Topology and its Applications 159 (2012) 823–832

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

# Generalized Dirichlet to Neumann operator on invariant differential forms and equivariant cohomology

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A R T I C L E  I N F O

*Article history:*
Received 23 May 2011
Received in revised form 21 November 2011
Accepted 22 November 2011

*MSC:*
58J32
57R19
55N91
57R91

*Keywords:*
Algebraic topology
Equivariant topology
Equivariant cohomology
Cup product (ring structure)
Group actions
Dirichlet to Neumann operator

A B S T R A C T

In recent work, Belishev and Sharafutdinov show that the generalized Dirichlet to Neumann (DN) operator $\Lambda$ on a compact Riemannian manifold $M$ with boundary $\partial M$ determines de Rham cohomology groups of $M$. In this paper, we suppose $G$ is a torus acting by isometries on $M$. Given $X$ in the Lie algebra of $G$ and the corresponding vector field $X_M$ on $M$, Witten defines an inhomogeneous coboundary operator $\mathrm{d}_{X_M} = \mathrm{d} + \iota_{X_M}$ on invariant forms on $M$. The main purpose is to adapt Belishev–Sharafutdinov's boundary data to invariant forms in terms of the operator $\mathrm{d}_{X_M}$ in order to investigate to what extent the equivariant topology of a manifold is determined by the corresponding variant of the DN map. We define an operator $\Lambda_{X_M}$ on invariant forms on the boundary which we call the $X_M$-DN map and using this we recover the $X_M$-cohomology groups from the generalized boundary data $(\partial M, \Lambda_{X_M})$. This shows that for a Zariski-open subset of the Lie algebra, $\Lambda_{X_M}$ determines the free part of the relative and absolute equivariant cohomology groups of $M$. In addition, we partially determine the ring structure of $X_M$-cohomology groups from $\Lambda_{X_M}$. These results explain to what extent the equivariant topology of the manifold in question is determined by $\Lambda_{X_M}$.



## 1. Introduction

The classical Dirichlet to Neumann (DN) operator $\Lambda_{cl} : C^\infty(\partial M) \longrightarrow C^\infty(\partial M)$ is defined by $\Lambda_{cl}\theta = \partial\omega/\partial\nu$, where $\omega$ is the solution to the Dirichlet problem

$$\Delta\omega = 0, \qquad \omega\big|_{\partial M} = \theta$$

and $\nu$ is the unit outer normal to the boundary. In the scope of inverse problems of reconstructing a manifold from the boundary measurements, the following question is of great theoretical and applied interest [7]: *to what extent are the topology and geometry of M determined by the DN operator*?

In this paper we are interested in the equivariant topology analogue of this question.

Much effort has been made to address this (non-equivariant) question. For instance, in the case of a two-dimensional manifold $M$ with a connected boundary, an explicit formula is obtained which expresses the Euler characteristic of $M$ in terms of $\Lambda_{cl}$ and the Euler characteristic completely determines the topology of $M$ in this case [6]. In the three-dimensional case [5], some formulas are obtained which express the Betti numbers $\beta_1(M)$ and $\beta_2(M)$ in terms of $\Lambda_{cl}$ and their operator on vector fields, $\vec{\Lambda} : C^\infty(T(\partial M)) \longrightarrow C^\infty(T(\partial M))$. This culminates in recent work of Belishev and Sharafutdinov [7] who prove that the real additive de Rham cohomology of a compact, connected, oriented smooth Riemannian manifold $M$ of

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0166-8641/$ – see front matter 
doi:10.1016/j.topol.2011.11.052