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Witten–Hodge theory for manifolds with boundary and equivariant cohomology

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ABSTRACT

We consider a compact, oriented, smooth Riemannian manifold M (with or without boundary) and we suppose G is a torus acting by isometries on M. Given X in the Lie algebra of G and corresponding vector field X_M on M, one defines Witten's inhomogeneous coboundary operator $\mathrm{d}_{X_M} = \mathrm{d} + \iota_{X_M} : \Omega_G^{\pm} \to \Omega_G^{\mp}$ (even/odd invariant forms on M) and its adjoint δ_{X_M}. Witten (1982) [18] showed that the resulting cohomology classes have X_M-harmonic representatives (forms in the null space of $\Delta_{X_M} = (\mathrm{d}_{X_M} + \delta_{X_M})^2$), and the cohomology groups are isomorphic to the ordinary de Rham cohomology groups of the set $N(X_M)$ of zeros of X_M. Our principal purpose is to extend these results to manifolds with boundary. In particular, we define relative (to the boundary) and absolute versions of the X_M-cohomology and show the classes have representative X_M-harmonic fields with appropriate boundary conditions. To do this we present the relevant version of the Hodge–Morrey–Friedrichs decomposition theorem for invariant forms in terms of the operators d_{X_M} and δ_{X_M}. We also elucidate the connection between the X_M-cohomology groups and the relative and absolute equivariant cohomology, following work of Atiyah and Bott. This connection is then exploited to show that every harmonic field with appropriate boundary conditions on $N(X_M)$ has a unique X_M-harmonic field on M, with corresponding boundary conditions. Finally, we define the X_M-Poincaré duality angles between the interior subspaces of X_M-harmonic fields on M with appropriate boundary conditions, following recent work of DeTurck and Gluck.

1. Introduction

Throughout we assume M to be a compact oriented smooth Riemannian manifold of dimension n, with or without boundary. For each k we denote by $\Omega^k = \Omega^k(M)$ the space of smooth differential k-forms on M. The de Rham cohomology of M is defined to be $H^k(M) = \ker \mathrm{d}_k / \operatorname{im} \mathrm{d}_{k-1}$, where d_k is the restriction of the exterior differential d to Ω^k. In other words it is the cohomology of the de Rham complex (Ω^*, d). If M has a boundary, then the relative de Rham cohomology $H^k(M, \partial M)$ is defined to be the cohomology of the subcomplex (Ω_D^*, d) where Ω_D^k is the space of *Dirichlet* k-forms—those satisfying $i^*\omega = 0$ where $i : \partial M \hookrightarrow M$ is the inclusion of the boundary.

Classical Hodge theory. Based on the Riemannian structure, there is a natural inner product on each Ω^k defined by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge (\star\beta), \tag{1.1}$$

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