# The Real Simple Lie Algebra sl(2,R):an Approximate Lie algebra for Abel's Equation

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***Abstract:***
The problem of classification of Abel's equation (when increased in order to the second) is considered. It is based on investigating Lie's invariance conditions to obtain Lie algebra .It has been proved that the real simple Lie algebra sl(2,R) can be realized as an approximate Lie algebra for this class of equations .

## ***1.Introduction***:

In [1] , M.Boyko had suggested an approach of description of integrable cases of the Abel equations. It is based on increasing of the order of equations up to the second one.
Abel's equation can readily be seen to arise in a variety of nonlinear problems [2]
Boyko had considered the following second order ODE.

$$y^{\bullet\bullet}(y^{\bullet}+f_0(y))=y^{\bullet}f_1(y)+y^{\bullet^2}f_2(y)+y^{\bullet^3}f_3(y)+y^{\bullet^4}f_4(y) \qquad (1)$$

which is related to the Abel's equation of the 2nd kind

$$p^{\bullet}(p+f_0(y))=p^3 f_4(y)+p^2 f_3(y)+p f_2(y)+f_1(y)$$

by the substitution $y^{\bullet}=p(y)$.

In the case when (1) admit two dimensional Lie algebra, then it is shown that eq.(1) is integrable in the frame work of the invariance Lie approach [1].

Invariance (Lie point analysis) of DE with respect to one-parameter groups of transformation in the space of independent and dependent variables carries important information, see [3] .

Many authors have studied what is called approximate symmetries [4,5] and properties of this kind of Lie point analysis are discussed by Gazizov [6] .It turns out that approximate symmetries form an approximate Lie algebra.

The plan of this paper is as follows:
In section (2) we calculate the characterizing system of equation (1) using Lie’s invariance criterion. Solving partially two equations of them ,we construct a basis to the real simple algebra sl(2,R) in section (3) .Finally in section (4) ,while testing the remaining determining

equations we see that sl(2,R) can be proved to be as an approximate Lie algebra for the class of equations.

## ***2.Preliminary group classification of equation (1)***

We are looking for infinitesimal operator of Lie point analysis of equation (1) in the form

$$Q = \xi(x,y)\partial x + \eta(x,y)\partial y \tag{2}$$

where $\xi$ and $\eta$ are arbitrary real-valued smooth functions .Operator (2) generates one-parameter invariance group of (1) , iff it's coefficients $\xi$ and $\eta$ satisfy the equation (Lie's invariance criterion ):

$$\eta\,[y^{\bullet\bullet} f_\circ'(y) - y^{\bullet 4} f_4'(y) - y^{\bullet 3} f_3'(y) - y^{\bullet 2} f_2'(y) - y^{\bullet} f_1'(y)] + \eta^{(1)}[y^{\bullet\bullet} - 4y^{\bullet 3} f_4 - 3y^{\bullet 2} f_3 - 2y^{\bullet} f_2 - f_1] + \eta^{(2)}(y^{\bullet} + f_\circ)\Big|_{eq.(1)} = 0 \tag{3}$$

where $y^{\bullet} = \dfrac{dy}{dx}$ , $f_i'(y) = \dfrac{df}{dy}$ , and

$$\eta^{(1)} = \eta_x + (\eta_1 - \xi_{xx})y^{\bullet} - \xi_y y^{\bullet 2}$$

$$\eta^{(2)} = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y^{\bullet} + (\eta_{yy} - 2\xi_{xy})(y^{\bullet})^2 - \xi_{yy}(y^{\bullet})^3 + (\eta_y - 2\xi_x)y^{\bullet\bullet} - 3\xi_y y^{\bullet} y^{\bullet\bullet}$$

After simplifying eq.(3) we represent it in the form of the following equations

$$f_\circ^2 \eta_{xx} - \eta_x f_1 = 0 \tag{4a}$$

$$f_4 \xi_y = 0 \tag{4b}$$

$$2\eta_y f_4 - 3\xi_x f_4 - \eta f_4' = 0 \tag{4c}$$

$$\eta_{yy} - (\eta_y - 2\xi_x)f_3 + (\eta_y - 2\xi_x)f_\circ f_4 + \eta f_\circ f_4' - \eta f_3' - \eta f_\circ' f_4 + \eta_x f_4 + (\eta_y - \xi_x) f_3 - 4(\eta_y - \xi_x)f_4 = 0 \tag{4d}$$

$$(2\eta_{xy} - \xi_{xx}) + 2f_\circ \eta_{yy} + (\eta_y - 2\xi_x)f_2 + (\eta_y - 2\xi_x)f_\circ f_3 + \eta f_\circ' f_3 - \eta f_2' - \eta f_\circ f_3' + \eta_x f_3 + (\eta_y - \xi_x)f_2 - 4\eta_x f_4 - 3(\eta_y - \xi_x)f_3 = 0 \tag{4e}$$

$$\eta_{xx} + 2(2\eta_{xy} - \xi_{xx})f_\circ + f_\circ^2 \eta_{yy} + (\eta_y - 2\xi_x)f_1 + (\eta_y - 2\xi_x)f_\circ f_2 + \eta f_\circ' f_2 - \eta f_1' - \eta f_\circ f_2' + \eta_x f_2 + (\eta_y - \xi_x)f_1 - 3\eta_x f_3 - 2(\eta_y - \xi_x)f_2 = 0 \tag{4f}$$

$$2f_\circ \eta_{xx} + f_\circ^2(2\eta_{xy} - \xi_{xx}) + (\eta_y - 2\xi_x)f_\circ f_1 + \eta f_\circ' f_1 - \eta f_\circ f_1' + \eta_x f_1 - 2\eta_x f_2 - (\eta_y - \xi_x)f_1 = 0 \tag{4g}$$

first we solve (4a) which gives that

$$\eta_x = F(y)\exp(\frac{f_1}{f_o^2}x) \tag{5}$$

$F$ is an arbitrary function.

According to eq.(5) the class of DEs (1) is divided into two subclasses when $f_1 = 0$ which leads to $\eta = F(y)x + G(y)$ and if $f_1 \neq 0$ which gives $\eta = F(y)\exp(\frac{f_1}{f_o^2}x) + G(y)$ ,$G$ integration constant.

From equation (4b) we have $f_4 = 0$ or $\xi = \xi(x)$.

## 3.The Lie algebra, representation of the group:.

A feature of Lie point analysis of a DE is that they constitute an algebra, a representation of the group. And since all the second order ODEs have eight Lie point analysis and so belong to the equivalence class of $y'' = 0$ [7] hence our equation will have the eight Lie point analysis $Q_1, Q_2, \ldots, Q_8$. Moreover, any 2nd order ODE is completely specified by three Lie point analysis [8]. The three Lie point analysis for eq.(1) will be obtained in the following result.

***Proposition .1***

sl(2,R) is a Lie algebra for equation (1) when $f_1 \neq 0$ and $f_4 \neq 0$

Proof .

Let $Q_i = \xi_i \partial x + (F_i \exp\{F_o x\} + G_i)\partial y$ , $i=1,2$ $\quad F_o = \frac{f_1}{f_o^2}$

It is clear that equation (1) is invariant under the operator $Q_3 = \partial x$
We have to satisfy the commutator relation of sl(2,R) ,
i.e.,

$[Q_1, Q_3] = -2Q_3$ , $[Q_1, Q_2] = 2Q_2$ and $[Q_2, Q_3] = 2Q_1$

$[Q_1, Q_3] = -\xi_1{}'\partial x - F_1 F_o \exp\{F_o x\}\partial y = -2\partial x$

This implies that, $\xi_1{}' = 2$ and $F_1 F_o \exp\{F_o x\} = 0$
which means that ,

$Q_1 = (2x + k_1)\partial x + G_1(y)\,\partial y$ , $k_1$ arb. Const.

To obtain : $Q_2$

$[Q_1, Q_2] = 2Q_2$

i.e.

$$(2x+k_1)\xi_2'-2\xi_2 = 2\xi_2 \tag{6}$$
and
$$(2x+k_1)F_2F_o\exp\{F_o x\}+G_1[(F_2'+F_2F_o'x)\exp\{F_o x\}+G_2']-G_1'[F_2\exp\{F_o x\}$$
$$+G_2] = 2(F_2\exp\{F_o x\}+G_2) \tag{7}$$
Eq. (6) leads to
$\xi_2 = k_2(2x+k_1)^2$ , $k_2$ arb. Const.

To satisfy the third relation

$$[Q_2, Q_3] = 2Q_1$$

$$[Q_2, Q_3] = -4k_2(2x+k_1)\partial x - F_2F_\circ\exp\{F_\circ x\}\partial y$$

$$2Q_1 = 2(2x+k_1)\partial x + 2G_1(y)\partial y$$
Thus
$$-4k_2(2x+k_1) = 2(2x+k_1) \rightarrow k_2 = -\frac{1}{2}$$
and
$$-F_2F_o\exp\{F_o x\} = 2G_1(y) \tag{8}$$
since $F_o \neq 0$, hence eq.(8) cannot be satisfied unless $F_2 = G_1 = 0$.This result , when implemented in eq.(7) leads to $G_2 = 0$ too .Thus we get the operators:
$$Q_1 = (2x+k_1)\partial x$$
$$Q_2 = -\frac{1}{2}(2x+k_1)^2\partial x$$
and $Q_3 = \partial x$ □

## ***4.Approximate Lie algebra:***

Since $\eta = 0$ and $\xi = \xi(x)$, for the previous discussion, the remaining characterizing system (4c-4g) will be reduced to the following equations:

$$\xi_x f_4 = \xi_x f_3 = \xi_{xx} + 3\xi_x f_2 = (4f_o f_2 - 3f_1 + 2f_2)\xi_x$$
$$= (3f_o^2 f_2 - 2f_o f_1 + f_1)\xi_x = 0 \tag{9}$$

For the operator $Q_3$, it is clear that $\xi_x = 0$ and consequently it all the equation (9) are satisfied. While, if $\xi_x \neq 0$ as in $Q_1$ and $Q_2$ then we have the restrictions:
$$f_3 = f_4 = 4f_o f_2 - 3f_1 + 2f_2 = 3f_o^2 f_2 - 2f_o f_1 + f_1 = 0$$

Solving the equation $\xi_{xx}+3\xi_x f_2=0$ yields that $f_2$ is constant$(\lambda)$. If $\lambda\neq 0$ then

$\xi(x)=\frac{c_1}{\lambda}\exp(\lambda x)+c_2$ or $\xi(x)=c_1(\frac{1}{\lambda}+x+\frac{\lambda}{2!}x^2+.......)+c_2$

This expansion, when truncated, will recover $Q_1$ and $Q_2$. Thus we have an approximate Lie algebra.

## 5.Conclusion:

The approach taken, is a combination of Lie point analysis and testing some types of Lie algebras .In this paper sl(2,R) has been chosen to be the representation. The result, we get, of realization of sl(2,R) is implemented to proof that this type of algebra can be an approximate one. One can use the same approach to test other types of algebras.

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