = (\Pi_1^V \cup \dots \cup \Pi_k^V, \Pi_1^{\mathcal{E}} \cup \dots \cup \Pi_k^{\mathcal{E}}).$$

Hence, $\Pi_k^{\mathcal{E}} = \{e_{i_1}, \dots, e_{i_{d_k}}\}$ such that $e_{i_j} = [v_{i_j}, v]$, $1 \leq j \leq d_k$. From definition (3.3) of the hypersolvable partition we have $\{v_{i_1}, \dots, v_{i_k}\}$ induced a complete subgraph of G_{k-1} and if we add the blocks $G_k \backslash G_{k-1} = (\Pi_k^V, \Pi_k^{\mathcal{E}})$, we obtain that $\{v_{i_1}, \dots, v_{i_k}, v_k\}$ is a complete subgraph of G_k and this finishes the proof. □

Lemma (3.9): (The complete property of $\mathbf{\Pi}_k^{\mathcal{E}}$)

Let G be a connected hypersolvable graph with a hypersolvable partition $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$. For $2 \leq k \leq \ell$, if $e_1, e_2 \in \Pi_k^{\mathcal{E}}$, then there exists a unique $e \in \Pi_1^{\mathcal{E}} \cup \ldots \cup \Pi_{k-1}^{\mathcal{E}}$ such that $\{e_1, e_2, e\}$ forms a triangle.

Proof: Since $e_1, e_2 \in \Pi_k^{\mathcal{E}}$, hence $|\Pi_k^{\mathcal{E}}| \geq 2$ and by applying definition (3.3) we have $V_k \backslash V_{k-1} = \{v\}$ such that $e_1 = [v_{i_1}, v]$ and $e_2 = [v_{i_2}, v]$. That is v_{i_1} and v_{i_2} form neighbors of v. But the set of all neighbors of v is a complete subgraph of G. Thus $[v_{i_1}, v_{i_2}]$ forms an edge in $\Pi_1^{\mathcal{E}} \cup \ldots \cup \Pi_{k-1}^{\mathcal{E}}$ and $\{e_1, e_2, e\}$ is a triangle. On the other hand, the graph G is simple, so e must be unique. □

Notation:

Since e must be unique, we will denoted it by $e_{i,j}$.

Lemma (3.10): (The solvable property of $\mathbf{\Pi}_k^{\mathcal{E}}$)

Under the hypotheses of lemma (3.2), if $e_1, e_2, e_3 \in \Pi_k^{\mathcal{E}}$, then $\{e_{1,2}, e_{1,3}, e_{2,3}\}$ is a triangle.

Proof: Since $e_1, e_2, e_3 \in \Pi_k^{\mathcal{E}}$, hence $|\Pi_k^{\mathcal{E}}| \geq 2$ and $V_k \backslash V_{k-1} = \{v\}$. Let $e_1 = [v_{i_1}, v]$, $e_2 = [v_{i_2}, v]$ and $e_3 = [v_{i_3}, v]$. It is clear that $e_{1,2} = [v_{i_1}, v_{i_2}]$, $e_{1,3} = [v_{i_1}, v_{i_3}]$, $e_{2,3} = [v_{i_2}, v_{i_3}]$. That is, $\{e_{1,2}, e_{1,3}, e_{2,3}\}$ forms a triangle. □

Definition (3.11):

Let G be a hypersolvable graph with hypersolvable partition. Define a *hypersolvable order* on G associated to an Hp $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$ and denoted by $\trianglelefteq$, as follows:

1. Put an arbitrary order on the vertices of Π_1^V.
2. If $v_i \in \Pi_i^V$ and $v_j \in \Pi_j^V$ such that; $i < j$, put $v_i \trianglelefteq v_j$.
3. If $e \in \Pi_i^{\mathcal{E}}$ and $e' \in \Pi_j^{\mathcal{E}}$ such that; $i < j$, put $e \trianglelefteq e'$.
4. If $e, e', e'' \in \Pi_k^{\mathcal{E}}$, set$e_{i_1} \trianglelefteq e_{i_2} \trianglelefteq e_{i_3} \Leftrightarrow e_{i_1,i_2} \trianglelefteq e_{i_1,i_3} \trianglelefteq e_{i_2,i_3}$, where;

$$\{e_{i_1}, e_{i_2}, e_{i_3}\} = \{e, e', e''\}.$$

If G is supersolvable, we will call $\trianglelefteq$, a supersolvable ordering.

Theorem (3.12): [17]

A graph $G = (V, \mathcal{E})$ is supersolvable if, and only if, there exists an ordering $v_1, v_2, \ldots, v_m$ of its vertices such that if $1 \leq i < j < k \leq m$, such that $[v_i, v_k] \in \mathcal{E}$ and $[v_j, v_k] \in \mathcal{E}$, then $[v_i, v_j] \in \mathcal{E}$. Equivalently, in the restriction of G to the vertices $v_1, \ldots, v_i$ the neighborhood of v_i is a clique.

Proposition (3.13):

Let $G = (V, \mathcal{E})$ be a supersolvable graph with a supersolvable partition $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$. Via a supersolvable ordering $\trianglelefteq$ on G, if $[v_i, v_k] \in \mathcal{E}$ and $[v_j, v_k] \in \mathcal{E}$, then $[v_i, v_j] \in \mathcal{E}$, where $1 \leq i < j < k \leq m$.

Proof: From definition (3.3) of the supersolvable partition $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$, we have v_i, v_j, v_k distributed among the blocks of Π^V as follows;

$$v_i \in \Pi_{i'}^V, v_j \in \Pi_{j'}^V \text{ and } v_k \in \Pi_{k'}^V;$$

with keeping in mind that, either $n' = n - 1$ or $n' = n$, for $n = i, j, k$, i.e. if $i = 1$ and $j = 2$, $i' = j' = 1$. Then $\Pi_{k'}^V \neq \Pi_{n'}^V$, $n = i, j$. Since $[v_i, v_k], [v_j, v_k] \in \Pi_{k'}^{\mathcal{E}}$, hence $[v_i, v_j] \in \Pi_{j'}^{\mathcal{E}}$. In fact, the neighborhood set of v_k is a clique of G. □

Remark (3.14):

Proposition (3.13) is a worth pointing out that the existence of the supersolvable ordering that induced from the structure of the supersolvable partition, forms a necessary and sufficient condition on a graph G to be a supersolvable graph, as shown in Stanly theorem (3.12).

Definition(3.15):

A simple connected graph $G = (V, \mathcal{E})$ with $|V| = m > 3$ is said to be generic if every $m - 1$-eadges of $\mathcal{E}$ cannot be an $m - 1$-cycle.

Proposition (3.16):

A connected graph $G = (V, \mathcal{E})$ with $|V| = m > 3$ is generic if, and only if, G has a generic hypersolvable partition Π^G with length $\ell(G) = m$.

Proof: It is clear that, If G is generic, then $\mathcal{E}$ contains no triangles. That is, one can simply construct a hypersolvable partition $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$ of length $\ell(G) = \ell$ by adding one edge at time. Thus Π^G is generic, since G contains no $m - 1$-cycles. Conversely, if G has a generic hypersolvable partition $\Pi = (\Pi^V, \Pi^{\mathcal{E}})$, then $\mathcal{E} = \bigcup_{k=1}^{\ell} \Pi_k^{\mathcal{E}}$, where $\Pi_k^{\mathcal{E}}$ is a singleton for $1 \leq k \leq \ell$ and from the properties of the generic partition, $\mathcal{E}$ contains no $m - 1$-cycles.

On the other hand, by contrary suppose $\ell(G) > m$ and let $e_{i,j}$ be the k^{th} edge of $\mathcal{E}$ via a hypersolvable order, $m < k \leq \ell$. Deduce that, the block $\Pi_k^{\mathcal{E}}$ is singular, since $rk\left(\Pi_k^{\mathcal{E}}\right) = m - 1$, for $m \leq k \leq \ell$. That is, if $V_k = \Pi_1^V \cup \ldots \cup \Pi_{m_k}^V$ is the subset of V that contains all the end points of the edges in $\Pi_1^{\mathcal{E}} \cup \ldots \cup \Pi_k^{\mathcal{E}}$, then $V_k = V$. But G is a connected graph, so without loss of generality we can construct the generic partition Π^G such that the graph $G' = (V, \Pi_1^{\mathcal{E}} \cup \ldots \cup \Pi_{m-1}^{\mathcal{E}})$ is connected. Thus there is a path of G'started at i and ending at j and the number of edges that this path passes through cannot exceed $m - 1$, since $i \neq j$. Also if we add the block $\Pi_m^{\mathcal{E}}$, then the subgraph $G" = (V, \Pi_1^{\mathcal{E}} \cup \ldots \cup \Pi_{m-1}^{\mathcal{E}} \cup \Pi_m^{\mathcal{E}})$ is a connected subgraph of G that forms an m-cycle and contains no $m - 1$-cycles. Let $\Pi_m^{\mathcal{E}} = \{e_{i',j'}\}$. It is clear that, $e_{i,j} \neq e_{i',j'}$, since G is a simple graph. i.e. $\Pi_1^{\mathcal{E}} \cup \ldots \cup \Pi_{m-1}^{\mathcal{E}} \cup \Pi_m^{\mathcal{E}}$is an m-cycle and if we add $\Pi_k^{\mathcal{E}} = \{e_{i,j}\}$, we will induce a new cycle of $\Pi_1^{\mathcal{E}} \cup \ldots \cup \Pi_m^{\mathcal{E}} \cup \Pi_k^{\mathcal{E}}$ with length less than m, which contradicts our assumption that G contains no $m - 1$-cycles. Therefore $\ell(G) = m$. □

Remark (3.17):

We mention that;

1. $rk(A_G) = |V| - 1$.
2. If $K \subseteq G$, then $rk(A_K) = 2$ if, and only if, K is a triangle of G.
3. $\chi(G, t) = \chi(A_G, t)$.

Proposition (3.18): [14]

A graph G is hypersolvable if, and only if, the graphic arrangement A_G is hypersolvable.

Corollary (3.19):

A graph $G = (V, \mathcal{E})$ is hypersolvable if, and only if, the graphic arrangement A_G has a hypersolvable partition.

Proof: From proposition (3.18), a graph $G = (V, \mathcal{E})$ is hypersolvable if, and only if, A_G is hypersolvable and by applying theorem (3.7), the graphic arrangement A_G is hypersolvable if, and only if, A_G has a hypersolvable partition. □

Remark (3.20):

The important points to note here are that:

1. If $G = (V, \mathcal{E})$ is a hypersolvable graph, then A_G has a partition $\Pi' = (\Pi_1, \dots, \Pi_\ell)$ induced from the hypersolvable partition $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$, as for $1 \le k \le \ell$, $H_{ij} \in \Pi_k$ if, and only if, $[i, j] \in \Pi_k^{\mathcal{E}}$.
2. If $[i_1, j_1]$, $[i_2, j_2]$ and $[i_3, j_3]$ form a triangle. Then the set $\{i_1, j_1, i_2, j_2, i_3, j_3\} = \{i, j, m\}$ contains just three vertices. Therefore, $\{H_{i_1 j_1}, H_{i_2 j_2}, H_{i_3 j_3}\} = \{H_{ij}, H_{im}, H_{jm}\}$ and $rk\{H_{ij}, H_{im}, H_{jm}\} = rk\{(x_1, \dots, x_r) | x_i = x_j = x_m\} = 2$.

Corollary (3.21):

Let A be a hypersolvable graphic arrangement with hypersolvable partition $\Pi' = \{\Pi_1, \dots, \Pi_\ell\}$. For $2 \le k \le \ell$, if $H_1, H_2, H_3 \in \Pi_k$, then;

$$rk\{H_1, H_2, H_3\} = 3.$$

Proof: By contrary, suppose $rk\{H_1, H_2, H_3\} = 2$. Since $H_1, H_2, H_3 \in \Pi_k$, then there exist $[i_1, j_1], [i_2, j_2], [i_3, j_3] \in \Pi_k^{\mathcal{E}}$ such that $[i_1, j_1], [i_2, j_2], [i_3, j_3]$ form a triangle. Therefore, the set $\{i_1, j_1, i_2, j_2, i_3, j_3\} = \{i, j, m\}$ contains just three vertices. Now, from the complete property of the block $\Pi_k^{\mathcal{E}}$, for the edges $[i, j], [i, m] \in \Pi_k^{\mathcal{E}}$, there exists an edge $[j, m] \in \Pi_k^{\mathcal{E}} \cup \dots \cup \Pi_{k-1}^{\mathcal{E}}$ such that $[i, j], [i, m], [j, m]$ form a triangle. Thus there are two edges from the vertex j to m and that contradicts our assumption that our graph contains no parallel edges. Thus, $rk\{H_1, H_2, H_3\} = 3$, as we claimed. □

Corollary (3.22):

Suppose we have the assumption of corollary (3.21). Then $|\Pi_2| = 1$ or 2.

Proof: By applying the fact that $\mathrm{rk}(\Pi_2) = 2$ and corollary (3.21), it is clear that $|\Pi_2| =$ 1 or 2. □

Theorem (3.23):

Let G be a hypersolvable graph with hypersolvable partition $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$. Then the partition $\Pi' = (\Pi_1, \dots, \Pi_\ell)$ that given in remark (3.20) is a hypersolvable partition of A_G.

Proof: We need to show that Π' satisfied the properties of the hypersolvable partition. So, the proof will be divided into the following steps:

HP1: Obviously, Π_1 is a singleton, since $\Pi_1^{\mathcal{E}}$ is a singleton.

HP2: For a given $2 \le k \le \ell$;

For the closed property of $\boldsymbol{\Pi_k}$: Let $H_1, H_2 \in \Pi_k$. From the construction of Π', let $[i_1, j_1], [i_2, j_2] \in \Pi_k^{\mathcal{E}}$ be their related edges and from definition (3.3), we have $\Pi_{m_k}^V = \{v\}$. So, $j_1 = v = j_2$. By contrary suppose that there exists a hyperplane $H \in \Pi_{k+1} \cup \dots \cup \Pi_\ell$ such that $rk\{H_1, H_2, H\} = 2$. That is there exists $[i_H, j_H] \in \Pi_{k+1}^{\mathcal{E}} \cup \dots \cup \Pi_\ell^{\mathcal{E}}$, the related edge to H such that $\{[i_1, v], [i_2, v], [i_H, j_H]\}$ form a triangle. Which contradicts the closed property of $\Pi_k^{\mathcal{E}}$. Therefore, there is no $H \in \Pi_{k+1} \cup \dots \cup \Pi_\ell$ such that $rk\{H_1, H_2, H\} = 2$.

For the complete property of $\boldsymbol{\Pi_k}$: Let $H_1, H_2 \in \Pi_k$ and let $[i_1, j_1], [i_2, j_2] \in \Pi_k^{\mathcal{E}}$ be their related edges. By applying the complete property of $\Pi_k^{\mathcal{E}}$, there exists $[i_H, j_H] \in \Pi_1^{\mathcal{E}} \cup \dots \cup \Pi_{k-1}^{\mathcal{E}}$ such that $\{[i_1, j_1], [i_2, j_2], [i_H, j_H]\}$ forms a triangle. That is there exists $H \in \Pi_1 \cup \dots \cup \Pi_{k-1}$, such that $rk\{H_1, H_2, H\} = 2$.

For the solvable property of Π_k: Suppose $H_1, H_2, H_3 \in \Pi_k$. From definition (3.3), let $\Pi^V_{m_k} = \{v\}$ and from the structure of Π', let $[i_1, v], [i_2, v], [i_3, v] \in \Pi^{\mathcal{E}}_k$ be their related edges. In fact, from corollary (3.21), $rk\{H_1, H_2, H_3\} = 3$ and $\{[i_1, v], [i_2, v], [i_3, v]\}$ cannot be a triangle. By applying definition (3.3), $\{[i_1, i_2], [i_1, i_3], [i_2, i_3]\}$ forms a triangle and their related hyperplanes;

$$H_{1,2}, H_{1,3}, H_{2,3} \in \Pi_1 \cup \dots \cup \Pi_{k-1};$$

Satisfied $rk\{H_{1,2}, H_{1,3}, H_{2,3}\} = 2$ and this complete our proof. □

Construction (3.24):

Suppose $G = (V, \mathcal{E})$ be a hypersolvable graph with hypersolvable partition $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$ such that $\ell > r = rk(G)$, (i.e. G not supersolvable). Let A_G be the graphic arrangement of G, and Π' be the induced hypersolvable partition given in remark (3.20).

Case I : If $\ell = r + 1$, then we have the cases as follows:

Case I.1 : If Π_ℓ be the singular block of Π'. Then, for each $H \in A_G$, put $\tilde{\alpha}_H = (\alpha_H, \lambda_H)$, where

$$\lambda_H = \begin{cases} 0 & \text{if} \quad H \in \Pi_1, \dots, \Pi_{\ell-1} \\ 1 & \text{if} \quad H \in \Pi_\ell \end{cases}.$$

Case I.2 : If $\Pi_{\ell-1}$ be the singular block Π. Then for each $H \in \Pi_1, \dots, \Pi_{\ell-1}$, put $\tilde{\alpha}_H = (\alpha_H, \lambda_H t)$, where

$$\lambda_H = \begin{cases} 0 & \text{if} \quad H \in \Pi_1, \dots, \Pi_{\ell-2} \\ 1 & \text{if} \quad H \in \Pi_{\ell-1} \end{cases}.$$

If $H_{i_1} \in \Pi_\ell$ the minimal hyperplane of Π_ℓ via the hypersolvable order, then for $k = i_1, i_1 + 1$ put $\tilde{\alpha}_{H_k} = (\alpha_{H_k}, \lambda_{H_k} t)$ where;

$$\lambda_{H_k} = \begin{cases} 0 & \text{if} \quad H_{i_1, i_1+1} \in \Pi_1, \dots, \Pi_{\ell-2} \\ 1 & \text{if} \quad H_{i_1, i_1+1} \in \Pi_{\ell-1} \end{cases}.$$

And for $k = i_1 + 2, \dots, i_1 + d_\ell$, put $\tilde{\alpha}_{H_k} = (\alpha_{H_k}, \lambda_{H_k} t)$ where;

$$\lambda_{H_k} = -\frac{(g_k \lambda_{H_{i_1},k} + h_k \lambda_{H_{i_1}})}{q_k}$$

and $(g_k, h_k, q_k) \in \mathbb{C}^3/\underline{0}$ such that, $g_k \alpha_{H_{i_1},k} + h_k \alpha_{H_{i_1}} + q_k \alpha_{H_k} = 0$.

Case II: If Π' has s-singular block. Then we will use iterated applications of the cases (I.1) and (I.2) as shown in construction ((4.1.19), [2]).

Proposition (3.25):

A graph $G = (V, \mathcal{E})$ is a generic graph if, and only if, its graphic arrangement A_G is generic.

Proof: By applying proposition (3.16) and theorem (3.23), our claim will be proved. □

Theorem (3.26):

The following assertions are equivalent:

1. G is supersolvable.
2. Π' is nice.
3. $NBC(A_G) = S(A_G)$ and for $1 \le k \le rk(G)$;

$$b_k = \sum_{i_1=1}^{\ell-k} \sum_{i_2=i_1+1}^{\ell-k+1} \dots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} d_{i_2} \dots d_{i_k}.$$

Proof: This is a direct result of theorem ((1.2), [3]) and theorem (3.8). □

Theorem (3.27):

If $rk(A_G) = m - 1 \leq \ell$, i.e. G is not supersolvable, then via a hypersolvable ordering on the hyperplanes of A_G, we have the following:

1. Recall the definition of $p(A_G)$ in corollary (3.15). Then $2 \leq p(A_G) \leq m - 2$ such that for each $1 \leq k \leq p(A_G)$, $NBC_k(A_G) = S_k(A_G)$ and $NBC_{p(A)+1}(A_G) = S_{p(A)+1}(A_G) \setminus S_{p(A_G)+1}(A_G) \cap BC_{p(A_G)+1}(A_G)$.
2. If $m = 4$, then $p(A_G) = 2$.

Proof: This is a direct application of corollary ((1.4), [3]) and proposition (3.18). □

Remark (3.28):

Let G be a graph with no triangles. It is clear that G is a hypersolvable graph and if $\Pi = (\Pi^V, \Pi^{\mathcal{E}})$ be any hypersolvable partition $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$ of G, then Π^G has an exponent vector $d = (1, \dots, 1)$ and the following theorem is to classify such graphs:

Theorem (3.29):

Let G be a hypersolvable graph with $m \geq 3$ and Hp, $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$ has an exponent vector $d = (1, \dots, 1)$. Then we have the following:

1. If $|\mathcal{E}| = \ell = m - 1$, then G is supersolvable and for $1 \leq j \leq m - 1$, $b_j = |NBC_j(A_G)| = \binom{m-1}{j}$, i.e. $\chi(G, t) = \sum_{j=0}^{m-1} \binom{m-1}{j} t^{m-j-1}$.
2. If $m(G) = |\mathcal{E}| = \ell = m$, then G is generic have just one m-cycle and for $1 \leq j \leq m - 1$, $b_j = |NBC_j(G)| = \binom{m}{j}$ and $b_{m-1} = m - 1$. With respect to a fixed hypersolvable ordering, the maximal hyperplane H of A_G will be deformed by Jambu's and Papadima's deformation method into $\widetilde{H}$ which is defined by the linear form: $\tilde{\alpha}_H = (\alpha_H, 1)$, where the other hyperplanes will be lifted by trivial lift.
3. $m(G) \leq rk(G) = m - 1 < |\mathcal{E}| = \ell$, then G is neither supersolvable nor generic and $b_j = |NBC_j(G)| \leq \binom{\ell}{j}$ for $1 \leq j \leq r$, and A_G will deformed by Jambu's and Papadima's deformation method as follows:
   - For $H \in \Pi_k$ and Π_k is non singular block of Π' will deformed into $\widetilde{H}$ by trivial lift, i.e. $\alpha_{\widetilde{H}} = (\alpha_H, 0)$.
   - For $H \in \Pi_k$ and Π_k is a singular block of Π' will deformed into $\widetilde{H}$ as $\alpha_{\widetilde{H}} = (\alpha_H, 1)$.

Proof: **For 1:** Since $|\mathcal{E}| = \ell = m - 1 = rk(G)$, hence G has no singular block. Therefore G is a supersolvable graph and from theorem ((1.3), [3]) is easy to check that, $b_j = \binom{m-1}{j}$, for $1 \leq j \leq rk(A_G) = m - 1$.

For 2: If $|\mathcal{E}| = \ell = m$, then by applying proposition (3.16), G has no singular block and it is a generic graph. According to theorem ((3.2.15), [2]) and proposition ((3.25), $b_j = \binom{m}{j}$, for $1 \leq j \leq rk(A_G) = m - 1$ and for $j = rk(A_G) = m - 1$, $b_{m-1}|NBC_{m-1}| = \binom{m-1}{m-2} = m - 1$. Where for the other claim proved only by applying construction ((3.24), case I.1), since $\ell(G) = m$.

For 3: Similarly, by applying theorem ((3.2.15), [2]) for the graphic arrangement A_G which is hypersolvable with exponent vector $d = (1, \dots, 1)$ and $m(G) \leq rk(A_G) = m - 1$. On the other hand, since $d = (1, \dots, 1)$, hence any singular block of Π' contains just one hyperplane and there is no collinear relation among the blocks of Π'. Thus by applying construction (3.24) our claimed is proven.□

Remark (3.30):

It is worth pointing out that if A_G is a graphic 3-arrangement associated to a graph $G = (V, \mathcal{E})$, then $|V| = 4$, i.e. the number of such arrangements is finite and we can illustrate all the results given in chapter one and two in this thesis.

If G is a complete 4-graph, then G is supersolvable. As an application of theorem (3.27), we easily compute the chromatic polynomial of G;

$$\chi(G,t) = t^3 + 6t^2 + 11t + 6.$$

Furthermore, all the other 4-graphs can be obtained from the complete graph G by deleting edges from $\mathcal{E}$. So, we can simply classify it up to isomorphism of graphs as follows:

1. If we remove just one edge from $\mathcal{E}$, then we will obtain six 4-graphs are isomorphic. Their bond lattices are isomorphic and the following figure is one of them:

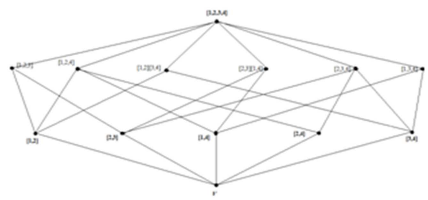


Figure (3.1)

Each one of them is a supersolvable graph and its chromatic polynomial is;

$$\chi(G,t) = t^3 + 5t^2 + 8t + 4.$$

2. If we remove two edges from $\mathcal{E}$, then we have the following:
   i. twelve isomorphic supersolvable 4-graphs with the same bonds lattices as given in the following figure:

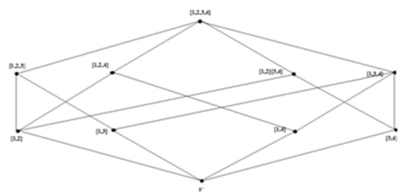


Figure (3.2)

Each one of them has a chromatic polynomial;

$$\chi(G,t) = t^3 + 4t^2 + 5t + 2.$$

   ii. three isomorphic generic 4-graphs with the same bonds lattices as given in the following figure:

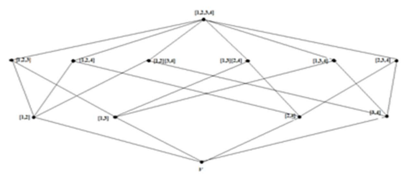


Figure (3.3)

Each one of them has a chromatic polynomial;

$$\chi(G,t) = t^3 + 4t^2 + 6t + 3.$$

3. If we remove three edges from Ɛ, then we have twenty isomorphic supersolvable 4-graphs with no triangles and they have same bonds lattices as shown in the following figure:

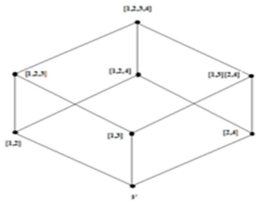


Figure (3.4)

Each one of them has a chromatic polynomial;

$$\chi(G,t) = t^3 + 3t^2 + 3t + 1.$$

Theorem (3.31):

If a graph $G = (V, \mathcal{E})$, has $|V| = 4$, then G is hypersolvable and either G is supersolvable with

$$\chi(G,t) = t^3 + (1 + d_2 + d_3)t^2 + (d_2 + d_3 + d_2 d_3)t + d_2 d_3;$$

or G is generic with

$$\chi(G,t) = t^3 + \binom{4}{1} t^2 + \binom{4}{2} t + 3 = t^3 + 4t^2 + 6t + 3;$$

where(d_1, d_2, d_3) is an exponent vector for a fixed hypersolvable partition of G.

Proof: This is a direct result to our classification given in remark (3.30). □

(4) THE HYPERSOLVABLE GRAPHIC MATROIDS

In this section we will study the hypersolvable (supersolvable) matroids associated to the hypersolvable (supersolvable) graph and we will defined the hypersolvable graphic matroid M_G.

Definition (4.1):[4], [6] and [18]

A "finite" *matroid* is a pair $M = (A, \Delta)$, where A is a finite set and Δ is a collection of subsets of A, satisfying the following axioms:

1. Δ is a non-empty (abstract) simplicial complex, i.e. $\Delta \neq \emptyset$ and if $\Delta' \in \Delta$ and $\Delta'' \subset \Delta'$, then $\Delta'' \in \Delta$.
2. Every induced subcomplex of Δ is a pure, i.e. if $B \subseteq A$, the maximal elements of $\Delta \cap 2^B$ have the same cardinality, where $2^B = \{C \subseteq A \mid C \subseteq B\}$.

The members of Δ are called *independent* sets of the matroid, the facets is said to be the bases of the matroid and we write $\upsilon \in M$ to mean $\upsilon \in A$. We call Δ a G-complex. Two matroids $M_1 = (A_1, \Delta_1)$ and $M_2 = (A_2, \Delta_2)$ are said to be *isomorphic* if there exists a bijection $\psi : A_1 \longrightarrow A_2$ such that $\{\upsilon_1, \dots, \upsilon_k\} \in \Delta_1$ if, and only if, $\{\psi(\upsilon_1), \dots, \psi(\upsilon_k)\} \in \Delta_2$.

A *circuit* $C \subseteq A$ is a minimal dependent set, i.e. C is not independent but becomes independent when we remove any point from it. If $B \subseteq A$, we define the rank of B by;

$$rk(B) = \max\{|B'| \,|\, B' \subseteq B \text{ and } B' \in \Delta\}.$$

In particular, $rk(\phi) = 0$ and we will define the following:

1. The *rank* of the matroid M itself by $rk(M) = rk(A) = \dim(\Delta) + 1 = |F|$, where F is a facet of M. The level of a matroid is $l(M) = |A| - rk(M)$-1.

2. A *k-flat* of M is a maximal subset of rank k. It has been noticed that, if B and B' are flats of a matroid M, then so is $B \cap B'$. We can defined the *closure* $\bar{B}$ of a subset $B \subseteq A$ to be the smallest flat containing B, i.e. $\bar{B} = \bigcap_{flats\ B' \supseteq B} B'$.
3. $L(M)$ for a matroid M to be *the poset of flats* of M, ordered by inclusions. Since $L(M)$ has a top element A, then $L(M)$ is a lattice, which we call the *lattice of flats* of M. It has been noticed that, $L(M)$ has a unique minimal element $\hat{0} = \emptyset$.
4. Define the *characteristic polynomial* $\chi_M(t)$ of M, by;

$$\chi_M(t) = \sum_{X \in L(M)} \mu(\hat{0}, x)\, t^{r - rk(X)};$$

where μ denotes the Möbius function of $L(M)$ and $r = rk(M)$.
5. Define the *Crapo's beta invariant*;

$$\beta(M) = (-1)^{rk(M)} \sum_{B \subseteq A} (-1)^{|B|}\, rk(B).$$

Definition (4.2):[4], [6] and [7]

A *broken circuit* of an ordered matroid $M_{\trianglelefteq}$, is a set $\bar{C} = C \backslash \upsilon$, where C is a circuit and υ is the minimal element of C via $\trianglelefteq$. The broken circuit complex (or BC-complex) which is defined by to be the simplicial complex;

$$NBC_{\trianglelefteq}(M) = \{B \subseteq A \mid B \text{ contains no broken circuit}\}.$$

For $0 \leq k \leq rk(M)$, set;

$$NBC_{\trianglelefteq}^{k}(M) = \{B \subseteq A \mid B \text{ contains no broken circuit and } |B| = k+1\};$$

to be the k^{th}- skeleton of $NBC_{\trianglelefteq}(M)$. It has been noticed that, if $f^{\Delta} = (f_0^{\Delta}, f_1^{\Delta}, \dots, f_{r-1}^{\Delta})$ be the f-vector of $NBC_{\trianglelefteq}(M)$, then $|NBC_{\trianglelefteq}^{k}(M)| = f_k^{\Delta}$ and by applying a result of Rota [15];

$$\chi_M(t) = f_{-1}^{\Delta} t^r - f_0^{\Delta} t^{r-1} + \cdots + (-1)^r f_{r-1}^{\Delta};$$

where $f_{-1}^{\Delta} = 1$.

The family of all subsets of $A/\{1\}$ that contains no broken circuits is called the *reduced broken circuit complex* of $M_{\trianglelefteq}$ and denoted by $\overline{NBC}_{\trianglelefteq}(M)$.

Definition (4.3): [20]

Let $A = \{H_1, \dots, H_n\}$ be a central r-arrangement of hyperplanes over $\mathbb{C}$. Define a matroid $M_A = (A, \Delta)$ on A by letting Δ to be the collection of all independent subarrangements of A. It has been noticed that, $L(A) \equiv L(M_A)$. Via a linear ordering $\trianglelefteq$, let:

$$NBC_{\trianglelefteq}(M_A) = \{B \subseteq A \mid B \text{ contains no broken circuit}\};$$

be the NBC-complex of M_A. Then;

$$\chi_M(t) = f_{-1}^{\Delta} t^r - f_0^{\Delta} t^{r-1} + \cdots + (-1)^r f_{r-1}^{\Delta};$$

where $r = rk(A) = \delta + 1$ and $f^{\Delta} = (f_0^{\Delta}, f_1^{\Delta}, \dots, f_{\delta}^{\Delta})$ be the f-vector of $NBC_{\trianglelefteq}(M_A)$ and $f_{-1} = 1$. Notice that, $h_r = \beta_{rk(M_A)}(A_{\Delta}) = f_{r-1}^{\Delta}$ is the type of the Cohen-Macaulay ring A_{Δ} and it has a minimal free resolution;

$$0 \longrightarrow M_r \longrightarrow M_{r-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow A_{\Delta} \longrightarrow 0;$$

where for $0 \leq k \leq r$, $rk(M_k) = \beta_k(A_{\Delta}) = |NBC_{\trianglelefteq}^{k}(M)| = f_{k-1}^{\Delta}$.

The matroid M_A is said to be hypersolvable (supersolvable) matroid if, A is hypersolvable (supersolvable) arrangement.

We concern with A is hypersolvable r-arrangement with Hp $\Pi = (\Pi_1, \dots, \Pi_\ell)$ and d-vector $d = (d_1, \dots, d_\ell)$. Let $NBC_{\trianglelefteq}(M_A)$ be the NBC-complex of the matroid M_A via the hypersolvable ordering $\trianglelefteq$ with f-vector, $f^{\Delta} = (f_0^{\Delta}, f_1^{\Delta}, \dots, f_{\delta}^{\Delta})$. That is, we shall give the no broken circuit subarrangements the degree lexicographic (DegLex) order with respect the hypersolvable ordering. Where, by $NBC_{\trianglelefteq}(M_A)|_i = \{S \in NBC_{\trianglelefteq}(M_A) | S \subseteq \Pi_i\}$ we denote the restriction of $NBC_{\trianglelefteq}(M_A)$ to Π_i, for $1 \leq i \leq \ell$.

For $1 \leq k \leq \ell$, let $S_{\Pi}^{k}(A) = \{S \subseteq A \mid S \text{ is a } k-\text{section of } \Pi\}$ and let $S_{\Pi}(A)|_k = \{\{H\} | H \in \Pi_k\}$ be the discrete 0-dimensional simplicial complex. Let $S_{\Pi}(A) = S_{\Pi}(A)|_1 * \cdots * S_{\Pi}(A)|_{\ell}$ be the multiple join of the complexes $S_{\Pi}(A)|_1, \dots, S_{\Pi}(A)|_{\ell}$. That is $S_{\Pi}(A) = \bigcup_{k=1}^{\ell} S_{\Pi}^{k}(A)$. We call $S_{\Pi}(A)$ a *hypersolvable partition complex* of the matroid M_{A} via the hypersolvable ordering. It has been noticed that, in general $S_{\Pi}(A)$ need not to be a subcomplex of the G-complex Δ of the matroid $S_{\Pi}(A)$. The important point to know here $NBC_{\trianglelefteq}(M_A)|_k = S_{\Pi}(A)|_k$ in general, for $1 \leq k \leq \ell$. But $NBC_{\trianglelefteq}(M_A) \neq S_{\Pi}(A)$ in general.

Definition (4.4):[16]

For any graph $G = (V, \mathcal{E})$, by a *graphic matroid* $M_G = (G, \Delta_G)$ on G, we mean the matroid that isomorphic to $M_{A_G} = (A_G, \Delta_{A_G})$ on the graphic arrangement A_G by letting $\Delta_G \equiv \Delta_{A_G}$, i.e. via an ordering $\trianglelefteq$ on the edges of $\mathcal{E}$, Δ_G will be the collection of all broken circuits and no broken circuits of G. If G is hypersolvable (supersolvable) graph, we will call M_G, a hypersolvable (supersolvable) graphic matroid on G. It has been noticed that, $L(M_{A_G})$ is the bond lattice $L(G)$ on G. Via a linear ordering $\trianglelefteq$ on $\mathcal{E}$, let:

$$NBC_{\trianglelefteq}(M_G) = \{B \subseteq A_G | B \text{ contains no broken circuit}\};$$

be the NBC-complex of M_G. Then;

$$\chi_G(t) = \chi_{M_G}(t) = f_{-1}^{\Delta_G} t^r - f_0^{\Delta_G} t^{r-1} + \cdots + (-1)^r f_{r-1}^{\Delta_G};$$

where $r = rk(A_G) = |V'| - 1 = \delta + 1$ and $f^{\Delta_G} = (f_0^{\Delta_G}, f_1^{\Delta_G}, \dots, f_{\delta}^{\Delta_G})$ be the f-vector of $NBC_{\trianglelefteq}(M_G)$ and $f_{-1} = 1$. Notice that, $h_r = \beta_{rk(M_G)}(\mathrm{A}_{\Delta_G}) = f_{r-1}^{\Delta_G}$ is the type of the Cohen-Macaulay ring A_{Δ_G} and it has a minimal free resolution,

$$0 \longrightarrow M_r \longrightarrow M_{r-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow \mathrm{A}_{\Delta_G} \longrightarrow 0,$$

where for $0 \leq k \leq r$, $rk(M_k) = \beta_k(\mathrm{A}_{\Delta_G}) = |NBC_{\trianglelefteq}^{k}(M)| = f_{k-1}^{\Delta_G}$.

We will apply Al-Ta'ai's and Ali's basic results in [4] by the following theorems:

Theorem (4.5):

Let $G = (V, \mathcal{E})$ be a hypersolvable graph with Hp $\Pi = (\Pi^V, \Pi^{\mathcal{E}})$, $\ell(G) = \ell$, $\Pi' = (\Pi_1, \dots, \Pi_{\ell})$ be the induced hypersolvable partition of its graphic arrangement A_G with exponent vector $d = (d_1, \dots, d_{\ell})$. Via a hypersolvable ordering $\trianglelefteq$ on G the following statements are equivalent:

1. G is supersolvable.
2. $NBC_{\trianglelefteq}(M_G) \equiv S_{\Pi}(A_G)$, i.e.; $NBC_{\trianglelefteq}(M_{A_G}) \equiv S_{\Pi}(A_G)|_1 * \cdots * S_{\Pi}(A_G)|_{\ell}$ is factored completely and $f_k^{\Delta_G} = \sum_{i_1=1}^{\ell-k} \sum_{i_2=i_1+1}^{\ell-k+1} \dots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} d_{i_2} \dots d_{i_k}$, for $0 \leq k \leq \ell - 1$,where $f^{\Delta_G} = (f_0^{\Delta_G}, f_1^{\Delta_G}, \dots, f_{\ell-1}^{\Delta_G})$ be the f-vector of $NBC_{\trianglelefteq}(M_G)$ and $\chi_G(t) = \chi_{M_G}(t) = f_{-1}^{\Delta_G} t^{\ell} - f_0^{\Delta_G} t^{\ell-1} + \cdots + (-1)^{\ell} f_{\ell-1}^{\Delta_G}$; where $f_{-1}^{\Delta_G} = 1$. $h_r = \beta_{rk(M_G)}(\mathrm{A}_{\Delta_G}) = f_{\ell-1}^{\Delta_G} = d_2 d_3 \dots d_{\ell}$ is the type of the Cohen-Macaulay ring A_{Δ_G} and it has a minimal free resolution,

$$0 \longrightarrow M_{\ell} \longrightarrow M_{\ell-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow \mathrm{A}_{\Delta_G} \longrightarrow 0,$$

where for $0 \leq k \leq \ell$, $rk(M_k) = \beta_k(\mathrm{A}_{\Delta_G}) = |NBC_{\trianglelefteq}^{k}(M)| = f_{k-1}^{\Delta_G}$.

3. $NBC_{\trianglelefteq}(M_G)$ is completely balanced.
4. The 1st-skeleton $NBC^1_{\trianglelefteq}(M_G)$ is a complete ℓ-partite graph.
5. The minimal broken circuits (under inclusion) are all of size two, i.e. every broken circuit of A_G contains 2-broken circuit.

Proof: **For$(\mathbf{1} \to \mathbf{2})$*:*** Since G is supersolvable, Then A_G is supersolvable arrangement and by applying is partitioned into ℓ classes $\Pi_1, \dots, \Pi_\ell$ such that theorem (3.26), $NBC(A_G) = S(A_G)$via a hypersolvable ordering $\trianglelefteq$ and for $1 \leq k \leq rk(G) = \ell$;

$$b_k = \sum_{i_1=1}^{\ell-k} \sum_{i_2=i_1+1}^{\ell-k+1} \dots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} d_{i_2} \dots d_{i_k};$$

forms the number of the k^{th} –NBC bases of A_G.That is, $NBC(A_G)$ induced an ordered subcomplex $NBC_{\trianglelefteq}(M_G)$ of Δ_G which is completely factored into $S_\Pi(A_G) = S_\Pi(A_G)|_1 * \cdots * S_\Pi(A_G)|_\ell$ and for $0 \leq k \leq \ell - 1$ the number of k^{th} –faces of $NBC_{\trianglelefteq}(M_G)$ is $f_k^{\Delta_G} = b_{k+1}$ and our claim is down.

For$(\mathbf{2} \to \mathbf{3})$*:* In fact $NBC_{\trianglelefteq}\left(M_{A_G}\right) \equiv S_\Pi(A_G)|_1 * \cdots * S_\Pi(A_G)|_\ell$, implies that the vertex set of $NBC_{\trianglelefteq}(M_G)$, A_G every facet of $NBC_{\trianglelefteq}(M_G)$ has exactly one vertex in every class.

For$(\mathbf{3} \to \mathbf{4})$*:* Since $NBC_{\trianglelefteq}(M_G)$ is cmpletely balanced, i.e. A_G is partitioned into ℓ classes $\Pi_1, \dots, \Pi_\ell$ such that every facet of $NBC_{\trianglelefteq}(M_G)$ has exactly one vertex in every class. Therefore, every 1-faces of any facets has exactly one vertex in two different classes. That is the graph $NBC^1_{\trianglelefteq}(M_G)$ can partitioned into ℓ classes $\Pi_1, \dots, \Pi_\ell$ such that the vertices in every edge are from different classes. Then $NBC^1_{\trianglelefteq}(M_G)$ is a complete ℓ-partite graph.

For$(\mathbf{4} \to \mathbf{5})$*:* If 1st-skeleton $NBC^1_{\trianglelefteq}(M_G)$ is a complete ℓ-partite graph, then the 0th-skeleton $NBC^0_{\trianglelefteq}(M_G) = A_G$ is partitioned into ℓ classes $\Pi_1, \dots, \Pi_\ell$ such that every 1-faces has exactly one vertex in two different classes. That is every facet (ℓ-NBC base) of $NBC_{\trianglelefteq}(M_G)$ has exactly one vertex in every class. Hence, every facets of Δ_G which is not of $NBC_{\trianglelefteq}(M_G)$, (i.e. the ℓ-broken circuit of Δ_G), has two vertices of Π_i forsome $2 \leq i \leq \ell$. Therefore, the minimal broken circuits are all of size two.

For$(\mathbf{5} \to \mathbf{1})$*:* Al-Tai' and Ali in [4], proved that via a hypersolvable ordering on the hyperplanes of a superslvable arrangement every broken circuit contains broken circuit of rank two and in [5], Björner and Ziegler showed that if there exists an ordering such that every broken circuit contains broken circuit of rank two, (i.e. via this ordering The minimal broken circuits (under inclusion) are all of size two), then $L(A_G)$ is supersolvable geometric lattice. Thus, by applying theorem (3.8), G is supersolvable. □

Theorem (4.6):

Let $G = (V, \mathcal{E})$ be a hypersolvable graph with Hp $\Pi = (\Pi^V, \Pi^{\mathcal{E}})$, $\ell(G) = \ell$, a hypersolvable ordering $\trianglelefteq$ on G, d-vector $d = (\mathrm{d}_1, \dots, \mathrm{d}_\ell)$, f-vector of Δ_G, $f = (f_0, f_1, \dots, f_{rk(G)-1})$ and f-vector of $NBC_{\trianglelefteq}(M_G)$, $f^{\Delta_G} = (f_0^{\Delta_G},\ f_1^{\Delta_G},\ \dots, f_{rk(G)-1}^{\Delta_G})$ such that $rk(G) = r = |V| - 1 = m - 1 < \ell$. Then:

1. For $2 \leq k \leq r$, $NBC^{k-1}_{\trianglelefteq}(M_G) \equiv NBC^{k-1}_{\trianglelefteq}\left(M_{A_G}\right) \subseteq S^k_\Pi(A_G)$ in general, i.e.
$$f_{k-1}^{\Delta_G} \leq \sum_{i_1=1}^{\ell-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \cdots d_{i_k}.$$
2. $NBC^1_{\trianglelefteq}(M_G) \equiv NBC^1_{\trianglelefteq}\left(M_{A_G}\right) = S^2_\Pi(A_G)$ is a complete ℓ-partite graph, i.e.
$$f_1^{\Delta_G} = \sum_{i_1=1}^{\ell-1} \sum_{i_2=i_1+1}^{\ell} d_{i_1} d_{i_2}.$$
3. There exists, $2 \leq p = p(A_G) \leq |V| - 2$ such that;

$$p = p(A_G) = \max\{k \mid S_\Pi^k(A_G) = NBC_{\trianglelefteq}^{k-1}(M_{A_G}) \equiv NBC_{\trianglelefteq}^{k-1}(M_G)\}$$

and $L_{p+1}(A)$ represents the first level in the bonds lattice $L(A_G)$ that the induced partition Π' from Π on A_G, has dependent sections among $(p+2)$-different blocks of Π' via the induced hypersolvable ordering. That is, for $1 \le k \le p$;

$$f_{k-1}^{\Delta_G} = \sum_{i_1=1}^{\ell-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \cdots d_{i_k}.$$

Proof: **For 1:** From theorem (2.5), since A_G be a hypersolvable arrangement , then for $2 \le k \le rk(A_G) = |V| - 1$, every k-NBC base of A_G must be an k-section of Π. That is the number of all $(k-1)^{\text{th}}$ faces of A_G can not exceed the number of k-sections of Π and our aim is hold.

For 2: By applying theorem ((1.4), [3]), $NBC_2(A_G) = S_2(A_G)$. Therefore, $NBC_{\trianglelefteq}^1(M_{A_G}) = S_\Pi^2(A_G)$ and $f_1^\Delta = \sum_{i_1=1}^{\ell-1} \sum_{i_2=i_1+1}^{\ell} d_{i_1} d_{i_2}$.

For 3: From theorem ((1.4), [3]), for $2 \le k \le p(A_G)$, $NBC_k(A_G) = S_k(A_G)$. That is, for $1 \le k \le p$;

$$NBC_{\trianglelefteq}^{k-1}(M_{A_G}) = S_\Pi^k(A_G);$$

and $f_{k-1}^\Delta = \sum_{i_1=1}^{\ell-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \cdots d_{i_k}$. □

Proposition (4.7):

Under the assumptions of theorem (4.6), For all $t_1, t_2 \in \mathbb{C} \setminus \{0\}$, Jambu's and Papadima's deformed arrangements $\widetilde{A_G}_{t_1}$ and $\widetilde{A_G}_{t_2}$ are L-equivalent and they have isomorphic matroids $M_{\widetilde{A_G}_{t_1}} = (\widetilde{A_G}_{t_1}, \tilde{\Delta}_{t_1})$ and $M_{\widetilde{A_G}_{t_2}} = (\widetilde{A_G}_{t_2}, \tilde{\Delta}_{t_2})$. That is, they have isomorphic partition complexes, i.e. $NBC_{\trianglelefteq}\left(M_{\widetilde{A_G}_{t_1}}\right) \cong NBC_{\trianglelefteq}\left(M_{\widetilde{A_G}_{t_2}}\right)$ via the hypersolvable ordering which give rise into isomorphic standard K-algebra $\mathrm{A}_{\Delta_{t_1}} \cong \mathrm{A}_{\Delta_{t_2}}$.

Proof: From theorem ((4.1.1), [1]), for $t_1, t_1 \in \mathbb{C} \setminus \{0\}$, $\widetilde{A_G}_{t_1}$ and $\widetilde{A_G}_{t_2}$are ℓ_2-equivalent and by applying theorem ((1.3), [3]), they are L-equivalent they have isomorphic matroids, i.e. $M_{\widetilde{A_G}_{t_1}} = (\widetilde{A_G}_{t_1}, \tilde{\Delta}_{t_1}) \cong M_{\widetilde{A_G}_{t_2}} = (\widetilde{A_G}_{t_2}, \tilde{\Delta}_{t_2})$ and isomorphic partition complexes, i.e. $NBC_{\trianglelefteq}\left(M_{\widetilde{A_G}_{t_1}}\right) \cong NBC_{\trianglelefteq}\left(M_{\widetilde{A_G}_{t_2}}\right)$ via equivalent hypersolvable orders which give rise into isomorphic standard K-algebras $\mathrm{A}_{\Delta_{t_1}} \cong \mathrm{A}_{\Delta_{t_2}}$. □

Theorem (4.8):

Under the assumptions of theorem (4.6), we have A_G and $\widetilde{A_G}_t$ are L_p-equivalent, for all $t \in \mathbb{C} \setminus \{0\}$. Thus Jambu's-Papadima's deformation preserves the lattice intersection pattern up to codimension p, then it destroyed all the dependent sections of rank greater than p among the blocks of the induced partition Π' from Π and replaced it by independent sections which add new faces of $NBC_{\trianglelefteq}(M_{A_G}) \equiv NBC_{\trianglelefteq}(M_G)$ to deform it into the partition complex $S_{\widetilde{\Pi'}_t}(\widetilde{A_G}_t)$ as follows:

i. For $0 \le j \le p$ and $1 \le k \le p$, Δ_j and $NBC_{\trianglelefteq}^{k-1}(M_{A_G})$ are invariant under the deformation, i.e. $\Delta_j^G = {\Delta_j}^t$, $f_j = f_j^t$, $NBC_{\trianglelefteq}^{k-1}(M_{A_G}) \cong S_{\widetilde{\Pi}_t}^k(\widetilde{A_G}_t)$ and $f_{k-1}^{\Delta_G} = f_{k-1}^{\Delta t}$,

ii. For $p+1 \le k \le r$, Jambu's-Papadima's deformation replaced $NBC_{\trianglelefteq}^{k-1}(M_G) \equiv NBC_{\trianglelefteq}^{k-1}(M_{A_G})$by $S_{\widetilde{\Pi}_t}^k(\widetilde{A_G}_t)$ by adding exactly;

$$\{\sum_{i_1=1}^{\ell-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \cdots d_{i_k}\} - f_{k-1}^{\Delta_G}.\ (k-1)\text{-faces}.$$

iii. For $r+1 \leq k \leq \ell$, Jambu's-Papadima's deformation adding;

$$\sum_{i_1=1}^{\ell-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \cdots d_{i_k},\ (k-1)\text{-faces..}$$

That is, the G-complex and NBC-complex of A_G embedded in the G-complex and partition complex of $\widetilde{A_G}_t$ respectively. Thus, for $0 \leq k \leq p+1$, $\mathrm{A}^k_{\Delta_G} \cong \mathrm{A}^k_{\tilde{\Delta}_t}$ and for $0 \leq k \leq p$, $M_k \cong M^t_k$.

Proof: By applying theorem ((1.4), [3]), A_G and $\widetilde{A_G}_t$ are ℓ_p-equivalent, for all $t \in C \setminus \{0\}$. That is Jambu's-Papadima's deformation method destroyed all the dependent sections among any $(p+2)$-different blocks of Π' and replaced it by independent sections which add new NBC bases of $\tilde{A}_t$. In fact, the one to one correspondence $\varphi : A_G \rightrightarrows \widetilde{A_G}_t$ which respects the lattice intersection pattern up to codimension two, $\varphi : L_2(A_G) \rightrightarrows L_2(\widetilde{A_G}_t)$, define a one to one correspondence $\varphi : \Pi' \rightrightarrows \widetilde{\Pi}'_t$ with respect the same hypersolvable ordering of A_G and $\widetilde{A_G}_t$, (i.e. φ gives each one of A_G and $\widetilde{A_G}_t$ the same d-vector). Therefore,

$$\varphi : S_{\Pi'}(A_G) \rightrightarrows S_{\widetilde{\Pi'}_t}\left(\widetilde{A_G}_t\right) \cdots\cdots (4.8.1)$$

form a one to one correspondence between the sets of all sections of Π' and $\widetilde{\Pi}'_t$, where it's restriction on $NBC_{\trianglelefteq}(M_{A_G})$;

$$\varphi_{NBC_{\trianglelefteq}(M_{A_G})} :\ NBC_{\trianglelefteq}(M_{A_G}) \rightrightarrows \varphi(NBC_{\trianglelefteq}(M_{A_G})) \hookrightarrow S_{\widetilde{\Pi'}_t}(\widetilde{A_G}_t);$$

define an injection between the broken circuits complexes. That is, the deformation start with embedding $NBC_{\trianglelefteq}(M_{A_G})$ as a subcomplex of $S_{\widetilde{\Pi'}_t}(\widetilde{A_G}_t)$since all the no broken circuits of A_G are invariant under the deformation, then we can constructs the broken circuit complex $S_{\widetilde{\Pi'}_t}(\widetilde{A_G}_t)$ of $\widetilde{A_G}_t$ by using the deformation method as follows:

For (i): For $0 \leq j \leq p$ and $1 \leq k \leq p$, the equivalent lattices pattern up to codimension p of A_G and $\widetilde{A_G}_t$ give rise into a bijection $\varphi_{(j)} : \Delta^G_j \rightrightarrows \Delta^t_j$ of $(j)^{\text{th}}$-faces of the G-complexes and the restriction of $\varphi_{(j)}$ on $NBC^{(k-1)}_{\trianglelefteq}\left(M_{A_G}\right)$ forms a bijection;

$$\varphi^{(k-1)}_{NBC^{(k-1)}_{\trianglelefteq}\left(M_{A_G}\right)} :\ NBC^{(k-1)}_{\trianglelefteq}\left(M_{A_G}\right) \rightrightarrows S^k_{\widetilde{\Pi}}(\widetilde{A_G}_t);$$

of broken circuit complexes which keeps $NBC^{(k-1)}_{\trianglelefteq}\left(M_{A_G}\right)$ invariant under the deformation.

For (ii): For $p+1 \leq k \leq r$, those faces which are the minimal non-faces of $NBC_{\trianglelefteq}(M_A)$ of dimension $(k-1)$ are deformed by jambu's-Papadima's deformation method, which added new $(k-1)$-faces by the restriction of (4.8.1) on $S^k_{\Pi'}(A_G) \setminus NBC^{(k-1)}_{\trianglelefteq}\left(M_{A_G}\right)$;

$$\varphi^{(k-1)}_{S^k_{\Pi'}(A_G)\setminus NBC^{(k-1)}_{\trianglelefteq}\left(M_{A_G}\right)} : S^k_{\Pi'}(A_G) \setminus NBC^{(k-1)}_{\trianglelefteq}\left(M_{A_G}\right) \rightrightarrows \varphi^{(k-1)}(S^k_{\Pi'}(A_G) \setminus NBC^{(k-1)}_{\trianglelefteq}\left(M_{A_G}\right)) \hookrightarrow S^k_{\widetilde{\Pi'}_t}(\widetilde{A_G}_t),$$

i.e. we add exactly $\left|S^k_{\Pi'}(A_G)\right| - |NBC^{(k-1)}_{\trianglelefteq}\left(M_{A_G}\right)|$, $(k-1)$-faces.

For (iii): For $r+1 \leq k \leq \ell$, Jambu's-Papadima's deformation method added all those faces of dimension $(k-1)$ by the isomorphism between the partition complexes, $\varphi_{(k-1)} : S^k_{\Pi'}(A_G) \rightrightarrows S^k_{\widetilde{\Pi'}_t}(\widetilde{A_G}_t)$, where the number of such faces is equal to $\left|S^k_{\Pi'}(A_G)\right|$.

Finally, for each $0 \leq k \leq p$, A_G and $\widetilde{A_G}_t$ have the same f_k, $f_k^{\Delta_G}$ and $H(\Delta, k+1)$ which is produced that for $0 \leq k \leq p+1$, $\mathrm{A}^k_{\Delta_G} \cong \mathrm{A}^k_{\tilde{\Delta}_t}$ and for $0 \leq k \leq p$, $M_k \cong M_k^t$ as free A-modules. □

Corollary (4.9):

Under the hypotheses of the theorem (4.6), If G is ℓ-generic, i.e. $m(G) = |V| = \ell$, then $p = |V| - 2$ and $NBC^{r-1}_{\trianglelefteq}(M_G) \equiv NBC^{r-1}_{\trianglelefteq}\left(M_{A_G}\right)$ has facets all $B \subseteq A$ with $B \cap \Pi'_1 \neq \emptyset$ and $|B| = r$, where $f^{\Delta_G}_{r-1} = |V| - 1$. Jambu's-Papadima's deformation letting $\Delta^G_{k-1} = NBC^{k-1}_{\trianglelefteq}(M_A)$ invariant for $1 \leq k \leq r-1$ and added just one $r-1$-face to deform $NBC^{r-1}_{\trianglelefteq}(M_G) \equiv NBC^{r-1}_{\trianglelefteq}\left(M_{A_G}\right)$ into $S^r_{\tilde{\Pi}_t}(\widetilde{A_G}_t)$. That is, the deformation added just one ℓ-face to deform $NBC_{\trianglelefteq}\left(M_{A_G}\right) \equiv NBC_{\trianglelefteq}(M_G)$ into the partition complex $S_{\tilde{\Pi}'_t}(\widetilde{A_G}_t)$. Therefore, for $0 \leq k \leq r$, $\mathrm{A}^k_{\Delta} \cong \mathrm{A}^k_{\tilde{\Delta}_t}$ and for $0 \leq k \leq r-1$, $M_k \cong M_k^t$, where there is a monomorphism $M_r \hookrightarrow M_r^t$.

Proof: This is an application of theorems (2.7), (3.29) and (4.8). □

Corollary (4.10):

Suppose we have the assumption of theorem (4.6) with $|V| = 4$, then G is generic, $p = 2$. Jambu's-Papadima's deformation keeps Δ^G_0, $NBC^0_{\trianglelefteq}(M_G)$, Δ^G_1 and $NBC^1_{\trianglelefteq}(M_G)$ unchanged and deform $NBC^2_{\trianglelefteq}(M_G)$ into $S^3_{\tilde{\Pi}_t}(\widetilde{A_G}_t)$ by adding just one 2-face. Then, Jambu's-Papadima's deformation destroys the dependent 4-section of A_G and replaced it by one 4-face to obtain $S^4_{\tilde{\Pi}_t}(\widetilde{A_G}_t)$, and;

$$\chi(A_G, t) = t^4 + 4t^3 + 6t^2 + 4t + 1;$$

That is for $0 \leq k \leq 3$, $\mathrm{A}^k_{\Delta} \cong \mathrm{A}^k_{\tilde{\Delta}_t}$ and for $0 \leq k \leq 2$, $M_k \cong M_k^t$, where there is a monomorphism $M_3 \hookrightarrow M_3^t$.

Proof: By applying theorem (3.31), if $|V| = 4$ then G is generic and by applying corollary (4.9), our aim is hold. □

Corollary (4.11):

Under the assumption of theorem (4.6), the reduced homology group,

$$\widetilde{H}_d(\Delta_G) \cong \begin{cases} \mathbb{Z}^q & \text{if } d = rk(M_G) - 1 = m - 2 \\ 0 & \text{if } d \neq rk(M_G) - 1 = m - 2 \end{cases};$$

where $q = (-1)^r \chi(\Delta_G)$ and $\overline{NBC}_{\trianglelefteq}\left(M_{A_G}\right)$ has top-dimensional reduced homology;

$$\widetilde{H}_{m-3}\left(\overline{NBC}_{\trianglelefteq}\left(M_{A_G}\right)\right) \cong \mathbb{Z}^{\beta\left(M_{A_G}\right)}, \text{ where } \beta(M) = (-1)^r \chi\left(\overline{NBC}_{\trianglelefteq}\left(M_{A_G}\right)\right).$$

Proof: By applying theorem ((7.7.2), [6]) and theorem ((7.8.2), [6]) since $rk(M_G) = m - 1$.

Example (4.12):

Let $G = (V, \mathcal{E})$ be a graph such that $V = \{1,2,3,4,5\}$ and $\mathcal{E} = \{[1,2], [1,5], [2,5], [2,3], [3,4], [4,5]\}$, as shown in the following figure:

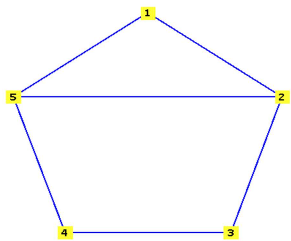


Figure (4.1)

The graph G is a hypersolvable graph with 5-vertices and Hp $\Pi = (\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5)$ with d-vector, $d = (1,2,1,1,1)$. The graphic matroid $M_G = (G, \Delta_G)$, is defined on G by letting $\Delta_G = \bigcup_{k=0}^{3} \Delta_{G_k}$, where;
$\Delta_{G_0} = \{1,2, \dots ,6\}$, $\Delta_{G_1} = \{\{i,j\} \mid 1 \leq i < j \leq 6\}$, $\Delta_{G_2} = \{\{i,j,k\} \mid 1 \leq i < j < k \leq 6\} \setminus \{1,2,3\}$ and $\Delta_{G_3} = \{\{i,j,k,p\} \mid 1 \leq i < j < k < p \leq 6\} \setminus \{1,2,3,4\}, \{1,2,3,5\}, \{1,2,3,6\}, \{3,4,5,6\}\}$. That is the f-vector of Δ is $f = (6,15,19,10)$.
In fact G is not supersolvable, since Π^{ε} has one singular block is Π_5. So A_G is not supersolvable arrangement and from theorem (4.5);

$$NBC_{\trianglelefteq}(M_G) \not\equiv S_{\Pi}(A_G).$$

By applying theorem (4.6), the broken circuit complex $NBC_{\trianglelefteq}(M_G)$ of G, is defined as; $NBC_{\trianglelefteq}(M_G) = \bigcup_{k=0}^{4} {NBC_{\trianglelefteq}}^{k}(M_G)$. That is the f-vector of $NBC_{\trianglelefteq}(M_G)$ is $f^{\Delta_G} = (1,6,14,16,7)$. And $\chi_G(t) = \chi_{M_G}(t) = t^4 + 6t^3 + 14t^2 + 16t + 7$. The type of the Cohen-Macaulay ring A_{Δ_G} is $h_4 = \beta_4(\mathrm{A}_{\Delta_G}) = f_3^{\Delta_G} = 7$ and it has a minimal free resolution;

$$0 \longrightarrow M_4 \longrightarrow M_3 \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow \mathrm{A}_{\Delta_G} \longrightarrow 0;$$

is completely determined by the f-vector of $NBC_{\trianglelefteq}(M_G)$, f^{Δ_G} .

By applying construction ((3.24), case (I.1)), the supersolvable Jambu's-Papadima's vertical deformation $\{\widetilde{A_{G_t}}\}_{t\in\mathbb{C}}$ of A_G in $\mathbb{C}^5 \times \mathbb{C} = \mathbb{C}^6$ is defined as;

$$Q\left(\widetilde{A_{G_t}}\right) = (x_1 - x_2)(x_1 - x_5)(x_2 - x_5)(x_2 - x_3)(x_3 - x_4)(x_4 - x_5 + x_6 t);$$

for each $t \in \mathbb{C}$ and by the same hypersolvable ordering of A_G let;

$$ {H_1}^t = Ker(x_1 - x_2), \qquad {H_2}^t = Ker(x_1 - x_5), {H_3}^t = Ker(x_2 - x_5),$$
$$ {H_4}^t = Ker(x_2 - x_3),\ {H_5}^t = Ker(x_3 - x_4) \text{ and } {H_6}^t = Ker(x_4 - x_5 + x_6 t);$$

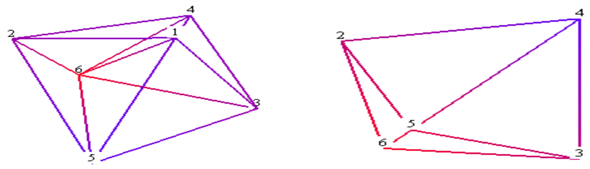
where the Hp, $\Pi_{\mathrm{t}} = (\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5) = (\{{H_1}^t\}, \{{H_2}^t, {H_3}^t\}, \{{H_4}^t\}, \{{H_5}^t\}, \{{H_6}^t\})$;
have the same d-vector, $d = (1,2,1,1,1)$ which shows that the blocks $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ must be kept unchanged by the trivial lift and the block Π_5 lifts to $\mathbb{C}^6$, by put $\tilde{\alpha}_H = (\alpha_H, \lambda_H t)$, where;

$$\lambda_H = \begin{cases} 0 & \text{if} \quad H \in \Pi_1, \Pi_2, \Pi_3, \Pi_4 \\ 1 & \text{if} \qquad H \in \Pi_5 \end{cases};$$

i.e.
$\tilde{\alpha}_{H_1} = (1,-1,0,0,0,0)$, $\tilde{\alpha}_{H_2} = (1,0,0,0,-1,0)$, $\tilde{\alpha}_{H_3} = (0,1,0,0,-1,0)$, $\tilde{\alpha}_{H_4} = (0,1,-1,0,0,0)$, $\tilde{\alpha}_{H_5} = (0,0,1,-1,0,0)$ and $\tilde{\alpha}_{H_6} = (0,0,0,1,-1,1)$. From theorem (4.8), $p = 3$. That is, $NBC_{\trianglelefteq}^{0}(M_{A_G})$, $NBC_{\trianglelefteq}^{1}(M_{A_G})$, $NBC_{\trianglelefteq}^{2}(M_{A_G})$ unchanged under the deformation, i.e. $f_3^{\Delta_G} = f_3^{\Delta_t}$. Jambu's-Papadima's deformation replaced $NBC_{\trianglelefteq}^{3}(M_{A_G})$ by $S_{\tilde{\Pi}_t}^{4}(\widetilde{A_{G_t}})$ and by adding exactly two 3-faces, $\{2^t, 4^t, 5^t, 6^t\}$ and $\{3^t, 4^t, 5^t, 6^t\}$. Jambu's-Papadima's deformation destroys all the dependent 5-section of A_G and replaced it by two 4-faces, $\{1^t, 2^t, 4^t, 5^t, 6^t\}$ and $\{1^t, 3^t, 4^t, 5^t, 6^t\}$ to obtain $S_{\tilde{\Pi}_t}^{5}(\widetilde{A_{G_t}})$ and we have A_G and $\widetilde{A_{G_t}}$ are L_3-equivalent. Thus, for $0 \leq k \leq 4$, $\mathrm{A}_{\Delta_G}^{k} \cong \mathrm{A}_{\tilde{\Delta}_t}^{k}$ and for $0 \leq k \leq 3$, $M_k \cong M_k^t$.

$$0 \longrightarrow M_5^t \longrightarrow M_4^t \longrightarrow M_3^t \longrightarrow M_2^t \longrightarrow M_1^t \longrightarrow M_0^t \longrightarrow \mathrm{A}_{\tilde{\Delta}_t} \longrightarrow 0.$$

Finally, the reduced broken circuit $\overline{NBC}_{\trianglelefteq}(M_G)$ of $M_{\trianglelefteq}$ was computed by applying definition (4.2), that is $f_{\overline{NBC}_{\trianglelefteq}(M_G)}^{\Delta_G} = (1,5,9,7,0)$. And figure (4.1) includes realization of $NBC_{\trianglelefteq}(M_G)$ and $\overline{NBC}_{\trianglelefteq}(M)$ complexes;



$NBC_{\trianglelefteq}(M_G)$ $\overline{NBC}_{\trianglelefteq}(M)$

Figure (4.2)

By a applying corollary (4.11), we have;

$$\widetilde{H}_d(\Delta_G) \cong \begin{cases} \mathbb{Z} & \text{if } d = 3 \\ 0 & \text{if } d \neq 3 \end{cases} \text{ and;}$$

$$\widetilde{H}_2\left(\overline{NBC}\left(M_{A_G}\right);\mathbb{Z}\right) \cong \mathbb{Z}^2;$$

where, $\beta(M) = (-1)^4\chi(\overline{NBC}_{\trianglelefteq}(M_{A_G}))$ and $\chi\left(\overline{NBC}_{\trianglelefteq}\left(M_{A_G}\right)\right) = 2$.

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حول الترتيبات البيانية القابلة للحل فوقياً

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المستخلص:
الهدف من هذا البحث هو دراسة الترتيبات البيانية القابلة للحل فوقياً، و التي قدمت أولاً من قبل Papadima و Suciu عام 2002. و لخدمة هذا الهدف عرفنا التجزئة القابلة للحل فوقياً (و رمزنا لها بالرمز Hp)، و الترتيب القابل للحل فوقيا للبيان، لغرض تقديم وجودهما كشرط ضروري و كافي لأن يكون أي بيان، قابل للحل فوقياً. من جهة أخرى، درسنا الماترويدات البيانية القابلة للحل فوقياً و قدمنا مقارنة بين الماترويد البياني القابل للحل فوقياً الذي يكون غير قابل للحل كلياً، والماترويد القابل للحل كلياً المشوه بواسطة طريقة التشويه التي قدمت من قبل Papadima و Jambu في (1998-2002). أخيراً، هذا البحث تضمن بعض التطبيقات والأمثلة التوضيحية.