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$_m$-Hypergeometric Solutions

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Abstract

In this paper we present approaches to find m - hypergeometric solutions for anti-difference equations, homogeneous linear recurrence equations with polynomial coefficients, and non-homogeneous linear recurrence equations with polynomial coefficients provided their leading and trailing coefficients are constant.

Keywords : Gosper's algorithm, hypergeometric solution, m - hypergeometric solution.

1. Introduction

Let m denotes a positive integer, $\mathbb{N}$ be the set of natural numbers, K be the field of characteristic zero, $K(n)$ be the field of rational functions over

K, $K[n]$ be the ring of polynomials over K. We assume the result of any gcd (greatest common divisor) computation in $K[n]$ as being normalized to a monic polynomial p, i.e., the leading coefficient of p being 1. Recall that a non-zero term t_n is called a hypergeometric term over K if there exist a rational function $r(n) \in K(n)$ such that

$$\frac{t_{n+1}}{t_n} = r(n).$$

Gosper's algorithm [3] (also see [4,6,7,9,11]) has been extensively studied and widely used to prove hypergeometric identities. Given a hypergeometric term t_n, Gosper's algorithm is a procedure to find a hypergeometric term z_n satisfying

$$z_{n+1} - z_n = t_n, \tag{1.1}$$

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if it exists, or confirm the nonexistence of any solution of (1.1). Gosper showed that any rational function $r(n)$ can be written in the following form, called the Gosper representation:

$$r(n) = \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)},$$

where a, b and c are polynomials over K and

$$\gcd(a(n), b(n+h)) = 1 \quad \text{for all } h \in \mathbb{N}.$$

Petkovšek [8] realized that the Gosper representation becomes unique, which is called the Gosper- Petkovšek representation, or GP representation, for short, if we further require that b, c are monic polynomials such that

$$\gcd(a(n), c(n)) = 1,$$

$$\gcd(b(n), c(n+1)) = 1.$$

In the same paper, Petkovšek presents algorithm **Hyper** to find all hypergeometric solutions of the recurrence

$$\sum_{i=0}^{d} p_i(n) \cdot z_{n+i} = 0,$$

where $p_0(n), p_1(n), \ldots, p_d(n)$ are given polynomials over K. In another paper, Petkovšek [9] generalizes Gosper's algorithm to find all hypergeometric solutions of the recurrence

$$\sum_{i=0}^{d} p_i(n) \cdot z_{n+i} = t_n, \tag{1.2}$$

where t_n is a given hypergeometric term over K and $p_0(n), p_d(n)$ are constant.

Recall that a non-zero term a_n is called an m-hypergeometric over K if there exist a rational function $w \in K(n)$ such that

$$\frac{a_{n+m}}{a_n} = w(n).$$

If $w(n) = f(n)/g(n)$, where $f(n), g(n) \in K[n]$, then the function $f(n)/g(n)$ is called the rational representation of $w(n)$. If additionally $\gcd(f(n), g(n)) = 1$ holds, then $f(n)/g(n)$ is called the reduced rational representation of $w(n)$. In [5], Koepf extends Gosper's algorithm to find m-hypergeometric solutions s_n of

$$s_{n+m} - s_n = a_n, \tag{1.3}$$

where a_n is a given m-hypergeometric term. In [10], Petkovšek and Bruno give the following lemma:

Lemma 1.1. *Let $w(n)$ be a non-zero rational function over K. Then there exist a non-zero constant $z \in K$ and monic polynomials a, b and c over K such that*

$$w(n) = z\frac{a(n)}{b(n)}\frac{c(n+m)}{c(n)}, \tag{1.4}$$

where

(i) $\gcd(a(n), b(n+mh)) = 1$ *for all* $h \in \mathbb{N}$,
(ii) $\gcd(a(n), c(n)) = 1$,
(iii) $\gcd(b(n), c(n+1)) = 1$.

The representation of $w(n)$ in (1.4) such that (i), (ii) and (iii) hold is called the m-Gosper-Petkovšek (in short: m GP) representation. The m GP representation is unique (The proof of this statement is analogous to the one given in [8] for the special case $m = 1$). Petkovšek and Bruno used Lemma 1.1 to describe an algorithm to find m-hypergeometric solutions of the recurrence

$$\sum_{i=0}^{d} p_i(n) \cdot s_{n+im} = 0, \tag{1.5}$$

where $p_0(n), p_1(n), \ldots, p_d(n)$ are given polynomials over K. Their algorithm reduces to algorithm **Hyper** when $m = 1$.

The contents of this paper are as follows: In Section 2, we extend Petkovšek's [9] and Paule-Strehl's [7] approaches for Gosper's algorithm to find m-hypergeometric solutions of the linear recurrence (1.3). In Section 3, we generalize algorithm **Hyper** to find m-hypergeometric solutions of the linear recurrence (1.5). Finally, In Section 4, we show how to generalize Petkovšek's approach [9] to find m-hypergeometric solutions of the recurrence

$$\sum_{i=0}^{d} p_i(n) \cdot s_{n+im} = a_n, \tag{1.6}$$

where a_n is a given m-hypergeometric term and $p_0(n), p_d(d)$ are constant.

2. Extension of Some Approaches of Gosper's Algorithm

In this section we extend Petkovšek's approach [9] and Paule-Strehl's approach [7] to find m-hypergeometric solutions of (1.3).

2.1 Extension of Petkovšek's Approach

In [9], Petkovšek give an approach for Gosper's algorithm. In this section we extend that approach to find m-hypergeometric solution for the recurrence (1.3). To do this we give the following results:

Lemma 2.1. *Let w be a rational function over K. Then there exist polynomials a,b,c over K such that*

$$w(n)=\frac{a(n)}{b(n)}\frac{c(n+m)}{c(n)}, \tag{2.1}$$

and

$$\gcd(a(n),b(n+mh))=1 \quad \textit{for all } h\in\mathbb{N}. \tag{2.2}$$

Proof. *The proof is analogous to the one given in* [3] *for the special case* $m=1$. □

The representation of $w(n)$ in (2.1) such that (2.2) holds is called the m-Gosper representation.

Lemma 2.2. *Let $a,b,c,A,B,C\in K[n]$ such that*

$$\gcd(a(n),c(n))=\gcd(b(n),c(n+m))=\gcd(A(n),B(n+mh))=1 \quad \textit{for all } h\in\mathbb{N}.$$

If

$$\frac{a(n)}{b(n)}\frac{c(n+m)}{c(n)}=\frac{A(n)}{B(n)}\frac{C(n+m)}{C(n)},$$

then $c(n)$ divides $C(n)$.

Proof. The proof is analogous to the one given in [8] for the special case $m = 1.$ □

Given an m-hypergeometric term a_n and suppose that there exists an m-hypergeometric solution s_n satisfying equation (1.3). By using (1.3) we get

$$\frac{s_n}{a_n} = \frac{s_n}{s_{n+m} - s_n} = \frac{1}{\frac{s_{n+m}}{s_n} - 1}.$$

Let $y(n) = {}^{s_n}\!/_{a_n}$. It follows that $y(n)$ is a rational function of n. Substituting $y(n)a_n$ for s_n in (1.3) to obtain

$$w(n)y(n+m) - y(n) = 1, \tag{2.3}$$

where $w(n) = {}^{a_{n+m}}\!/_{a_n}$ is a rational function of n. Let

$$y(n) = {}^{f(n)}\!/_{g(n)}\ , \tag{2.4}$$

and (2.4) into (2.3) to obtain

$$\frac{a(n)}{b(n)}\frac{c(n+m)}{c(n)} = \frac{(f(n)+g(n))}{f(n+m)}\frac{g(n+m)}{g(n)},$$

By Lemma 2.2, $g(n) \mid c(n)$, so $c(n)$ is a suitable denominator for $y(n)$.

Write $y(n) = {}^{v(n)}\!/_{c(n)}$,

where $v(n)$ is an unknown polynomial, and substitute this together with (2.1) into (2.3) to obtain

$a(n)v(n+m) = (v(n)+c(n))b(n).$ This shows that $b(n)$ divides $v(n+m)$, hence we have

$$y(n) = \frac{b(n-m)x(n)}{c(n)}, \tag{2.5}$$

where $x(n)$ is a polynomial in n. Substitution of (2.1) and (2.5) into (2.3) shows that $x(n)$ satisfies

$$a(n)x(n+m)-b(n-m)x(n)=c(n).$$

Now if such a polynomial solution $x(n)\in K[n]$ exists, then

$$s_n=\frac{b(n-m)x(n)}{c(n)}a_n$$

is an m-hypergeometric solution of (1.3). □

2.2 Extension of Paule-Strehl's Approach

In [7], Paule and Strehl give an approach for Gosper's algorithm. In this section we extend that approach to find m-hypergeometric solution for the recurrence (1.3). Given an m-hypergeometric term a_n and suppose that there exists an m-hypergeometric solution s_n satisfying equation (1.3). Let $F(n)/G(n)$ be the reduced rational representation of s_{n+m}/s_n. Then s_n can be written as

$$s_n=\frac{G(n)}{F(n)-G(n)}a_n. \tag{2.6}$$

By using (2.6) we get

$$w(n)=\frac{F(n)}{G(n+m)}\frac{F(n+m)-G(n+m)}{F(n)-G(n)}, \tag{2.7}$$

where $w(n)=a_{n+m}/a_n$ is a rational function of n. The right hand side of (2.7) is very close to the m GP representation, but in general there is no guarantee to have $\gcd(F(n),G(n+mj))=1$ for all $j\geq 1$. To overcome this problem consider the m GP representation for $F(n)/G(n+m)$,

$$\frac{F(n)}{G(n+m)}=\frac{A(n)}{B(n)}\frac{C(n+m)}{C(n)}, \tag{2.8}$$

say, for polynomials $A, B, C \in K[n]$. Write $w(n)$ in an m GP representation as

$$w(n) = \frac{a(n)}{b(n)} \frac{c(n+m)}{c(n)},$$

Then (2.7) turns into a true GP representation, namely

$$\frac{a(n)}{b(n)} \frac{c(n+m)}{c(n)} = \frac{A(n)}{B(n)} \frac{C(n+m)(F(n+m) - G(n+m))}{C(n)(F(n) - G(n))}.$$

Since the m GP representation is unique, we get

$$a(n) = A(n), \quad b(n) = B(n),$$

and

$$c(n) = C(n)(F(n) - G(n)). \tag{2.9}$$

Equation (2.9) can be rewritten as

$$c(n) = A(n) \frac{G(n+m)C(n+m)}{B(n)} - B(n-m) \frac{G(n)C(n)}{B(n-m)}.$$

It follows that

$$c(n) = a(n) \frac{G(n+m)C(n+m)}{b(n)} - b(n-m) \frac{G(n)C(n)}{b(n-m)}.$$

which shows that $x(n) = {G(n)C(n)}/{b(n-m)}$ is a solution to the equation

$$a(n)x(n+m) - b(n-m)x(n) = c(n).$$

Note that $x(n)$ is a polynomial, since $b(n-m) = B(n-m)$ divides $G(n)$ by the properties of the m GP representation applied to equation (2.8). Now if such a polynomial solution $x(n) \in K[n]$ exists, then

$$s_n = \frac{b(n-m)x(n)}{c(n)} a_n \tag{2.10}$$

is the m-hypergeometric solution of (1.3), otherwise no m-hypergeometric solution s_n of (1.3) exists.

□

The following example is a generalization to the example given in [5]:

Example 2.1. *Let m be any positive integer and let $a_n = (n+m)\left(\frac{n+m}{m}\right)!$. Then*

$$w(n) = \frac{a_{n+m}}{a_n} = \frac{n+2m}{m}\frac{n+2m}{n+m}.$$

Hence $a(n) = n+2m, \quad c(n) = n+m.$ The constant polynomial $x(n) = 1$ is a solution for the equation

$$(n+2m)x(n+m) - mx(n) = n+m.$$

Therefore, according to equation (2.10), $s_n = (n+m)\left(\frac{n}{m}\right)!$.

□

3. Generalization of the Algorithm Hyper

Let K_0 be a field of characteristic zero and K an extension field of K_0. In this section we generalize the algorithm **Hyper** to find m-hypergeometric solutions s_n over K for the recurrence (1.5). We assume that there exist algorithms for finding integer roots of polynomials over K and for factoring polynomials over K into irreducible factors over K. Now we consider the second-order recurrence

$$p_2(n)s_{n+2m} + p_1(n)s_{n+m} + p_0(n)s_n = 0. \tag{3.1}$$

Assume that s_n is an m-hypergeometric solution of (3.1). Then there is a rational function $R(n)$ such that $s_{n+m} = R(n)s_n$. Substituting this into (3.1) we obtain

$$p_2(n)R(n+m)R(n) + p_1(n)R(n) + p_0(n) = 0.$$

Write $R(n)$ in m GP representation

$$R(n) = z\frac{a(n)}{b(n)}\frac{c(n+m)}{c(n)}.$$

Then

$$z^2 p_2(n)a(n+m)a(n)c(n+2m) + zp_1(n)b(n+m)a(n)c(n+m) + p_0(n)b(n+m)b(n)c(n) = 0.$$

(3.2)

From this equation we immediately get that $a(n) \mid p_0(n)$ and that $b(n) \mid p_2(n-m)$. We can cancel $a(n)b(n+m)$ from the coefficients of (3.2) to obtain

$$z^2 \frac{p_2(n)}{b(n+m)}a(n+m)c(n+2m) + zp_1(n)c(n+m) + \frac{p_0(n)}{a(n)}b(n)c(n) = 0. \qquad (3.3)$$

To determine the value of z, we consider the leading coefficient of the left-hand side in (3.3) and find z that satisfies a quadratic equation with known coefficients. So given the choice of $a(n)$ and $b(n)$, there are at most two choices for z. When we choose $a(n)$, $b(n)$, and z, we can use the algorithms in [1,2,8] to determine any non-zero polynomial solution $c(n)$ of (3.3). If yes, we have found an m-hypergeometric solution of (3.1). If (3.3) has no non-zero polynomial solution for every choice of $a(n), b(n)$ and z, then (3.1) has no m-hypergeometric solution. □

The above algorithm can be easily generalized to recurrences of arbitrary order

Example 3.1. *Let* $m = 2$ *and let*

$$(n-1)s_{n+4} - (n^2+3n-2)s_{n+2} + 2n(n+1)s_n = 0.$$

Then $p_2(n) = n-1$, $p_1(n) = -(n^2+3n-2)$, $p_0(n) = 2n(n+1)$. *The monic factors of* $p_0(n)$ *are* 1, n, $n+1$ *and* $n(n+1)$ *and those of* $p_2(n-2)$ *are* 1 *and* $n-3$. *Taking* $a(n) = b(n) = 1$ *yields* $-z+2=0$. *The recurrence* (3.3) *is*

$$2(n-1)c(n+4) - (n^2+3n-2)c(n+2) + n(n+1)c(n) = 0,$$

with polynomial solution $c(n) = 1$. *This gives* $R(n) = 2$ *and* $s_n = 2^{n/2}$.

□

4. Generalization of Petkovšek's Approach

In [9], Petkovšek generalized Gosper's algorithm to find hypergeometric solutions for the recurrence (12). In this section we generalize that approach to find m-hypergeometric solutions s_n for the recurrence (1.6). Given an m-hypergeometric term a_n and suppose that there exists an m-hypergeometric solution s_n of equation (1.6). Then the left hand-side of (1.6) can be written as a rational function multiple of s_n. Let $y(n) = s_n / a_n$. Then $y(n)$ is a rational function of n. Substituting $y(n)a_n$ for s_n in (1.6) to obtain

$$\sum_{i=0}^{d} p_i(n)\, y(n+mi) \prod_{j=0}^{i-1} w(n+mj) = 1, \tag{4.1}$$

where $w(n) = a_{n+m} / a_n$ is a rational function of n. Hence the problem of finding m-hypergeometric solutions of (1.6) is reduced to the problem of finding rational solutions of (4.1). Write $w(n)$ in m-Gosper representation

$$w(n)=\frac{a(n)}{b(n)}\frac{c(n+m)}{c(n)}, \tag{4.2}$$

and

$$y(n)=\frac{f(n)}{g(n)c(n)}, \tag{4.3}$$

where $f(n)$ and $g(n)$ are two unknown relatively prime polynomials. Using (4.2) and (4.3) in (4.1) gives

$$\sum_{i=0}^{d} p_i(n)\, f(n+mi)\prod_{\substack{j=0\\ j\neq i}}^{d} g(n+mj)\prod_{j=0}^{i-1} a(n+mj)\ \prod_{\mathrm{j=i}}^{\mathrm{d\text{-}1}} \mathrm{b(n+mj)}$$

$$= c(n)\prod_{j=0}^{d-1} b(n+mj)\prod_{j=0}^{d} g(n+mj), \tag{4.4}$$

All terms in (4.4) except the one with $i=0$ are divisible by $g(n)$, so

$$g(n)\ \Big|\ p_0(n)\prod_{j=1}^{d} g(n+mj)\prod_{j=0}^{d-1} b(n+mj). \tag{4.5}$$

Similarly, looking at the term with $i=d$ and substituting $n-md$ for n, we find that

$$g(n)\ \Big|\ p_d(n-md)\prod_{j=1}^{d} g(n-mj)\prod_{j=1}^{d} a(n-mj). \tag{4.6}$$

Using (4.5) and (4.6), and by the fact that $p_0(n)$ and $p_d(n)$ are constant, one can show by induction that for every $l\in\mathbb{N}$, $g(n)$ divides a product of factors of the form $g(n+mj)$ and $b(n+mi)$ where $mj\geq l$ and $mi\geq 0$, as well as a product of factors of the form $g(n-mj)$ and $a(n-mi)$ where $mj\geq l$ and $mi\geq 1$. Since K has characteristic zero, there is a large enough l such that $g(n)$ is relatively prime with $g(n+mj)$ where $mj\geq l$ and for $mj\leq -l$. From the properties of $a(n)$ and $b(n)$ it follows that $g(n)$ is a constant. Therefore we may write $y(n)=q(n)/c(n)$ where $q(n)$ is a polynomial satisfying

$$\sum_{i=0}^{d} p_i(n)q(n+mi)\prod_{j=0}^{i-1} a(n+mj)\prod_{j=i}^{d-1} b(n+mj) = c(n)\prod_{j=0}^{d-1} b(n+mj) \qquad (4.7)$$

Looking at the term with $i=d$ in (4.7) and substituting $n-md$ for n, we find that $q(n)$ is divisible by $b(n-m)$, so we seek $y(n)$ in the form $y(n) = {b(n-m)x(n)}/{c(n)}$, where $x(n)$ is

a polynomial satisfying

$$\sum_{i=0}^{d} p_i(n)x(n+mi)\prod_{j=0}^{i-1} a(n+mj)\prod_{j=i-1}^{d-2} b(n+mj) = c(n)\prod_{j=0}^{d-2} b(n+mj) \qquad (4.8)$$

Finding m-hypergeometric solutions of (1.6) is therefore equivalent to finding polynomial solutions of (4.8). The relation between them is that if $x(n)$ is a polynomial solution of (4.8) then

$$s_n = \frac{b(n-m)x(n)}{c(n)} a_n$$

is an m-hypergeometric solution of (1.6), and vice versa.

□

Algorithm 4.1.

INPUT : $\{p_i(n)\}_{i=0}^{d} \in K[n]$ *such that* $p_0(n)$ *and* $p_d(n)$ *are constants and* $w(n) \in K(n)$ *such that* ${a_{n+m}}/{a_n} = w(n)$ *for all* $n \in \mathbb{N}$.

OUTPUT: *an* m*-hypergeometric solution* s_n *of* (1.6) *if it exists, otherwise* "*no* m*-hypergeometric solution of* (1.6) *exists*".

(1) *Compute the polynomials* $a(n), b(n)$ *and* $c(n) \in K[n]$ *such that* $w(n) = \frac{a(n)}{b(n)} \frac{c(n+m)}{c(n)}$ *is an* m*-Gosper representation.*

(2) *If equation* (4.8) *can be solved for the polynomial* $x(n)$, *then return* $s_n = \frac{b(n-m)x(n)}{c(n)} a_n$, *otherwise return "no* m*-hypergeometric solution of* (1.6) *exists*". □

Example 4.1. *Let* $m = 2$ *and let*

$$a_n = \frac{3}{4}\binom{2n+8}{n+4}\frac{11n^4 + 78n^3 + 185n^2 + 174n + 52}{(2n+1)(2n+3)(2n+5)(2n+7)}.$$

Then

$$w(n) = \frac{a_{n+2}}{a_n} = \frac{a(n)}{b(n)}\frac{c(n+2)}{c(n)},$$

where

$$a(n) = 4(2n+1)(2n+3),$$
$$b(n) = (n+5)(n+6),$$
$$c(n) = 11n^4 + 78n^3 + 185n^2 + 174n + 52.$$

We want find all 2*-hypergeometric solutions of*

$$s_{n+4} - 8s_{n+2} + 4s_n = a_n. \tag{4.9}$$

By (4.8), $x(n)$ *is a polynomial satisfies*

$$16(2n+1)(2n+3)(2n+5)(2n+7)x(n+4) - 32(2n+1)(2n+3)(n+5)(n+6)x(n+2)$$
$$+ 4(n+3)(n+4)(n+5)(n+6)x(n) = (11n^4 + 78n^3 + 185n^2 + 174n + 52)(n+5)(n+6).$$

Using the algorithm of [1,2,8] *it can be shown that the only polynomial solution of this equation is*

$$x(n) = \frac{1}{12}(n+1)(n+2).$$

Therefore

$$s_n = \frac{b(n-2)x(n)}{c(n)} a_n = \binom{2n}{n}$$

is the only 2 *-hypergeometric solution of* (4.9). □

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الحلول m-الهايبرجيومترية

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الخلاصة

في هذا البحث نقدم اساليب لايجاد الحلول الهايبرجيومترية لمعادلات الفروقات، المعادلات التكرارية الخطية المتجانسة ذات معاملات متعددات حدود والمعادلات التكرارية الخطية الغير متجانسة ذات معاملات متعددات حدود الاول والاخير ثابت. بشرط أن يكون المعامل

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