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On Finding Hypergeometric Solutions for Recurrences and Rational Solutions via gcd

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Abstract

In this paper, we consider the problem of finding hypergeometric solutions for recurrences of arbitrary order with the additional restriction that the leading and trailing coefficients are constant. Also we consider the problem of finding rational solutions of the linear difference equation with polynomial coefficients without any restriction. We give an explicit formula for a universal denominator of a linear difference equation with polynomial coefficients. These approaches do not require any factorization, but only gcd (greatest common divisor) computations.

Keywords: **Gosper's algorithm, hypergeometric solution, rational solution, universal denominator.**

1. Introduction

Let N be the set of nonnegative integers, K be a field of characteristic zero, $K(n)$ be the field of rational functions over K, and $K[n]$ be the ring of polynomials over K. As usual, we assume that subject to normalization the gcd of two polynomials always takes a value as a monic polynomial, namely, polynomials with the leading coefficient being 1. Recall that a nonzero term t_n is called a hypergeometric term over K if there exists a rational function $r \in K(n)$ such that

$$\frac{t_{n+1}}{t_n} = r(n).$$

If $r(n) = a(n)/b(n)$, where $a(n), b(n) \in K[n]$, then the function $a(n)/b(n)$ is called a rational representation of the rational function $r(n)$. If additionally $\gcd(a(n), b(n)) = 1$ holds, then $a(n)/b(n)$ is called the reduced rational representation of $r(n)$.

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In [11], Petkovšek presents algorithm Hyper to find all hypergeometric solutions of the recurrence

$$\sum_{i=0}^{d} p_i(n) \cdot z_{n+i} = 0,$$

where $p_0(n), p_1(n), \ldots, p_d(n)$ are given polynomials over K. In the same paper, he gives an algorithm to find all solutions that are linear combinations of hypergeometric terms of the recurrence

$$\sum_{i=0}^{d} p_i(n) \cdot z_{n+i} = t_n, \tag{1.1}$$

where t_n is a given hypergeometric term over K. In another paper, Petkovšek [12] generalizes Gosper's algorithm [5] (also see [6,7,8,9,10,13]) to find all hypergeometric solutions of the recurrence (1.1), provided that $p_0(n)$ and $p_d(n)$ are constant.

Consider the linear difference equation

$$\sum_{i=0}^{d} p_i(n) y(n+i) = p(n), \tag{1.2}$$

where $p_0(n), p_1(n), \ldots, p_d(n), p(n) \in K[n]$ are given polynomials such that $p_0(n) \neq 0, p_d(n) \neq 0$. Recall that $g(n) \in K[n]$ is a universal denominator for (1.2) if and only if for every solution $y(n) \in K(n)$ to (1.2) there exists a $f(n) \in K[n]$ such that $y(n) = f(n)/g(n)$. The common idea of all algorithms that compute all rational solutions of the linear difference equation (1.2), is to construct a universal denominator $g(n)$. After constructing $g(n)$ one can substitute $y(n) = f(n)/g(n)$, where $f(n)$ is unknown polynomial, in (1.2) and then use one of the algorithms given in [1,4,11] to find all polynomial solutions $f(n)$ of the resulting equation which yields all rational solutions $f(n)/g(n)$. Therefore computing the universal denominator is a key step for computing rational solutions (see [2,3,14]). Recall that the dispersion $\text{dis}(a(n), b(n))$ of the polynomials $a(n), b(n) \in K[n]$ is the greatest nonnegative integer k (if it exists) such that $a(n)$ and $b(n+k)$ have a nontrivial common divisor, i.e.,

$$\text{dis}(a,b) = \max\{k \in N \mid \deg\gcd(a(n), b(n+k)) \geq 1\}.$$

If k does not exist then we set $\text{dis}(a,b) = -1$. Abramov [3] gave an algorithm to compute a universal denominator $G(n)$ for (1.2). Define

$$L = \text{dis}(p_d(n-d), p_0(n)) = \max\{k \in N \mid \deg\gcd(p_d(n-d), p_0(n+k)) \geq 1\}. \tag{1.3}$$

The explicit formula for Abramov's universal denominator is

$$G(n)=\gcd\left([p_0(n+L)]^{\underline{L+1}},[p_d(n-d)]^{\underline{L+1}}\right).$$

For more details about Abramov's universal denominator see [14].

The main result of this paper is the observation that if we express the rational functions in terms of their reduced rational representations, then finding hypergeometric solutions reduces to finding polynomial solutions. Also we give an explicit formula for a universal denominator of the linear difference equation (1.2).

2. Gosper's Algorithm for Recurrences of Arbitrary Order

In this section we consider the problem of finding hypergeometric solutions z_n for the recurrence (1.1) where t_n is a given hypergeometric term, provided that $p_0(n)$ and $p_d(n)$ are nonzero constant. If (1.1) has a hypergeometric solution, then the left hand-side of (1.1) can be written as a rational function multiple of z_n. Let $y(n)=z_n/t_n$. It follows that $y(n)$ is a rational function of n. Substituting $y(n)t_n$ for z_n in (1.1) results in

$$\sum_{i=0}^{d} p_i(n)\cdot y(n+i)\cdot\prod_{j=0}^{i-1} r(n+j)=1, \tag{2.1}$$

where $r(n)=t_{n+1}/t_n$ is a rational function of n. Hence we need to find rational solutions of (2.1).

Theorem 2.1. Let $r(n)$ and $y(n)$ in equation (2.1) be in terms of their reduced rational representations:

$$r(n)=\frac{a(n)}{b(n)},\quad y(n)=\frac{f(n)}{g(n)}. \tag{2.2}$$

Let k_0 be defined by

$$k_0=\mathrm{dis}(a(n-1),b(n))=\max\{k\in N \mid \deg\gcd(a(n-1),b(n+k))\geq 1\}. \tag{2.3}$$

Then

$$g(n)\mid \gcd\left(\prod_{j=0}^{k_0} b(n+j)^{2^j},\prod_{j=0}^{k_0} a(n-j-1)^{2^j}\right).$$

Proof. Using $y(n)$, defined in (2.2), in (2.1) gives

$$\sum_{i=0}^{d} p_i(n)\cdot f(n+i)\cdot\prod_{\substack{j=0\\ j\neq i}}^{d} g(n+j)\cdot\prod_{j=0}^{i-1} a(n+j)\cdot\prod_{j=i}^{d-1} b(n+j)=\prod_{j=0}^{d-1} b(n+j)\cdot\prod_{j=0}^{d} g(n+j)\cdot \tag{2.4}$$

All terms in this equation except the one with $i=0$ are divisible by $g(n)$, so

$$g(n) \mid p_0(n) \cdot \prod_{j=1}^{d} g(n+j) \cdot \prod_{j=0}^{d-1} b(n+j). \tag{2.5}$$

Similarly, looking at the term with $i = d$ and substituting $n-d$ for n, we find that

$$g(n) \mid p_d(n) \cdot \prod_{j=1}^{d} g(n-j) \cdot \prod_{j=1}^{d} a(n-j). \tag{2.6}$$

Shifting n by 1 in (2.5) yields

$$g(n+1) \mid p_0(n) \cdot \prod_{j=2}^{d+1} g(n+j) \cdot \prod_{j=1}^{d} b(n+j). \tag{2.7}$$

By multiplying (2.5) and (2.7) it follows that

$$g(n) \mid p_0(n)^2 \cdot \prod_{j=2}^{d} g(n+j)^2 \cdot g(n+d+1) \cdot b(n) \cdot \prod_{j=1}^{d-1} b(n+j)^2 . b(n+d).$$

By induction we get that for $k \geq 1$:

$$g(n) \mid p_0(n)^{2^{k-1}} \cdot \prod_{j=k}^{d+k-1} g(n+j)^{\gamma_j} \cdot \prod_{j=0}^{d+k-2} b(n+j)^{\beta_j},$$

where the γ's are positive integers and the β's are defined by

$$\beta_j = \left\{ \begin{array}{ll} 2^j, & j = 0,1,\ldots,k-2 \\ 2^{k-1}, & j = k-1,k,\ldots,d-1 \\ 2^{k-1} - 2^{j-d}, & j = d,d+1,\ldots,d+k-2. \end{array} \right\}. \tag{2.8}$$

Since K has characteristic zero, there is a large enough k such that

$$\gcd(g(n), g(n+j)) = 1,$$

for $j \geq k$. It follows that

$$g(n) \mid p_0(n)^{2^{k-1}} \cdot \prod_{j=0}^{d+k-2} b(n+j)^{\beta_j},$$

for all large enough k. Analogously, from (2.6) we get

$$g(n) \mid p_d(n)^{2^{k-1}} \cdot \prod_{j=0}^{d+k-2} a(n-j-1)^{\beta_j},$$

for all large enough k, where the β's are defined as in (2.8). Since $p_0(n)$ and $p_d(n)$ are constants, it follows that

$$g(n) \mid \gcd\left(\prod_{j=0}^{d+k-2} b(n+j)^{\beta_j}, \prod_{j=0}^{d+k-2} a(n-j-1)^{\beta_j}\right),$$

for all large enough k. By using the definition of the dispersion we obtain

$$g(n) \mid \gcd\left(\prod_{j=0}^{k-1} b(n+j)^{\beta_j}, \prod_{j=0}^{k-1} a(n-j-1)^{\beta_j}\right),$$

for $k > k_0$. The rest of the proof follows when k goes to infinity in this equation and by the definition of the dispersion.
□

Now set

$$g(n) = \gcd\left(\prod_{j=0}^{k_0} b(n+j)^{2^j}, \prod_{j=0}^{k_0} a(n-j-1)^{2^j}\right) \quad \textbf{(2.9)}$$

in equation (2.4). If equation (2.4) can be solved for $f \in K[n]$ then

$$z_n = \frac{f(n)}{g(n)} t_n$$

is a hypergeometric solution of (1.1), otherwise no hypergeometric solution of (1.1) exists.

Algorithm 2.1.
INPUT : $\{p_i(n)\}_{i=0}^{d} \in K[n]$ **such that $p_0(n)$ and $p_d(n)$ are nonzero constant and $r(n) \in K(n)$**
such that $t_{n+1}/t_n = r(n)$ for all $n \in N$.
OUTPUT : **a hypergeometric solution z_n of (1.1) if it exists, otherwise "no hypergeometric solution of (1.1) exists".**

(1) Decompose $r(n)$ into a/b where a,b are two relatively prime polynomials.

(2) Compute k_0 as in (2.3).

(3) If $k_0 \geq 0$ then compute

$$g(n) = \gcd\left(\prod_{j=0}^{k_0} b(n+j)^{2^j}, \prod_{j=0}^{k_0} a(n-j-1)^{2^j}\right),$$

otherwise $g(n) = 1$.

(4) If equation (2.4) can be solved for $f \in K[n]$ then return $z_n = \frac{f(n)}{g(n)} t_n$, otherwise return "no hypergeometric solution of (1.1) exists".

□

Example 2.1. We want to find all hypergeometric solutions of

$$2z_{n+2} - 8z_{n+1} - z_n = t_n, \tag{2.10}$$

where $t_n = 4\binom{2n}{n+2} - 5\binom{2n}{n}$, then

$$r(n) = \frac{t_{n+1}}{t_n} = \frac{a(n)}{b(n)},$$

where $a(n) = (4n+2)(n^2+21n+30)$, $b(n) = (n+3)(n^2+19n+10)$, and then $k_0 = 0$. From (2.9),
$g(n) = n^2 + 19n + 10$. By (2.4) $f(n)$ is a polynomial which satisfies

$$\begin{aligned} 8(2n+1)(2n+3)f(n+2) - 16(2n+1)(n+4)f(n+1) - (n+3)(n+4)f(n) \\ = (n+3)(n+4)(n^2+19n+10). \end{aligned}$$

The only polynomial solution of this equation is $f(n) = -(n+1)(n+2)$. Thus $z_n = \frac{f(n)}{g(n)} t_n = \binom{2n}{n}$ is the only hypergeometric solution of (2.10).

□

3. Rational Solutions of Linear Difference Equations

In this section we consider the problem of finding rational solutions of the linear difference equation (1.2) by giving an explicit formula for a universal denominator for (1.2).

Theorem 3.1. Let $y(n)$ in equation (1.2) be defined as in equation (2.2) and let L be defined as in (1.3), then

$$g(n) \mid \gcd\left(p_0(n) \prod_{j=1}^{L} p_0(n+j)^{2^{j-1}}, p_d(n-d) \prod_{j=1}^{L} p_d(n-d-j)^{2^{j-1}} \right).$$

Proof. Using $y(n)$, defined in (2.2), in (1.2) gives

$$\sum_{i=0}^{d} p_i(n) \cdot f(n+i) \cdot \prod_{\substack{j=0 \\ j \neq i}}^{d} g(n+j) = p(n) \cdot \prod_{j=0}^{d} g(n+j). \tag{3.1}$$

All terms in this equation except the one with $i=0$ are divisible by $g(n)$, so

$$g(n) \mid p_0(n)\cdot\prod_{j=1}^{d} g(n+j)\cdot \tag{3.2}$$

Similarly, looking at the term with $i=d$ and substituting $n-d$ for n, we find that

$$g(n) \mid p_d(n-d)\cdot\prod_{j=1}^{d} g(n-j)\cdot \tag{3.3}$$

Shifting n by 1 in (3.2) yields

$$g(n+1) \mid p_0(n+1)\cdot\prod_{j=2}^{d+1} g(n+j)\cdot \tag{3.4}$$

By multiplying (3.2) and (3.4) it follows that

$$g(n) \mid p_0(n)p_0(n+1)\cdot\prod_{j=2}^{d} g(n+j)^2\cdot g(n+d+1)\cdot$$

By induction we get that for $k\geq 1$:

$$g(n) \mid p_0(n)\cdot\prod_{j=1}^{k-1} p_0(n+j)^{2^{j-1}}\cdot\prod_{j=k}^{d+k-1} g(n+j)^{\gamma_j},$$

where the γ's are positive integers. Since K has characteristic zero, there is a large enough k such that

$$\gcd(g(n), g(n+j)) = 1,$$

for $j\geq k$. It follows that

$$g(n) \mid p_0(n)\cdot\prod_{j=1}^{k-1} p_0(n+j)^{2^{j-1}},$$

for all large enough k. Analogously, from (3.3) we get

$$g(n) \mid p_d(n-d)\cdot\prod_{j=1}^{k-1} p_d(n-d-j)^{2^{j-1}},$$

for all large enough k. Therefore

$$g(n) \mid \gcd\left(p_0(n)\prod_{j=1}^{k-1} p_0(n+j)^{2^{j-1}}, p_d(n-d)\prod_{j=1}^{k-1} p_d(n-d-j)^{2^{j-1}}\right),$$

for all large enough k. The rest of the proof follows when k goes to infinity in this equation and by the definition of the dispersion.

□

Now set

$$g(n)=\gcd\left(p_0(n)\prod_{j=1}^{L}p_0(n+j)^{2^{j-1}},p_d(n-d)\prod_{j=1}^{L}p_d(n-d-j)^{2^{j-1}}\right)$$

in equation (3.1). If equation (3.1) can be solved for $f\in K[n]$ then $y(n)=f(n)/g(n)$ solves (1.2), otherwise no rational solution of (1.2) exists.

Algorithm 3.1.
INPUT : nonzero polynomials $p_0(n)$, $p_d(n)$.
OUTPUT : a polynomial $g(n)$.

(1) Compute L as in (1.3).

(2) If $L\geq 0$ then compute

$$g(n)=\gcd\left(p_0(n)\prod_{j=1}^{L}p_0(n+j)^{2^{j-1}},p_d(n-d)\prod_{j=1}^{L}p_d(n-d-j)^{2^{j-1}}\right),$$

otherwise $g(n)=1$.

□

Example 3.1. We want to find a rational solution of

$$\begin{aligned}&(2n^3+13n^2+22n+8)y(n+3)-(2n^3+11n^2+18n+9)y(n+2)\\&+(2n^3+n^2-6n)y(n+1)-(2n^3-n^2-2n+1)y(n)=0.\end{aligned}\tag{3.5}$$

We have $p_0(n)=-(n-1)(2n-1)(n+1)$, $p_d(n)=(n+4)(2n+1)(n+2)$, then $L=2$ and then $g(n)=(n-1)(n+1)n$. By (3.1), $f(n)$ is a polynomial which satisfies

$$\begin{aligned}&n(2n+1)(n+2)(n+1)f(n+3)-n(2n+3)(n+3)(n+1)f(n+2)+n(2n-3)\\&\cdot(n+3)(n+2)f(n+1)-(2n-1)(n+3)(n+1)(n+2)f(n)=0.\end{aligned}$$

The polynomial $f(n)=Cn(2n-3)$ is a solution of this equation. Thus $y(n)=\dfrac{f(n)}{g(n)}=C\dfrac{2n+1}{n^2-1}$ is a rational solution of (3.5).

□

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حول ايجاد الحلول الهايبرجيومترية للمعادلات التكرارية
والحلول النسبية باستخدامgcd

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الملخص

في هذا البحث نتأمل مسألة ايجاد الحلول الهايبرجيومترية للمعادلات التكرارية من أي درجة مع قيود اضافية هي ان المعاملات القيادية والمتأخرة تكون ثابتة. كذلك نتأمل مسألة ايجاد الحلول النسبية لمعادلات الفروقات الخطية ذات معاملات متعددات حدود بدون قيود. نعطي صيغة صريحة للمقام الشامل لمعادلة الفروقات الخطية ذات معاملات متعددات حدود. هذه الاساليب لا تتطلب تحليل للعوامل، لكن فقط حسابات القاسم المشترك الأعظم.

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