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Explicit Parameter-dependent Representations of Periodic Solutions for a Class of Nonlinear Systems

J. Al-Ameri Mohammed [*,**] **I. Tyukin** [***,****]

[*] *University of Leicester, Department of Mathematics, UK (e-mail: jmkaa2@le.ac.uk).*
[**] *University of Basrah, Department of Mathematics, College of Science, Iraq*
[***] *University of Leicester, Department of Mathematics, UK (e-mail: I.Tyukin@le.ac.uk)*
[****] *Saint-Petersburg State Electrotechnical University, Department of Automation and Control Processes, Russia.*

Abstract: We propose a method for deriving computationally efficient representations of periodic solutions of parameterized systems of nonlinear ordinary differential equations. These representations depend on parameters of the system explicitly, as quadratures of parameterized computable functions. The method applies to systems featuring both linear and nonlinear parametrization, and time-varying right-hand-side; it opens possibilities to invoke scalable parallel computations for numerical evaluation of solutions for various parameter values. Application of the method to parameter estimation problems is illustrated with constructing an algorithm for state and parameter estimation for the Morris-Lecar system.

Keywords: Parameter estimation, Nonlinear Parametrization, Adaptive observers, Time-varying systems.

1. INTRODUCTION

The problem of state and parameter estimation of systems of ordinary differential equations (ODEs) has been in the focus of attention for many decades. Many frameworks for addressing this problem have been developed to date, including but not limited to shooting methods (Bock et al., 2007), sensitivity functions (Banks et al., 2012), splines (Zhan and Yeung, 2011) and adaptive observers (Bastin and Gevers, 1988), (Marino, 1990), (Besançon, 2000), (Farza et al., 2009), (Tyukin et al., 2013), (Tyukin, 2011) (see also (Ljung, 1987), (Soderstrom and Stoica, 1988) for system-identification take on the problem).

Notwithstanding significant progress in this area in both theoretical and applied directions, there is a fundamental yet practical issue with this problem affecting further progress. The issue is that in general it is difficult if not impossible to express observed quantities as explicit known functions of parameters and initial conditions or their quadratures. Thus sequential numerical approximation of solutions over time is typically involved in the estimation process. The problem, however, is that this process is slow and does not scale well with computational resources available. At the same time there are problems such as e.g. real-time estimation of kinetic parameters of neural membranes (Prinz et al., 2003) that do require fast estimation of model parameters. Hence new approaches are needed.

Here we provide a method enabling us to address the above computational bottle-neck of the problem for a class of systems with nonlinear parameterziation. The main idea of the method is to present an observed quantity as an integral that is explicitly a) computable and b) dependent on the parameters entering the original ODE model nonlinearly. Doing so enables to benefit from computational advantages of prefix sum algorithms (Blelloch, 1990) and thus alleviating the issues of scalability and real-time. Our preliminary work in this direction (Tyukin et al., 2016) showed that employing the tools of adaptive observer design (Marino, 1990) provides a feasible solution for a relevant class of systems. In this work, employing observer structure (Hammouri and de Morales, 1990), we extend this idea to a significantly broader class of systems and provide the required representations as well as sufficient conditions for their existence.

The paper is organized as follows. In Section 2 we provide formal statement of the problem, including the definition of the considered class of systems and general technical assumptions. This is followed by presentation of main results in Section 3. In Section 4 we illustrate the method with examples, and Section 5 concludes the work.

2. PROBLEM FORMULATION

2.1 System definition

Consider the following class of nonlinear systems

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$$\begin{aligned} \dot{x} &= F(y,t)x + \Psi(y,t)\theta + g(y,\lambda,t) \\ y(t) &= C_1^T x;\ x(t_0) = x_0, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$ are the state and the output of the system, respectively, $F(y,t) \in \mathbb{R}^{n\times n}$ is a known matrix dependent on y and t; $\lambda \in \Omega_\lambda$, $\Omega_\lambda \subset \mathbb{R}^p$, $\theta \in \Omega_\theta$, $\Omega_\theta \subset \mathbb{R}^m$ are parameters, and $C_1 \in \mathbb{R}^n$: $C_1 = \mathrm{col}(1, 0, \cdots, 0)$. Other technical assumptions are detailed in Assumption 1 below.

Assumption 1. The following properties hold for (1):

(1) the solution of (1) is defined for all $t \geq t_0$, and it is T-periodic, $T > 0$;
(2) the function F is continuous, bounded, and $F(y(\cdot), \cdot)$ is T-periodic;
(3) exact values of parameters λ and θ are unknown;
(4) the values of $y(t)$ for $t \in [t_0, t_0 + T]$ are available and known;
(5) the function $\Psi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{n\times m}$ is such that $\Psi(y(\cdot), \cdot)$ is T-periodic and is in $L_\infty[t_0, \infty) \cap \mathcal{C}^0$;
(6) the function $g : \mathbb{R} \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^n$ is such that $g(y(\cdot), \lambda, \cdot)$ is T-periodic and is in $L_\infty[t_0, \infty) \cap \mathcal{C}^1$ for all $\lambda \in \Omega_\lambda$;
(7) the observability Gramian matrix
$$G(T, t_0) = \int_{t_0}^{t_0+T} \Phi_A(s, t_0)CC^T\Phi_A^T(s, t_0)ds,$$
$$C \in \mathbb{R}^{n+m},\ C = \mathrm{col}(1, 0, \ldots, 0),$$
where $\Phi_A(t, t_0)$, is the normalized (i.e. $\Phi_A(t_0, t_0) = I_{n+m}$) fundamental solution matrix of
$$\begin{aligned} \dot{x} &= A(y(t), t)x, \\ A(y(t), t) &= \begin{pmatrix} F(y(t), t) & \Psi(y(t), t) \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{2}$$
is of full-rank, i.e $\mathrm{rank}(G(T, t_0)) = n + m$.

The class of equations (1) accommodates a broad set of technical and natural systems ranging from models of (Bastin and Dochain, 1990), dynamics of populations (Jing and Chen, 1984), and neural membranes (Morris and Lecar, 1981). In case the solutions are periodic it also may, after suitable modifications (Tyukin et al., 2016), include systems

$$\begin{aligned} \dot{x} &= F(y,t)x + \Psi(y,t)\theta + g(y,q,\lambda,t) \\ \dot{q} &= \upsilon(y,\lambda,t)q + \omega(y,\lambda,t) \\ y &= C_1^T x;\ x(t_0) = x_0,\ q(t_0) = q_0, \end{aligned}$$

in which the functions $\upsilon(y(\cdot), \lambda, \cdot)$, $\omega(y(\cdot), \lambda, \cdot)$ are continuous.

For notational convenience (cf. (Torres et al., 2012)), in what follows, we will combine the state variable x and parameters θ entering the right-hand-side of (1) linearly into a single variable χ and rewrite the system accordingly:

$$\begin{aligned} \dot{\chi} &= A(y,t)\chi + \begin{pmatrix} g(y,\lambda,t) \\ 0 \end{pmatrix} \\ y(t) &= C^T\chi;\ \chi(t_0) = \chi_0. \end{aligned} \tag{3}$$

In (3) $\chi = (x, \theta)$ is the combined state vector, matrix $A(y, t)$ is defined as in (2), and $C \in \mathbb{R}^{n+m}$ is $C = \mathrm{col}(1, 0, \cdots, 0)$. Let us now proceed with the formal definition of the problem considered in this contribution.

2.2 Problem statement

Consider system (3), and suppose that the values of $y(t)$ for $t \in [t_0, t_0 + T]$ are known and available *a-priori.* These values will depend on the parameters λ and initial condition χ_0 which themselves are assumed to be *unknown.* The question is if there exists an operator $\mathcal{F}$ mapping $y(\cdot)$ over $[t_0, t_0+T]$ into an efficiently computable quantity that does depend on the parameters λ explicitly?

Formally we are seeking to find an $\mathcal{F}(\lambda, [y], t)$ such that

$$\begin{aligned} & C^T\chi(t; t_0, \chi_0, \lambda) = \mathcal{F}(\lambda, [y], t) \\ & \mathcal{F}(t, \lambda, [y]) = \pi(t, \lambda, [y]) + \int_{t_0}^{t} p(\tau, \lambda, y(\tau), [y])d\tau \\ & \forall\ t \in [t_0, t_0 + T],\ \lambda \in \Omega_\lambda \end{aligned} \tag{4}$$

in which the functionals π and p are known and computable, e.g. in quadratures. The functionals π, p must not depend on χ_0 as a parameter, but nevertheless have to ensure the required representation (4). When such a representation is found one can employ numerous off-line numerical optimization techniques to infer the values of λ, θ, and initial conditions from the values of y in the interval $[t_0, t_0 + T]$. We will illustrate this step with an example in Section 4 in which the Nelder-Mead algorithm (Nelder and Mead, 1965) will be used for this purpose.

3. MAIN RESULT

The problem of existence of representations (4) in the context of parameter estimation is hardly viable without assessing parameter identifiability (Distefano and Cobelli, 1980) of (3). The corresponding sufficient conditions are derived below.

3.1 Indistinguishable parametrizations of (3)

We begin with the following technical lemma.

Lemma 1. Consider the following class of system

$$\begin{aligned} \dot{\chi} &= A_0(t)\chi + u(t) + d(t), \\ y &= C\chi,\quad \chi(t_0) = \chi_0,\quad \chi_0 \in \mathbb{R}^\ell \end{aligned} \tag{5}$$

where

$$A_0(t) = \left(\begin{array}{c|cccc} \alpha_1(t) & \beta_2(t) & \beta_3(t) & \cdots & \beta_\ell(t) \\ \hline \alpha_2(t) & & & & \\ \vdots & & A_0^*(t) & & \\ \alpha_\ell(t) & & & & \end{array}\right)$$

and $u, d, \alpha : \mathbb{R} \to \mathbb{R}^\ell$, $\beta : \mathbb{R} \to \mathbb{R}^{\ell-1}$, $u \in \mathcal{C}^1$, $d, \alpha, \beta \in \mathcal{C}, \alpha = \mathrm{col}(\alpha_1(t), \alpha_2(t), \ldots, \alpha_\ell(t)), \beta = (\beta_2(t), \beta_3(t), \ldots, \beta_\ell(t))$.
and assume that solutions of (5) are globally bounded in forward time.

Let, in addition:

1) $u, \dot{u}, d, \alpha, \beta$ be bounded: $\max\{\|u(t)\|, \|\dot{u}(t)\|\} \leq B$, $\|d(t)\| \leq \triangle_\xi$, $\|\alpha(t)\| \leq M_1$, $\|\beta(t)\| \leq M_2$ for all $t \geq t_0$.

2) there exist a $b : \mathbb{R} \to \mathbb{R}^{\ell-1}, b \in \mathcal{C}, \|b(t)\| \leq M_3$ such that the zero solution of the system

$$\dot{z} = \Lambda(t)z,\qquad \Lambda(t) = A_0^*(t) - b(t)\beta(t),$$

is uniformly exponentially stable, and let $\Phi_\Lambda(t, t_0)$ be the corresponding fundamental solution: $\Phi_\Lambda(t_0, t_0) = I_\ell$.

Then the following statements hold:

1) If the solution of (5) is globally bounded for all $t \geq t_0$ then, for T sufficiently large, there are $k_1, k_2 \in \mathcal{K}$: $\|y(t)\|_{\infty,[t_0,t_0+T]} \leq \epsilon \Rightarrow \exists \quad t'(\epsilon, x_0) \geq t_0$: $\|h(\tau) + u_1(\tau)\|_{\infty,[t',t_0+T]} \leq k_1(\epsilon) + k_2(\triangle_\xi)$,

where $h(t) = \beta(t)z$,

$$\begin{array}{l} \dot{z} = \Lambda(t)z + Gu, \\ G = \begin{pmatrix} -b(t) & I_{\ell-1} \end{pmatrix}, \quad z(t_0) = 0, \end{array} \tag{6}$$

2) If $d(t) \equiv 0$, then $y(t) = 0$ for all $t \in [t_0, t_0 + T]$ implies existence of $P \in \mathbb{R}^{\ell-1}$:

$$\beta(t)\Phi_\Lambda(t, t_0)P + h(t) + u_1(t) = 0 \tag{7}$$

for all $t \in [t_0, t_0 + T]$.

The proof of the lemma is proved in the Appendix.

According to Lemma 1 the set of parameters:

$$\begin{aligned} \mathcal{E}(\lambda) = &\{\lambda' \in \mathbb{R}^p | \; \exists \, p \in \mathbb{R}^{\ell-1} : \; \eta(t, p, \lambda', \lambda) = 0, \\ &\forall t \in [t_0, t_0 + T]\} \end{aligned} \tag{8}$$

where

$$\begin{aligned} &\eta(t, p, \lambda', \lambda) = \\ &\quad \beta(t)\Phi_\Lambda(t, t_0)p + g_1(y(t), \lambda', t) - g_1(y(t), \lambda, t) + \\ &\beta(t)\int_{t_0}^{t} \Phi_\Lambda(t, \tau)G(\tau)\begin{pmatrix} g(y(\tau), \lambda', \tau) - g(y(\tau), \lambda, \tau) \\ 0 \end{pmatrix} d\tau, \end{aligned}$$

and Λ is defined as in (6), contains parameters λ' producing measurements $y(t) = C^T\chi(t; t_0, \chi_0, \lambda')$ that are indistinguishable from $C^T\chi(t; t_0, \chi_0, \lambda)$ on the interval $[t_0, t_0 + T]$. If the set $\mathcal{E}(\lambda)$ contains more than one element then the system (3) may not be uniquely identifiable on $[t_0, t_0 + T]$. Notwithsdanding existence and possible utility of systems that are not uniquely identifiable, we will nevertheless focus on systems (3) that are uniquely identifiable on $[t_0, t_0 + T]$. Thus we assume that the following holds:

Assumption 2. For every $\lambda \in \Omega_\lambda$, the set $\mathcal{E}(\lambda)$ consists of just one element.

3.2 Auxiliary observer in the differential form

In addition to (3) consider the following *auxiliary* system:

$$\begin{aligned} \dot{\hat{\chi}} &= A(y(t), t)\hat{\chi} + \begin{pmatrix} g(y(t), \lambda', t) \\ 0 \end{pmatrix} - R^{-1}C(C^T\hat{\chi} - y), \\ \dot{R} &= -\delta R - A(y(t), t)^T R - RA(y(t), t) + CC^T \end{aligned} \tag{9}$$

$\hat{\chi}(t_0) = \hat{\chi}_0 \in \mathbb{R}^{n+m}, \; R(t_0) \in \mathbb{R}^{(n+m)\times(n+m)}$

where $\hat{\chi} \in \mathbb{R}^{n+m}$ is the observer's state, $R(t_0)$ is a positive-definite symmetric matrix, and $\delta \in \mathbb{R}_{>0}$ is a positive parameter. Solutions of (9) are defined for all $t \geq t_0$ (see items (1), (2) in Assumption 1), and hence, (Hammouri and de Morales, 1990), $R(t)$ is given by

$$\begin{aligned} R(t) = &e^{-\delta(t-t_0)}\Phi_A(t_0, t)^T R(t_0)\Phi_A(t_0, t) + \\ &\int_{t_0}^{t} e^{-\delta(t-s)}\Phi_A(s, t)^T CC^T\Phi_A(s, t)ds \end{aligned} \tag{10}$$

It is clear that $R(t)$ is non-singular for all $t \geq t_0$, symmetric, and positive-definite. Furthermore, if the value of the parameter $\delta > 0$ is chosen so that

$$\|e^{-\delta(t-t_0)/2}\Phi_A(t_0, t)\| \leq De^{-a(t-t_0)}, \; a > 0, \tag{11}$$

then $R(t)$ is bounded. In what follows the following additional assumption is instrumental:

Assumption 3. There exist $t_1 \geq t_0$ and $\alpha(\delta) > 0$ such that

$$\phi(t, \delta) = \int_{t_0}^{t} e^{-\delta(t-s)}\Phi_A(s, t)^T CC^T\Phi_A(s, t)ds \geq \alpha(\delta)I_{n+m}$$

for all $t \geq t_1$.

The next theorem specifies asymptotic behaviour of the observer system (9) (adapted from (Hammouri and de Morales, 1990)).

Theorem 2. Consider (9) and suppose that $\delta > 0$ be chosen so that both (11) and Assumption 3 hold, and $\lambda' = \lambda$. Then there exists a $t_2 \geq t_0$, such that:

$$\|\hat{\chi}(t; \hat{\chi}_0) - \chi(t; \chi_0)\| \leq ke^{-\delta(t-t_0)}$$

for all $t \geq t_2$, where k is a constant dependent on δ, t_0, χ_0 and the initial state $\hat{\chi}_0$ of the observer system (9).

Theorem 2 states the variable $\hat{\chi}(t)$ asymptotically tracks $\chi(t)$, and that the difference between the two converges to zero exponentially. Here, however, we are interested in establishing finite-time relationships (4). To do so we need another technical result establishing sufficient conditions for the existence of unique periodic solutions of R. The result is provided in Lemma 3.

Lemma 3. Consider (9) with $A(y(t), t)$ being T-periodic. Then, for sufficiently large $\delta > 0$, there exists a unique symmetric $R(t_0)$ ensuring that the function $R(t)$ defined by (10) is T-periodic. If, in addition, (11) and Assumption 3 hold then $R(t_0)$ is positive-definite.

Proof. Consider $R(t + T)$ and its derivative wrt. t:

$$\begin{aligned} \dot{R}(t + T) = &-\delta R(t + T) - A(y(t + T), t + T)^T R(t + T) \\ &-R(t + T)A(y(t + T), t + T) + CC^T \end{aligned}$$

Since $A(y(t+T), t+T) = A(y(t), t)$ for all $t \geq t_0$, we have

$$\begin{aligned} \dot{R}(t + T) = &-\delta R(t + T) - A(y(t), t)^T R(t + T) \\ &-R(t + T)A(y(t), t) + CC^T \end{aligned} \tag{12}$$

Denoting $E(t) = R(t + T) - R(t)$ and invoking (12) we obtain:

$$\dot{E} = -\delta E - A(y(t), t)^T E - EA(y(t), t). \tag{13}$$

If $R(t_0) = R(t_0 + T)$ then $E(t) = \mathbf{0}$ is the unique $(n + m)$ zero matrix solution of (13). This implies that $R(t) = R(t+T)$ for all $t \geq t_0$. Let us show that such $R(t_0)$ exists.

For $R(t_0) = R(t_0 + T)$ to hold $R(t_0)$ must satisfy

$$\begin{aligned} R(t_0) = &e^{-\delta T}\Phi_A(t_0, t_0 + T)^T R(t_0)\Phi_A(t_0, t_0 + T) + \\ &\int_{t_0}^{t_0+T} e^{-\delta(T+t_0-s)}\Phi_A(s, t_0 + T)^T CC^T\Phi_A(s, t_0 + T)ds. \end{aligned} \tag{14}$$

Let us rewrite (14) as:

$$R(t_0) - H_1R(t_0)H_2 = B \tag{15}$$

where

$$\begin{aligned} H_1 &= e^{-\delta T/2}\Phi_A(t_0, t_0 + T)^T, \\ H_2 &= e^{-\delta T/2}\Phi_A(t_0, t_0 + T) \end{aligned}$$

and

$$B = \int_{t_0}^{t_0+T} e^{-\delta(T+t_0-s)}\Phi_A(s, t_0+T)^T CC^T\Phi_A(s, t_0+T)ds.$$

The matrices H_1, H_2 are non-singular by construction, and hence (15) is equivalent

$$H_1^{-1}R(t_0) - R(t_0)H_2 = H_1^{-1}B. \tag{16}$$

Moreover, $H_1 = H_2^T$. The latter implies that if $R(t_0)$ is a solution of (15) then so is $R(t_0)^T$:

$$R(t_0)^T = (H_1 R(t_0) H_2)^T + B^T = H_2^T R(t_0)^T H_1^T + B$$

Further, (16) is the Sylvester equation; it has a unique solution if the spectra of $(n+m)\times(n+m)$ matrices H_1^{-1} and H_2 are disjoint (i.e. H_1^{-1} and H_2 have no common eigenvalues).

Note that

$$H_1^{-1} = e^{\delta T/2}(\Phi_A(t_0, t_0+T)^T)^{-1},$$

and let $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_{n+m}$ and $\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_{n+m}$ be the eigenvalues of $(\Phi_A(t_0, t_0+T)^T)^{-1}$ and $\Phi_A(t_0, t_0+T)$, respectively. The moduli of eigenvalues α_i of H_1^{-1} and eigenvalues β_i of the matrix H_2 are:

$$|\alpha_i| = e^{\delta T/2}|\tilde{\alpha}_i|,$$
$$|\beta_i| = e^{-\delta T/2}|\tilde{\beta}_i|$$

for all $i = 1, 2, \cdots, n+m$. Denote

$$\alpha_{\max} = \max_i\{|\tilde{\alpha}_i|\}, \quad \alpha_{\min} = \min_i\{|\tilde{\alpha}_i|\}$$
$$\beta_{\max} = \max_i\{|\tilde{\beta}_i\}, \quad \beta_{\min} = \min_i\{|\tilde{\beta}_i\}$$

Given $\alpha_{\min} \neq 0$ one can pick the value of δ so large that

$$e^{\delta T} > \frac{\beta_{\max}}{\alpha_{\min}}.$$

Doing so implies that

$$e^{\delta T/2}\alpha_{\min} > e^{-\delta T/2}\beta_{\max}$$

This, in turn, results in

$$|\alpha_i| > |\beta_j|, \forall\, i, j = 1, \ldots, n+m.$$

Hence

$$\alpha_i \neq \beta_j, \forall\, i, j = 1, \ldots, n+m,$$

and there is a symmetric matrix $R(t_0)$ satisfying (16) and, consequently, (14).

Finally, let us show that if (11) and Assumption 3 hold then the corresponding $R(t_0)$ is positive-definite. Let N be a non-negative integer. Given that $R(t_0) = R(t_0+NT)$ we see that

$$R(t_0) = e^{-\delta NT}\Phi_A(t_0, t_0+NT)^T R(t_0)\Phi_A(t_0, t_0+NT)$$
$$+\phi(t_0+NT, \delta). \tag{17}$$

According to (11) the norm

$$\|e^{-\delta NT}\Phi_A(t_0, t_0+NT)^T R(t_0)\Phi_A(t_0, t_0+NT)\|$$

can be made arbitrarily small if N is large enough. At the same time, Assumption 3 guarantees that $\phi(t_0+NT, \delta) \geq \alpha(\delta)$ in (17) for all N that are sufficiently large. Since the value of N in (17) can be chosen arbitrary large we conclude that $R(t_0)$ is positive-definite too. □.

3.3 Integral parametrization of periodic solutions of (3)

For notational convenience, let us rewrite auxiliary observer equations (9) as:

$$\dot{\hat{\chi}} = (A(t) - R^{-1}CC^T)\hat{\chi} + \begin{pmatrix} g(y(t), \lambda', t) \\ 0 \end{pmatrix} + R^{-1}Cy(t)$$
$$\dot{R} = -\delta R - A(y(t), t)^T R - RA(y(t), t) + CC^T$$
$$\hat{\chi}(t_0) = \hat{\chi}_0 \in \mathbb{R}^{n+m}, \; R(t_0) \in \mathbb{R}^{(n+m)\times(n+m)} \tag{18}$$

and additionally consider dynamics of the linear part of the first equation:

$$\xi = \left(A(y(t), t) - R^{-1}(t)CC^T \right)\xi. \tag{19}$$

Let $\Phi(t, s)$ be the normalized fundamental solution matrix of (19), i.e. $\Phi(t, t) = I_{n+m}$ and $\Phi(s, t) = \Phi(t, s)^{-1}$.

Theorem 4. Consider system (18) and suppose that Assumptions 1 and 2 hold. In addition, suppose that condition (11) hold and the values of δ and the initial condition $R(t_0)$ in (18) are chosen such that $R(t) > 0$ is T-periodic.

Consider the function $\hat{y} : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}$:

$$\hat{y}(\lambda', t) = C^T\big(\Phi(t, t_0)\hat{\chi}_0 + \textstyle\int_{t_0}^t \Phi(t, \tau)\times$$
$$\left(R^{-1}(\tau)Cy(\tau) + \begin{pmatrix} g(y(\tau), \lambda', \tau) \\ 0 \end{pmatrix} \right) d\tau\big) \tag{20}$$

where

$$\hat{\chi}_0 = (I_{n+m} - \Phi(t_0+T, t_0))^{-1}\textstyle\int_{t_0}^{t_0+T}\Phi(t_0+T, \tau)\times$$
$$\left(R^{-1}(\tau)Cy(\tau) + \begin{pmatrix} g(y(\tau), \lambda', \tau) \\ 0 \end{pmatrix} \right) d\tau. \tag{21}$$

Then

$$\hat{y}(\lambda', t) = C\chi(t; t_0, \chi_0, \lambda) \;\forall\; t \in [t_0, t_0+T] \Leftrightarrow \lambda = \lambda'.$$

Proof. *Sufficiency*, i.e. implication ⇒. Assumption 1 implies that Assumption 3 holds along the solution of (18). This together with condition (11) assure that there are positive constants $\rho, D > 0$ such that

$$\|\Phi(t, t_0)\| \leq De^{-\rho(t-t_0)}$$

Hence the matrix $I_{n+m} - \Phi(t_0+T, t_0)$ has no zero eigenvalues, and its inverse matrix, $(I_{n+m} - \Phi(t_0+T, t_0))^{-1}$, exists. Thus $\hat{y}(\lambda', t)$ described by (20), (21) is defined for all $t \in [t_0, t_0+T]$. Periodicity of $R(t)$ implies that

$$\hat{\chi}(t; t_0, \hat{\chi}_0, \lambda') = \Phi(t, t_0)\hat{\chi}_0 + \tag{22}$$
$$\int_{t_0}^t \Phi(t, \tau)\left(R^{-1}(\tau)Cy(\tau) + \begin{pmatrix} g(y(\tau), \lambda', \tau) \\ 0 \end{pmatrix} \right) d\tau$$

with $\hat{\chi}_0$ defined by (21) is the unique asymptotically stable periodic solution of the $\hat{\chi}$-subsystem in (18). On this solution we have: $CC^T\hat{\chi}(t) - Cy(t) = 0$ for all $t \in [t_0, t_0+T]$. Thus

$$\dot{\hat{\chi}} = A(y(t), t)\hat{\chi} + \begin{pmatrix} g(y(t), \lambda', t) \\ 0 \end{pmatrix}, \; C^T\hat{\chi}(t) = y(t).$$

Consider $e = \hat{\chi} - \chi$:

$$\dot{e} = A(y(t), t)e + \begin{pmatrix} g(y(t), \lambda', t) \\ 0 \end{pmatrix} - \begin{pmatrix} g(y(t), \lambda, t) \\ 0 \end{pmatrix}$$

According to Lemma 1 and Assumption 2 the set of indistinguishable parametrizations $\mathcal{E}(\lambda)$ of (3) comprises of a single element, and hence $\lambda' = \lambda$.

Necessity, ⇐. Let $\lambda = \lambda'$. According to assumptions of the theorem dynamics of $\hat{\chi} - \chi$ satisfies (19). The zero solution of the latter is globally asymptotically stable, and hence $\lim_{t\to\infty}\hat{\chi}(t) - \chi(t) = 0$. Noticing that (22) is the unique exponentially stable periodic solution of the $\hat{\chi}$-subsystem in (18) we obtain that $\hat{\chi}(t; t_0, \hat{\chi}_0, \lambda') = \chi(t; t_0, \chi_0, \lambda)$ for all $t \in [t_0, t_0 + T]$, and hence $\hat{y}(\lambda', t) = C^T \chi(t; t_0, \chi_0, \lambda)$. □

4. EXAMPLE

Consider the following simple point model of neural membrane activity (Morris and Lecar, 1981):

$$\begin{aligned} \dot{x} &= g_{Ca} m_\infty(x)(x - E_{Ca}) + g_K q(x + E_K) \\ &\quad + g_L(x + E_L) + I \\ \dot{q} &= \frac{-1}{\tau(x)} q + \frac{\omega_\infty(x)}{\tau(x)} \\ y &= x, \end{aligned} \tag{23}$$

$$\begin{aligned} m_\infty(x) &= 0.5\left(1 + \tanh\left(\frac{x - V_1}{V_2}\right)\right) \\ \omega_\infty(x) &= 0.5\left(1 + \tanh\left(\frac{x + V_3}{V_4}\right)\right) \\ \tau(x) &= T_0\left(\cosh\left(\frac{x + V_3}{2V_4}\right)\right). \end{aligned}$$

Here x is the measured voltage, q is the recovery variable. Parameters E_{Ca}, E_K, E_L are the Nernst potentials of which the nominal values are assumed to be *known*: $E_{Ca} = 55.17, E_K = -110.14, E_L = 49.49$; other parameters may vary from one cell to another and thus are considered *unknown*.

Assume that the model operates in the oscillatory regime which corresponds to periodic solutions of (23). For practically relevant values of T_0, V_3, V_4 the integral

$$\int_{t_0}^{t_0+T} -\frac{1}{\tau(s)} ds < 0$$

where T is the period of oscillations. Given that $x(\cdot)$ is T-periodic, the variable q can be expressed as:

$$q(t) = e^{\int_{t_0}^{t} -\frac{1}{\tau(x(s))} ds} q_0 + \int_{t_0}^{t} e^{\int_{z}^{t} -\frac{1}{\tau(x(s))} ds} \frac{\omega_\infty(x(z))}{\tau(x(z))} dz$$

$$q_0 = \left(1 - e^{\int_{t_0}^{t} -\frac{1}{\tau(x(s))} ds}\right)^{-1} \times$$

$$\int_{t_0}^{t_0+T} e^{\int_{z}^{t_0+T} -\frac{1}{\tau(x(s))} ds} \frac{\omega_\infty(x(z))}{\tau(x(z))} dz.$$

Denoting $g(t, \lambda, [y]) = g_{Ca} m_\infty(x)(x - E_{Ca}) + g_K q(x + E_K)$, $\Psi(t, y) = (y(t), 1)$, and combining parameters as $\theta = (g_L, I)$, $\lambda = (V_1, V_2, V_3, V_4, T_0, g_{Ca}, g_K)$ we can rewrite (23) in the form of equation (3) with

$$A(y(t), t) = \begin{pmatrix} 0 & y(t) & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For this system and given nominal parameter values, the period of oscillations is $T = 15.1692$, and hence for convenience the integration interval is chosen as $[0, 15.1692]$. In what follows, numerical evaluation of integrals and solutions of all auxiliary differential equations was performed on equi-spaced grids with the step size of 0.0002.

According to Theorem 4, explicit parameter-dependent representation of the observed quantity, $\hat{y}(\lambda, t)$, is defined by (20), where $C = (1, 0, 0)$, $\chi = col(x, \theta)$, and the fundamental solution (3×3)-matrices $\Phi(t, t_0)$ and $\Phi_A(t, t_0)$ are computed for the linear systems $\dot{\chi} = (A(y(t), t) - R^{-1}(t)CC^T)\chi$, $\dot{R} = -\delta R - A(y(t), t)^T R - RA(y(t), t) + CC^T$, and $\dot{\chi} = A(y(t), t)\chi$, respectively, by the Improved Euler method for $t \in [0, 15.1692]$. The value of δ was set as $\delta = 2$, and numerical approximations of matrices $\Phi_A(t, t_0)$

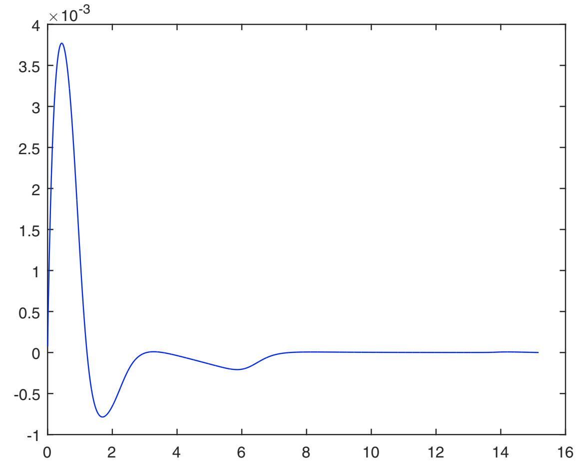

Fig. 1. The values of relative error $e(t) = (\hat{y}(\lambda, t) - y(t))/\|y\|_{\infty,[t_0, t_0+\infty]}$ as a function of t.

were used to compute the matrices $R(t)$ in accordance with equation (10). The value of $R(t_0)$ in (10) was chosen to be the unique solution of the Sylvester equation (16) (see Lemma 3).

Figure 1 shows the relative error, $e(t) = (\hat{y}(\lambda, t) - y(t))/\|y\|_{\infty,[t_0, t_0+\infty]}$, between the proposed numerical representation (20) and simulated $y(t)$ (Runge-Kutta, step size 0.0002) for nominal parameter values.

The parameterized representations were later used, in combination with the NelderMead algorithm (Nelder and Mead, 1965) to recover the values of parameters λ and θ. Results are provided in Table 1 and Figure 2 for parameters.

Table 1. True (first row) and Estimated (second row) of λ and θ, and the value of x_0

Vector $\lambda = (V_1, V_2, V_3, V_4, T_0, g_{Ca}, g_K)$

V_1	V_2	V_3	V_4	T_0	g_{Ca}	g_K
-1	15	-10	14.5	3	-1.1	-2
-0.9999	14.9999	-10	14.5	3	-1.1	-2

Vector $\theta = (g_L, I)$ and x_0

g_L	I	x_0
-0.5	10	21.96388
-0.49982	9.99345	21.96166

The process took less than 10 minutes on a standard PC in Matlab R2015a.

Table 2. Time for 1000 evaluations of y

Eq. (20)	Improved Euler method	Ratio
2.21311 minutes	10.43818 minutes	4.71652

In order to assess potential computational advantage of the proposed integral form of equation (24) we compared the time required for 1000 evaluations of $y(t)$ in Matlab a) expressed as in (20) and b) computed by the Improved Euler method over the interval $[t_0, t_0 + T]$. The parameter values for both cases were kept identical and did not change from one trial to the other. The results are summarized in Table 2. This experiment shows that evaluation of the proposed representation, (20), in Matlab on CPU is approximately and on average 5 times faster than the Improved Euler integration.

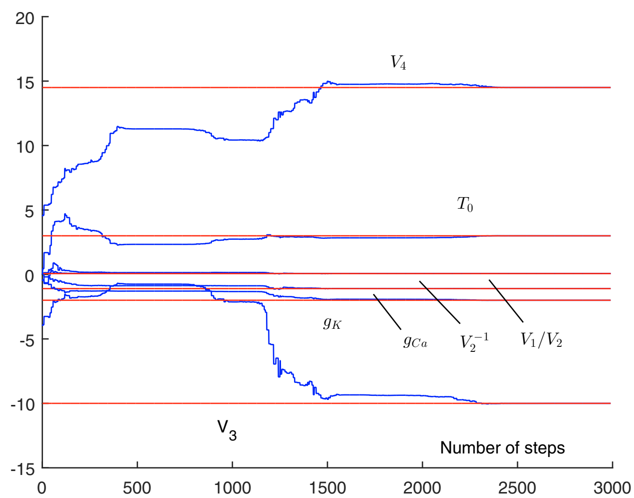

Fig. 2. Estimates and true values of V_1, V_2, V_3, V_4, T_0, g_{Ca} and g_K.

5. CONCLUSION

The work presented a method for computationally efficient and explicit parameter-dependent representation of periodic solutions of systems of nonlinear ODEs. The method is rooted in the ideas from adaptive observers theory and is an extension of our earlier work (Tyukin et al., 2016) in which linear part of the system was supposed to be time-invariant. Here we extended this earlier result to systems with time-varying linear parts. Similar extension can be carried out for other observer structures, including e.g. (Loria et al., 2009), followed by replacement of condition (7) in Assumption 1 with the requirement of persistency of excitation of corresponding terms.

The computational advantage of the method is due to the possible parallel implementation of calculations that the proposed representations offer. In addition to offering scalability and making use of parallel computations, the method offers reduction of dimensionality of the problem due to incorporating linearly parameterized part of the model into internal variables of the proposed representations. These internal variables are uniquely determined by parameters entering the model nonlinearly and are computed as a part of the representation.

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APPENDIX

Lemma 5. Consider $\dot{y} = k(t)y + u(t) + d(t)$, $k, u, d : \mathbb{R}_{\geq t_0} \to \mathbb{R}$, $u \in \mathcal{C}^1, d \in \mathcal{C}^0$, and let $\max\{|u(t)|, |\dot{u}(t)|\} \leq B, |d(t)| \leq \triangle_\xi$, $|k(t)| \leq \rho$. Finally, let T, ϵ be a non-negative real numbers such that $T > \sqrt{\epsilon}$. Then

$$\|y\|_{\infty,[t_0,t_0+T]} \leq \epsilon \Rightarrow$$
$$\|u\|_{\infty,[t_0,t_0+T)} \leq 2\sqrt{\epsilon}(\ e^{\rho\sqrt{\epsilon}} + B) + \triangle_\xi, \quad \forall\ t \leq t_0 + T.$$

Proof. Let L be an arbitrary element of $[0, T]$. Note that for all $t \geq t_0 + L$ the variable $y(t)$ can be expressed as:

$$y(t) = y(t-L)e^{\int_{t-L}^{t} k(\tau)d\tau} + \int_{t-L}^{t} e^{\int_{\tau}^{t} k(\tau_1)d\tau_1}(u(\tau)+d(\tau))d\tau.$$

According to the mean value theorem there is a $\tau' \in [t - L, t]$:

$$y(t) - y(t-L)e^{\int_{t-L}^{t} k(\tau)d\tau} = L\ e^{\int_{\tau'}^{t} k(\tau_1)d\tau_1}(u(\tau') + d(\tau'))$$
$$\Rightarrow \quad y(t)e^{-\int_{\tau'}^{t} k(\tau_1)d\tau_1} - y(t-L)e^{\int_{t-L}^{\tau'} k(\tau)d\tau}$$
$$= L\ (u(\tau') + d(\tau')) \Rightarrow |y(t)e^{-\int_{\tau'}^{t} k(\tau_1)d\tau_1}| +$$
$$|y(t-L)e^{\int_{t-L}^{\tau'} k(\tau)d\tau}| \geq L\ |u(\tau') + d(\tau')|$$

Given that:

$$\begin{aligned} |e^{\int_{t}^{\tau'} k(\tau_1)d\tau_1}| &\leq |e^{\int_{t}^{\tau'} \rho d\tau_1}| \triangleq e^{\rho(\tau'-t)} \\ \Rightarrow |e^{-\int_{\tau'}^{t} k(\tau_1)d\tau_1}| &\leq e^{\rho(\tau'-t)} \leq e^{\rho L} \\ \Rightarrow |e^{\int_{t-L}^{\tau'} k(\tau_1)d\tau_1}| &\leq |e^{\int_{t-L}^{\tau'} \rho d\tau_1}| \triangleq e^{\rho(\tau'-(t-L))} \\ &\leq e^{\rho(\tau'-(t-L))} \leq e^{\rho L} \end{aligned}$$

and invoking the mean value theorem we conclude that $\exists\ \tau'' \in [\tau', t]$:

$$\begin{aligned} |u(t)| &= |u(\tau) - u(\tau) + u(t)| \\ &= |u(\tau') + u'(\tau'')(t - \tau') - d(\tau') + d(\tau')| \\ &\leq |u(\tau') + d(\tau')| + BL + \triangle_\xi \end{aligned}$$
$$\Rightarrow |u(\tau') + d(\tau')| \geq |u(t)| - BL - \triangle_\xi$$

Hence

$$|u(t)| \leq BL + \triangle_\xi + \frac{2\ \epsilon}{L}\ e^{\rho L}, \quad \forall\ t \leq t_0 + L$$

Given that L can be chosen arbitrary in the interval $[0, T]$, we let $L = \sqrt{\epsilon}$, and thus $|u(t)| \frac{2\ \epsilon}{\sqrt{\epsilon}}\ e^{\rho\sqrt{\epsilon}} + \ \leq\ BL + \triangle_\xi$. Finally, given that $|\dot{u}(t)| \leq B|$ for all $t \in [t_0, t_0 + T]$ including in the interval $[t_0, t_0 + \sqrt{\epsilon}]$, we conclude that

$$|u(t)| \leq 2\sqrt{\epsilon}(\ e^{\rho\sqrt{\epsilon}} + B) + \triangle_\xi, \quad \forall\ t \leq t_0 + T.$$

Proof of Lemma 1

Let us rewrite the system as

$$\begin{aligned} \dot{y} &= C^T\dot{\chi} = \dot{\chi}_1 \\ &= \alpha_1(t)\chi_1 + \beta(t)\tilde{\chi} + u_1(t) + d_1(t) \end{aligned} \tag{24}$$

$$\begin{aligned} \dot{\tilde{\chi}} &= A_0^*(t)\tilde{\chi} + \tilde{\alpha}(t)\chi_1 + b(t)u_1(t) - b(t)u_1(t) + \\ &\quad \tilde{u}(t) + \tilde{d}(t) \\ &= A_0^*(t)\tilde{\chi} + \tilde{\alpha}(t)y + b(t)u_1(t) + G(t)u(t) + \tilde{d}(t) \end{aligned} \tag{25}$$

where $G(t) = (-b(t) \quad I_{\ell-1})$, $\tilde{\alpha}(t) = \mathrm{col}(\alpha_2(t), \ldots, \alpha_\ell(t))$, $\beta(t) = (\beta_1(t), \ldots, \beta_\ell(t))$, $\tilde{d}(t) = \mathrm{col}(d_2(t), \ldots, d_\ell(t))$ and $\tilde{\chi} = \mathrm{col}(\chi_2, \ldots, \chi_\ell)$.

Let $\|y(t)\|_{\infty,[t_0,t_0+T]} \leq \epsilon$ and denote $e(t) = \beta(t)\tilde{\chi} + u_1(t)$. According to Lemma 5, there are $v_1, v_2 \in \mathcal{K}$ such that $\|e(t)\| = \|\beta(t)\tilde{\chi} + u_1(t)\| \leq v_1(\epsilon) + v_2(\triangle_\xi)$ for all $t \in [t_0, t_0 + T]$.

Using the notation above one obtains:

$$\begin{aligned} \dot{\tilde{\chi}} &= (A_0^*(t) - b(t)\beta(t))\tilde{\chi} + \tilde{\alpha}(t)y + G(t)u(t) + \\ &\quad b(t)e(t) + \tilde{d}(t) \\ &= \Lambda(t)\tilde{\chi} + \tilde{\alpha}(t)y + G(t)u(t) + b(t)e(t) + \tilde{d}(t). \end{aligned} \tag{26}$$

Therefore,

$$\begin{aligned} \|u_1(t) + h(t)\| &= \|u_1(t) + \beta(t)\tilde{\chi} - \beta(t)\tilde{\chi} + h(t)\| \\ &\leq \|u_1(t) + \beta(t)\tilde{\chi}\| + \|\beta(t)\tilde{\chi} - h(t)\|. \end{aligned}$$

The solutions of (6) and (26) are:

$$\begin{aligned} z(t) &= \Phi_\Lambda(t, t_0)z_0 + \int_{t_0}^{t} \Phi_\Lambda(t,\tau)G(\tau)u(\tau)d\tau \\ &= \int_{t_0}^{t} \Phi_\Lambda(t,\tau)G(\tau)u(\tau)d\tau, \end{aligned}$$

$$\begin{aligned} \tilde{\chi}(t) &= \Phi_\Lambda(t, t_0)\tilde{\chi}_0 + \int_{t_0}^{t} \Phi_\Lambda(t,\tau)(\tilde{\alpha}(\tau)y(\tau) + G(\tau)u(\tau) + \\ &\quad b(\tau)e(\tau) + \tilde{d}(\tau))d\tau. \end{aligned}$$

Hence

$$\begin{aligned} &\textstyle\int_{t_0}^{t} \Phi_\Lambda(t,\tau)G(\tau)u(\tau)d\tau = \tilde{\chi}(t) - \Phi_\Lambda(t, t_0)\tilde{\chi}_0 - \\ &\quad \textstyle\int_{t_0}^{t} \Phi_\Lambda(t,\tau)(\tilde{\alpha}(\tau)y(\tau) + b(\tau)e(\tau) + \tilde{d}(\tau))d\tau \end{aligned}$$

Since the system $\dot{z} = \Lambda(t)z$ is uniformly exponentially stable, there are $D, k \in \mathbb{R}_{>0}$ such that $\|\Phi(t, t_0)\| \leq De^{-k(t-t_0)}$.

Therefore,

$$\begin{aligned} &\|\beta(t)\tilde{\chi}(t) - \beta(t)z(t)\| = \|\beta(t)\Phi_\Lambda(t, t_0)\tilde{\chi}_0 + \\ &\quad \beta(t) \textstyle\int_{t_0}^{t} \Phi_\Lambda(t,\tau)(\tilde{\alpha}(\tau)y(\tau) + b(\tau)e(\tau) + \tilde{d}(\tau))d\tau\| \\ &\quad \leq M_2De^{-k(t-t_0)}\|\tilde{\chi}_0\| + \frac{DM_2}{k}\ (1 - e^{-k(t-t_0)})(M_1\epsilon + \\ &\quad M_3(v_1(\epsilon) + v_2(\triangle_\xi)) + \triangle_\xi) \end{aligned}$$

Noticing that $h(t) = \beta(t)\int_{t_0}^{t} \Phi_\Lambda(t,\tau)G(\tau)u(\tau)d\tau$, denoting $\kappa(\epsilon) = \dfrac{DM_2}{k}(M_1\epsilon + M_3v_1(\epsilon)) + v_1(\epsilon)$, $\kappa_2(\triangle_\xi) = \dfrac{DM_2}{k}(\triangle_\xi + M_3v_2(\triangle_\xi)) + v_2(\triangle_\xi)$, and letting

$$t'(\epsilon, \chi_0) = t_0 + \frac{1}{k}\ln\left(\frac{DM_2\|\chi_0\|}{\epsilon}\right)$$

we conclude that there is a $t'(\epsilon, \chi_0) \geq t_0$ such that

$$\begin{aligned} \|u_1(t) + h(t)\|_{\infty,[t,t_0+T]} &\leq \kappa(\epsilon) + \epsilon + \kappa_2(\triangle_\xi) \\ &\triangleq \kappa_1(\epsilon) + \kappa_2(\triangle_\xi) \end{aligned}$$

for all $t \in [t'(\epsilon, \chi_0), t_0 + T] : T$ is sufficiently large to satisfy $t_0 + T > t'(\epsilon, \chi_0)$.

Finally, let $d(t) \equiv 0$. Then $y \equiv 0 \Rightarrow \dot{y} \equiv 0$, and hence (24) implies that $\beta(t)\tilde{\chi} + u_1(t) \equiv 0$. On the other hand,

$$\begin{aligned} e(t) =& \beta(t)\tilde{\chi} + u_1(t) = \beta(t)\Phi_\Lambda(t, t_0)P + \\ & \beta(t)\int_{t_0}^{t} \Phi_\Lambda(t,\tau)G(\tau)u(\tau)d\tau + u_1(t) \\ =& \beta(t)\Phi_\Lambda(t, t_0)P + h(t) + u_1(t). \end{aligned}$$

Thus there is a $P \in \mathbb{R}^{\ell-1}$ such that (7) holds.