# Nonpolynomial Spline Method of Singularly Perturbed Boundary Value Problems

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## Abstract

This paper presents the application of nonpolynomial spline method for finding the numerical solution of singularly perturbed boundary value problems. Three numerical examples are considered to demonstrate the usefulness of the method and to show that the method converges with sufficient accuracy to the exact solutions.

**Keywords:** nonpolynomial spline method, ,singularly perturbation, truncation error, exact solution.

# طريقة الشرائح لغير متعددات الحدود لمسائل قيم حدودية ذات الاضطراب المنفرد

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## المستخلص

في هذا البحث نقدم تطبيق طريقة الشرائح(السبلاين) لغير متعددات الحدود على مسائل قيم حدودية ذات الاضطراب المنفرد . ثم قمنا بحل ثلاث امثلة لنرى فائدة الطريقة ومدى تقاربها مع الحل الحقيقي.

## 1. Introduction

Singular perturbation problems containing a small perturbation parameter $\varepsilon$ , arise very frequently in many branches of applied mathematics such as, fluid dynamics, quantum mechanics, chemical reactor theory, elasticity, aerodynamics, and the other domain of the great world of fluid motion [1-3] .

A well known fact is that the solution of such problems has a multiscale character ,i.e. there are thin transition layers where the solution varies very rapidly, while away from the layer the solution behaves regularly and varies slowly. Numerically, the presence of the perturbation parameter leads to difficulties when classical numerical techniques are used to solve such problems, this is due to the presence of the boundary layers in these problems. We consider a second order singularly perturbed boundary problem[4-5]:

$$\varepsilon y'' + p(x)y' + q(x)y = r(x) \ , x \in [a, b] \tag{1}$$

with the boundary conditions

$$y(a) = \alpha \quad and \quad y(b) = \beta \ , \tag{2}$$

where $\epsilon$ is a small positive parameter $0 < \varepsilon \ll 1$, $\alpha$ and $\beta$ are given constants, $p(x)$ , $q(x)$ and $r(x)$ are assumed to be sufficiently continuously differentiable functions.

The nonpolynomial spline method[6-12] developed in this paper has lower computational cost and its only requires solving $n + 1$ linear or non-linear equations.

## 2. Derivation of the Method

We divide the interval $[a, b]$ into $n + 1$ equal subintervals using the point

$$x_i = a + ih, \qquad i = 0,1,2, \dots, n, n + 1,$$

with

$$a = x_0, \ b = x_{n+1} \ and \ \ h = \frac{b - a}{n + 1}$$

Where arbitrary positive integer.

Let $y(x)$ be the exact solution and $y_i$ be an approximation to $y(x_i)$ obtained by the non polynomial $P_i(x)$ passing through the points $(x_i, y_i)$ and $(x_{i+1}, y_{i+1})$, we do not only require that $P_i(x)$ satisfies interpolatory conditions at $x_i$ and $x_{i+1}$but also the continuity of first derivative at the common nodes $(x_i, y_i)$ are fulfilled . We write $P_i(x)$ in the form

[7-8]:

$$P_i(x) = a_i \sin \tau(x - x_i) + b_i \cos \tau(x - x_i) + c_i(x - x_i) + d_i \ , i = 0,1,2, \dots, n + 1 \qquad (3)$$

where $a_i, b_i, c_i$ *and* $d_i$ are constants and $\tau$ is free parameter to be determined later .

A non-polynomial function $P(x)$ of class $C^2[a, b]$ interpolates $y(x)$ at the grid points $x_i, i = 0,1,2, \dots, n + 1$ depends on a parameter $\tau$ , and reduces to ordinary spline $P(x)$ in $[a, b]$ as $\tau \to 0$.

To derive expression for the coefficient of Eq. (3) in term $y_i, y_{i+1}, D_i, D_{i+1}, S_i$ and $S_{i+1}$,we first define:

$$P_i(x_i) = y_i \ , \quad P_i(x_{i+1}) = y_{i+1} \ , \qquad P'_i(x_i) = D_i \ , \qquad P'_i(x_{i+1}) = D_{i+1} \ ,$$
$$P''_i(x_i) = S_i, \ \ P''_i(x_{i+1}) = S_{i+1}. \qquad (4)$$

From algebraic manipulation, we get the following expression:

$$a_i = h^2 \frac{-S_{i+1} + S_i \cos(\theta)}{\theta^2 \sin(\theta)}, \quad b_i = -h^2 \frac{S_i}{\theta^2},$$

$$c_i = \frac{y_{i+1} - y_i}{h} - \frac{h(S_{i+1} - S_i)}{\theta^2}, \qquad d_i = y_i + \frac{h^2 S_i}{\theta^2}, \qquad (5)$$

where $\theta = \tau h$ and $i = 0,1,2, \dots, n$.

We applying the first derivative at $(x_i, y_i)$, that is $P'_{i-1}(x_i) = P'_i(x_i)$ ,gives the following consistency relation for $i = 1, \dots n$ :

$$y_{i-1} - h^2 S_{i-1} \left(\frac{1}{\theta \sin \theta} - \frac{1}{\theta^2}\right) - 2y_i - 2h^2 S_i \left(\frac{1}{\theta^2} - \frac{\cos \theta}{\theta \sin \theta}\right) + y_{i+1} - h^2 S_{i+1} \left(\frac{1}{\theta \sin \theta} - \frac{1}{\theta^2}\right) = 0 \qquad (6)$$

which can further be written as ,

$$y_{i-1} - 2y_i + y_{i+1} = h^2[\alpha(S_{i-1} + S_{i+1}) + 2\beta S_i] \qquad (7)$$

where

$\alpha = (\frac{1}{\theta \sin\theta} - \frac{1}{\theta^2})$ ,

$\beta = (\frac{1}{\theta^2} - \frac{\cos\theta}{\theta \sin\theta})$

## 3. Truncation Error

Now the corresponding truncation error associated with (6)

$$T = y_{i-1} - 2y_i + y_{i+1} - h^2(\alpha y''_{i-1} + 2\beta y''_i + \alpha y''_{i+1})$$

Applying Taylors theorem and simplify we get

$$T = h^2(1 - 2\alpha - 2\beta)y''_i + \frac{h^2}{12}(1 - 12\alpha)y^{iv}_i + \frac{h^6}{360}(1 - 30\alpha)y^{vi}_i + \cdots \qquad , i = 1,2, \ldots n \tag{8}$$

## 4. Non-Polynomial Spline Solutions

In this section, a nonpolynomial spline approximation to equation(1), use $P_i(x)$ and $P_{i-1}(x)$ at the node $x_i$ implies,

$$P_{i-1}(x_i) = \frac{y_i - y_{i-1}}{h} + hS_i\left(\frac{1}{\theta^2} - \frac{\cos\theta}{\theta \sin\theta}\right) + hS_{i-1}\left(\frac{1}{\theta \sin\theta} - \frac{1}{\theta^2}\right) \tag{9}$$

$$P_i(x_i) = \frac{y_{i+1} - y_i}{h} + h\, S_i\left(-\frac{1}{\theta} + \frac{\cos\theta}{\theta \sin\theta}\right) + hS_{i+1}\left(-\frac{1}{\theta \sin\theta} + \frac{1}{\theta^2}\right) \tag{10}$$

where $i = 1,2, \ldots, n$ .

Using equation (9) and (10) in equation (1) we get

$$S_i\left(\varepsilon + h\frac{p_i \cos\theta}{\theta \sin\theta} - h\frac{p_i}{\theta^2}\right) + S_{i+1}h\left(\frac{p_i}{\theta^2} - \frac{p_i}{\theta \sin\theta}\right) = r_i - q_i y_i - \frac{p_i}{h}(y_{i+1} - y_i) \tag{11}$$

$$S_i\left(\varepsilon + h\frac{p_i}{\theta^2} - h\frac{p_i \cos\theta}{\theta \sin\theta}\right) + S_{i-1}h\left(\frac{p_i}{\theta \sin\theta} - \frac{p_i}{\theta^2}\right) = r_i - q_i y_i - \frac{p_i}{h}(y_i - y_{i-1}) \tag{12}$$

For $i = 1,2, \ldots, n$ . Addition of equation (11)and (12)we get the following equation:

$$S_{i-1}h\left(\frac{p_i}{\theta\, sin\,\theta}-\frac{p_i}{\theta^2}\right)+2\varepsilon\, S_i+S_{i+1}h\left(\frac{p_i}{\theta^2}-\frac{p_i}{\theta\, sin\,\theta}\right)=$$
$$2(r_i-g_iy_i)-\frac{p_i}{h}(y_{i+1}-y_{i-1}) \qquad (13)$$

Elimination of $S_{\rm i}$ between equation (13)and equation(6) yields the following equation,

$$\left(hp_i-\frac{\varepsilon\theta^2\, sin\,\theta}{sin\,\theta-\theta\, cos\,\theta}\right)y_{i+1}+\left(-hp_i+\frac{\varepsilon\theta^2\, sin\,\theta}{sin\,\theta-\theta\, cos\,\theta}\right)y_{i-1}+$$
$$\left(2h^2p_i-\frac{2\theta^2\, sin\,\theta}{sin\,\theta-\theta\, cos\,\theta}\right)y_i+S_{i+1}\left(p_ih^3\left(\frac{1}{\theta^2}-\frac{1}{\theta\, sin\,\theta}\right)+\varepsilon h^2\left(\frac{sin\,\theta-\theta}{sin\,\theta-\theta\, cos\,\theta}\right)\right)+$$
$$S_{i-1}\left(p_ih^3\left(\frac{1}{\theta\, sin\,\theta}-\frac{1}{\theta^2}\right)+\varepsilon h^2\left(\frac{sin\,\theta-\theta}{sin\,\theta-\theta\, cos\,\theta}\right)\right)-2h^2r_i=0 \qquad ,\ i=\quad 1,2,\dots n$$
$$(14)$$

An explicit expression can be obtained for $S_{i-1}$ in terms $y_{i-1}$ and $y_i$ by eliminating $S_i$ between equation (11) with $i$ replaced by $i-1$ and (12) manly,

$$A_iS_{i-1}=\left(r_{i-1}-q_{i-1}y_{i-1}-\frac{p_{i-1}}{h}(y_i-y_{i-1})\right)\left(1+p_ih\left(\frac{1}{\theta}-\frac{cos\,\theta}{\theta\, sin\,\theta}\right)\right)-$$
$$\left(r_i-q_iy_i-\frac{p_i}{h}(y_i-y_{i-1})\right)\left(p_{i-1}h\left(\frac{1}{\theta}-\frac{1}{\theta\, sin\,\theta}\right)\right) \qquad (15)$$

where

$$A_i=\left(\varepsilon+p_ih\left(\frac{1}{\theta^2}-\frac{cos\,\theta}{\theta\, sin\,\theta}\right)\right)\left(1+p_{i-1}h\left(\frac{cos\,\theta}{\theta\, sin\,\theta}-\frac{1}{\theta^2}\right)\right)+p_ip_{i-1}h^2\left(\frac{1}{\theta\, sin\,\theta}-\frac{1}{\theta^2}\right)^2$$
$$(16)$$

Similarly $S_{i+1}$ can be obtain in term of $y_{i+1}$ and $y_i$ from equation (11) and equation (12) with replaced by $i+1$ the resulting expression being

$$B_iS_i=\left(r_i-q_{i+1}y_{i+1}-\frac{p_{i+1}}{h}(y_{i+1}-y_i)\right)\left(1+p_ih\left(\frac{cos\,\theta}{\theta\, sin\,\theta}-\frac{1}{\theta}\right)\right)-(r_i-q_iy_i-$$
$$\frac{p_i}{h}(y_{i+1}-y_i)(p_{i+1}h\left(\frac{1}{\theta\, sin\,\theta}-\frac{1}{\theta^2}\right)) \qquad (17)$$

for $i=1,2,\dots,n$ where

$$B_i=\left(\varepsilon+p_{i+1}h\left(\frac{1}{\theta^2}-\frac{\cos\theta}{\theta\sin\theta}\right)\right)\left(1+p_ih\left(\frac{\cos\theta}{\theta\sin\theta}-\frac{1}{\theta^2}\right)\right)+p_ip_{i+1}h^2(\ (\frac{1}{\theta\sin\theta}-\frac{1}{\theta^2})^2 \tag{18}$$

$$y_{i+1}[\left(hp_i+\frac{\varepsilon\theta^2\sin\theta}{(\sin\theta-\theta\cos\theta)}\right)A_iB_i+(C_i\left(-q_{i+1}-\frac{p_{i+1}}{h}\right)+E_i\frac{p_ip_{i+1}}{h})A_i]+$$
$$y_i\left[h^2q_i-\frac{2\varepsilon\theta^2\sin\theta}{(\sin\theta-\theta\cos\theta)}\right)A_iB_i+\left(D_i\frac{-p_{i-1}}{h}+F_ip_{i-1}\left(q_i+\frac{p_i}{h}\right)\right)B_i+\left(C_i\frac{p_{i+1}}{h}+\right.$$
$$\left.E_iq_{i+1}\left(q_i-\frac{p_i}{h}\right)\right)A_i]+y_{i-1}\left[-p_i+\frac{\varepsilon\theta^2\sin\theta}{(\sin\theta-\theta\cos\theta)}\right)A_iB_i+(D_i\left(-q_{i-1}+\frac{p_{i-1}}{h}\right)-$$
$$F_i\frac{p_ip_{i+1}}{h})B_i]+r_{i+1}(A_iC_i)+r_i[-2A_iB_ih^2-B_iF_ip_{i+1}-A_iE_ip_{i+1}]+r_{i-1}B_iD_i=0 \tag{19}$$

Where $A_i$ and $B_i$ are given by equation (16)and (17) and $C_i, D_i, E_i$ and $F_i$ are given the following:

$$C_i=\left[p_ih^3\left(\frac{1}{\theta^2}-\frac{1}{\theta\sin\theta}\right)+\varepsilon h^2\left(\frac{\sin\theta-\theta}{\sin\theta-\theta\cos\theta}\right)\right][1+p_ih\left(\frac{\cos\theta}{\theta\sin\theta}-\frac{1}{\theta^2}\right)]$$

$$D_i=\left[p_ih^3\left(\frac{1}{\theta^2}-\frac{1}{\theta\sin\theta}\right)+\varepsilon h^2\left(\frac{\sin\theta-\theta}{\sin\theta-\theta\cos\theta}\right)\right][1+p_ih\left(\frac{1}{\theta^2}-\frac{\cos\theta}{\theta\sin\theta}\right)]$$

$$E_i=\left[[p_ih^3\left(\frac{1}{\theta^2}-\frac{1}{\theta\sin\theta}\right)+\varepsilon h^2\left(\frac{\sin\theta-\theta}{\sin\theta-\theta\cos\theta}\right)\right][h\left(\frac{1}{\theta\sin\theta}-\frac{1}{\theta^2}\right)]$$

$$F_i=\left[p_ih^3\left(\frac{1}{\theta^2}-\frac{1}{\theta\sin\theta}\right)+\varepsilon h^2\left(\frac{\sin\theta-\theta}{\sin\theta-\theta\cos\theta}\right)\right][h\left(\frac{1}{\theta^2}-\frac{1}{\theta\sin\theta}\right)] \tag{20}$$

for $i=1,2,\dots,n$

Formula (19) is counterpart of formula (6) when the first derivative term is present .

## 5. Numerical Results

We solve three singular perturbed problems using different values of $h$ and $\varepsilon$. The numerical solutions are computed and compared with the exact solutions at grade points. All calculations are implemented by Maple 13.

**Example 1[13]**: Consider the following equation with constant coefficients

$$-\varepsilon y''+y=-((\cos^2(\pi x)+2\varepsilon\pi^2\cos(2\pi x))\ , \qquad 0\le x\le 1$$

$$y(0)=y(1)=0,$$

the exact solution is given by

$$y(x) = \frac{exp(-\frac{1-x}{\sqrt{\varepsilon}} + exp(-x/\sqrt{\varepsilon}}{1 + exp\left(-\frac{1}{\sqrt{\varepsilon}}\right)} - cos^2(\pi x)$$

the numerical result of the example are presented in table 1 and figure 1 for different values subinterval $N$ and $\varepsilon = 1/64$. Figure 2 show the physical behavior of the numerical solutions for different values of $\varepsilon$ .

**Example 2[1]**:Consider the following equation with variable coefficients:

$$\varepsilon y'' + \big(1 + x(1-x)\big)y = 1 + x(1-x) + \left(2\sqrt{\varepsilon} - x^2(1-x)\right)exp\left(-\frac{1-x}{\sqrt{\varepsilon}}\right) + (2\sqrt{\varepsilon} - x(1-x)^2)exp\left(-\frac{x}{\sqrt{\varepsilon}}\right)$$

The exact solution is given by:

$$y(x) = 1 + (x-1)\,exp\left(-\frac{1}{\sqrt{\varepsilon}}\right) - x\,exp(-\frac{1-x}{\sqrt{\varepsilon}})$$

The numerical result of the example are presented in table 2 and figure 3 for different values subinterval $N$ and $\varepsilon = 1/32$. Figure 4 show the physical behavior of the numerical solutions for different values of $\varepsilon$ .

**Example 3[1]:** Consider the following equation with variable coefficients

$$\varepsilon y'' + xy' - y = -(1 + \varepsilon\pi^2)\cos(\pi x) - (\pi x)sin(\pi x)$$

$$y(-1) = -1, \qquad y(1) = 1,$$

The exact solution is given by:

$$y(x) = \cos(\pi x) + x + \frac{x\,erf\left(\frac{x}{\sqrt{2\varepsilon}}\right) + \sqrt{2\varepsilon/\pi}\,exp(-\frac{x^2}{2\varepsilon})}{erf\left(\frac{1}{\sqrt{2\varepsilon}}\right) + \sqrt{2\varepsilon/\pi}\,exp(-\frac{1}{2\varepsilon})}.$$

The numerical result of the example are presented in table 3 and figure 5 for different values subinterval $N$ and $\varepsilon = 1/64$. Figure 6 show the physical behavior of the numerical solutions for different values of $\varepsilon$ .

**Table 1:** Numerical solution of Example 1 at different value of subintervals.

| $x$ | $Numerical\ Sol.$ | | | | Exact Sol. |
|---|---|---|---|---|---|
| | N=16 | N=32 | N=64 | N=128 | |
| 1/16 | -0.1740613181 | -0.1740535992 | -0.1740522915 | -0.1740517442 | -0.1740519068 |
| 2/16 | -0.228908954 | -0.2284334136 | -0.2283165751 | -0.2282870599 | -0.2282776718 |
| 3/16 | -0.1911307967 | -0.1898286086 | -0.1895029921 | -0.1894212971 | -0.1893944408 |
| 4/16 | -0.0929919425 | -0.09063332439 | -0.09004269872 | -0.08989483097 | -0.0898457280 |
| 5/16 | 0.03086117788 | 0.03431676118 | 0.03518261484 | 0.03539918112 | 0.0354714190 |
| 6/16 | 0.1474006708 | 0.1518063028 | 0.1529105397 | 0.1531865969 | 0.1532788852 |
| 7/16 | 0.2293560424 | 0.2344040967 | 0.2356695326 | 0.2359858163 | 0.2360916672 |
| 8/16 | 0.2587638902 | 0.2640387549 | 0.2653611037 | 0.2656915887 | 0.2658022288 |
| 9/16 | 0.2293560426 | 0.2344040966 | 0.2356695325 | 0.2359858164 | 0.2360916672 |
| 10/16 | 0.1474006708 | 0.1518063027 | 0.1529105395 | 0.1531865969 | 0.1532788852 |
| 11/16 | 0.03086117786 | 0.03431676129 | 0.03518261487 | 0.03539918123 | 0.0354714190 |
| 12/16 | -0.09299194243 | -0.09063332432 | -0.09004269853 | -0.08989483104 | -0.0898457280 |
| 13/16 | -0.1911307968 | -0.1898286086 | -0.1895029920 | -0.1894212969 | -0.1893944408 |
| 14/16 | -0.2289028956 | -0.2284334135 | -0.2283165754 | -0.2282870598 | -0.2282776718 |
| 15/16 | -0.1740613181 | -0.1740535993 | -0.1740522913 | -0.1740517437 | -0.1740519068 |

**Table 2:** Numerical solution of Example 2 at different value of subintervals.

| $x$ | $Numerical\ Sol.$ | | | | Exact Sol. |
|---|---|---|---|---|---|
| | N=16 | N=32 | N=64 | N=128 | |
| 1/16 | 0.2694507465 | 0.2686645346 | 0.2684693382 | 0.2684203316 | 0.2684044067 |
| 2/16 | 0.4671405863 | 0.4659161991 | 0.4656121409 | 0.4655357779 | 0.4655109999 |
| 3/16 | 0.6108517734 | 0.6094094686 | 0.6090511884 | 0.6089611707 | 0.6089320119 |
| 4/16 | 0.7136795557 | 0.7121501159 | 0.7117700682 | 0.7116745389 | 0.7116436520 |
| 5/16 | 0.7851097856 | 0.7835628984 | 0.7831783980 | 0.7830816983 | 0.7830504955 |
| 6/16 | 0.8318053349 | 0.8302703669 | 0.8298887324 | 0.8297927183 | 0.8297617754 |
| 7/16 | 0.8581620967 | 0.8566433164 | 0.8562656400 | 0.8561705976 | 0.8561399960 |
| 8/16 | 0.8666777544 | 0.8651658059 | 0.8647898043 | 0.8646951755 | 0.8646647168 |
| 9/16 | 0.8581620967 | 0.8566433163 | 0.8562656399 | 0.8561705979 | 0.8561399960 |
| 10/16 | 0.8318053350 | 0.8302703669 | 0.8298887327 | 0.8297927184 | 0.8297617754 |
| 11/16 | 0.7851097854 | 0.7835628984 | 0.7831783978 | 0.7830816986 | 0.7830504955 |
| 12/16 | 0.7136795558 | 0.7121501157 | 0.7117700683 | 0.7116745385 | 0.7116436520 |
| 13/16 | 0.6108517732 | 0.6094094688 | 0.6090511883 | 0.6089611705 | 0.6089320119 |
| 14/16 | 0.4671405862 | 0.4659161995 | 0.4656121412 | 0.4655357776 | 0.4655109999 |
| 15/16 | 0.2694507467 | 0.2686645349 | 0.2684693378 | 0.2684203316 | 0.2684044067 |

**Table 3:** Numerical solution of Example 3 at different value of subintervals.

| $x$ | $Numerical\ Sol.$ | | | | Exact Sol. |
|---|---|---|---|---|---|
| | **N=64** | **N=128** | **N=512** | **N=1024** | |
| -7/8 | -0.9190258015 | -0.9219199438 | -0.9237709378 | -0.9240415171 | -.9243033330 |
| -6/8 | -0.6984560039 | -0.7035617854 | -0.7068294729 | -0.7073106180 | -.7077785740 |
| -5/8 | -0.3722043319 | -0.3783607642 | -0.3822747958 | --0.3828522232 | -.3834150957 |
| -4/8 | 0.009537748283 | 0.0038557192113 | 0.0003641358348 | -0.0001416429488 | -0.6318936e-3 |
| -3/8 | 0.3881155740 | 0.3845924696 | 0.3827696718 | 0.3825442896 | 0.3823396261 |
| -2/8 | 0.7079360848 | 0.7077704103 | 0.7085372990 | 0.7087543527 | .7090026623 |
| -1/8 | 0.9384379783 | 0.9408096810 | 0.9435055183 | 0.9440534564 | 0.9446355666 |
| 0 | 1.092840722 | 1.095498023 | 1.098469883 | 1.099075370 | 1.099715459 |
| 1/8 | 1.188481264 | 1.190838603 | 1.193508694 | 1.194053107 | 1.194635567 |
| 2/8 | 1.204935832 | 1.206883138 | 1.208472633 | 1.208735746 | 1.209002662 |
| 3/8 | 1.131158356 | 1.132521790 | 1.132618238 | 1.132502994 | 1.132339626 |
| 4/8 | 0.9998408258 | 1.000964472 | 1.000152356 | 0.9998016554 | .9993681064 |
| 5/8 | 0.8674062236 | 0.8685359797 | 0.8674978662 | 0.8670873438 | 0.8665849039 |
| 6/8 | 0.7927591290 | 0.7938125624 | 0.7929781403 | 0.7926384601 | .7922214264 |
| 7/8 | 0.8258581569 | 0.8265503836 | 0.8261169714 | 0.8259288994 | 0.8256966672 |

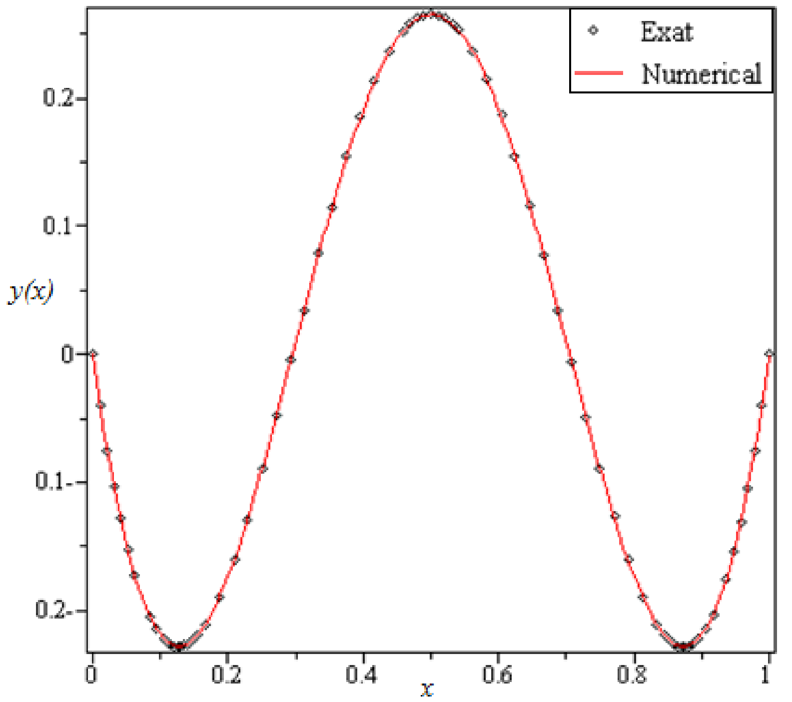


**Figure 1:** Comparison of exact and numerical solutions of Example 1 for $N = \varepsilon = 64$.

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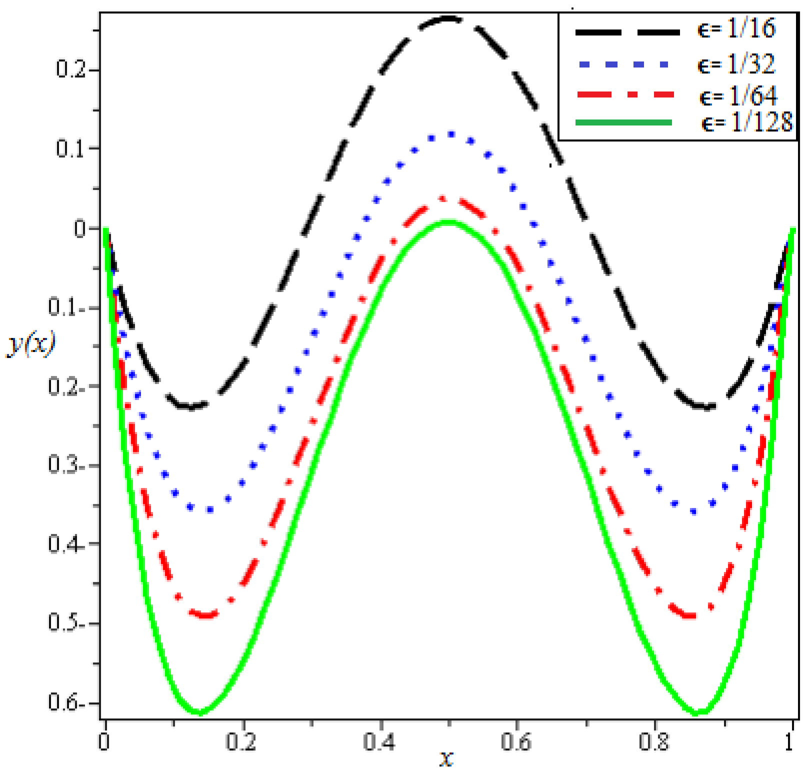


**Figure 2:** Numerical behavior of numerical solutions of Example 1 at different values of $\varepsilon$.

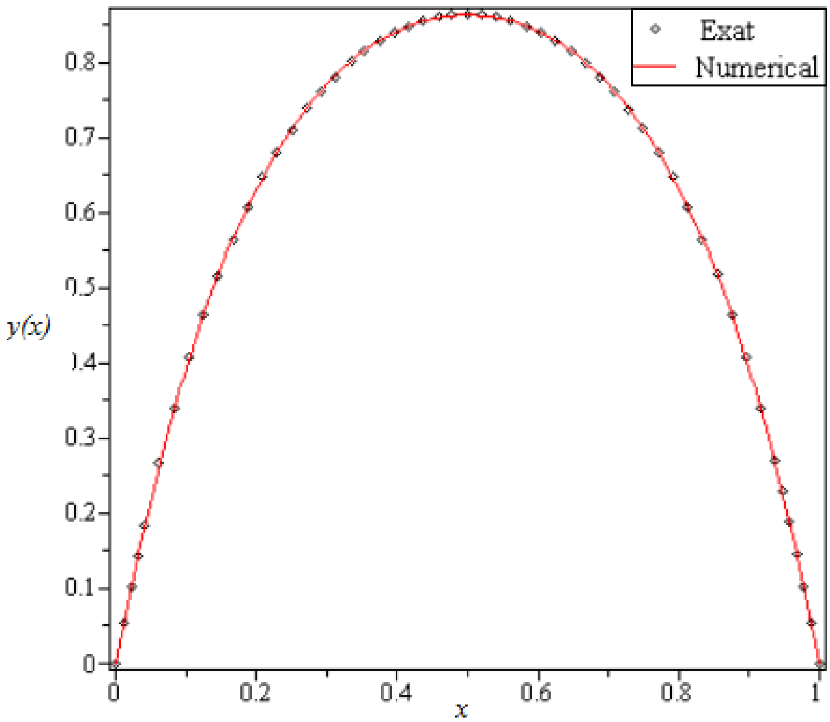


**Figure 3:** Comparison of exact and numerical solutions of example 2 for $N = 32$ $and$ $\varepsilon =$ 64.

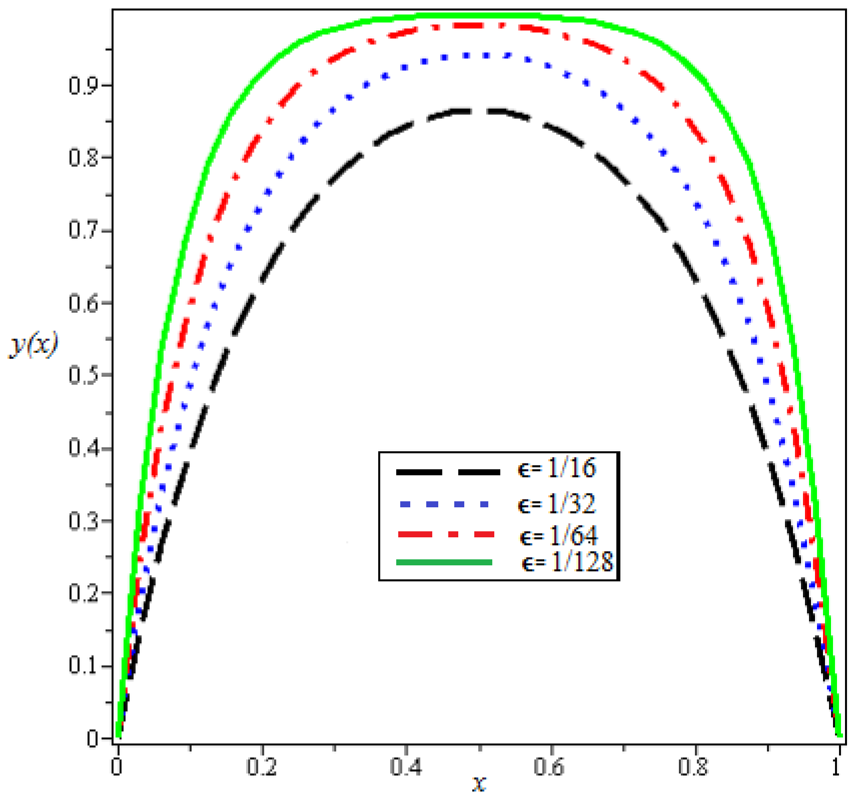

**Figure 4:** Numerical behavior of numerical solutions of Example 2 at different values of $\varepsilon$.

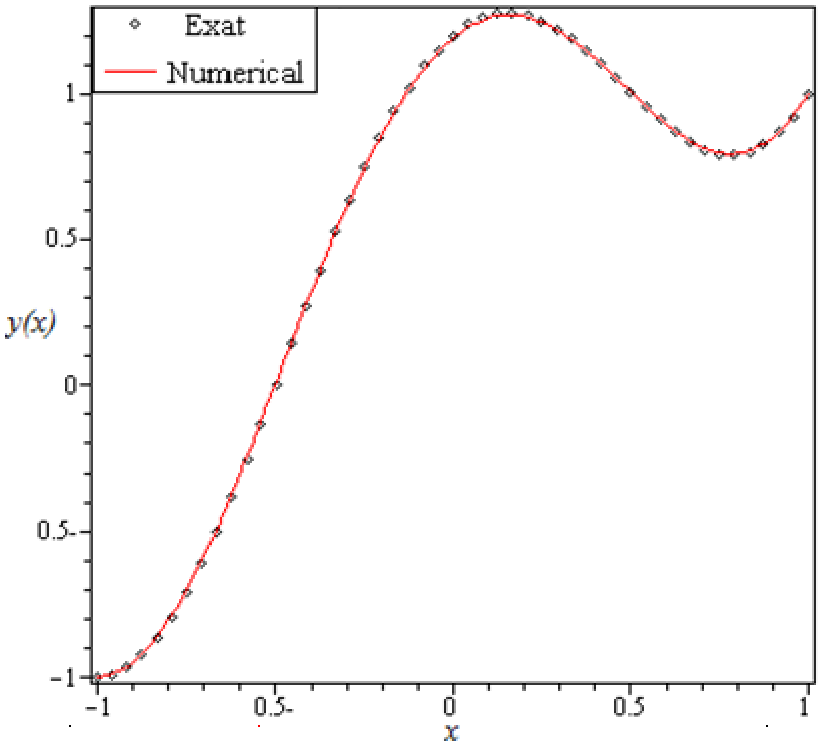


**Figure 5:** Comparison of exact and numerical solutions of example 3 for $N = \varepsilon = 64$.

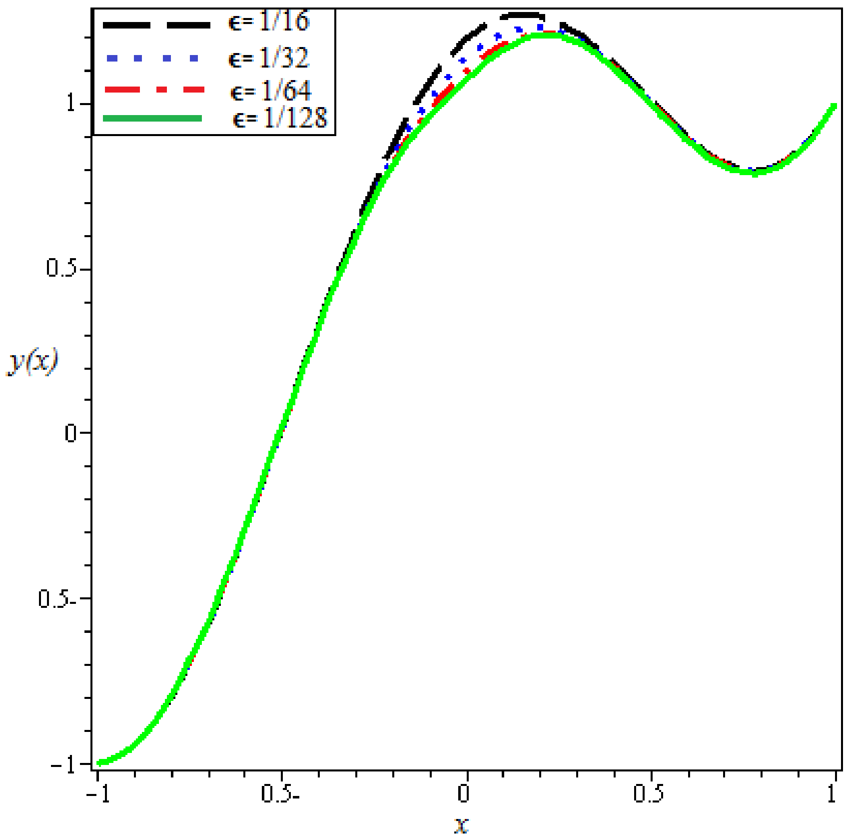


**Figure 6:** Numerical behavior of numerical solutions of Example 3 at different values of $\varepsilon$.

## 6. Conclusion

In this paper , a numerical technique for singularly perturbed boundary value problems using Nonpolynomial Spline functions is derived. Simplicity of the adaptation of Nonpolynomial Spline and obtaining acceptable solutions can be noted as advantages of given numerical methods. The method is tested on three problems and the results obtained are very encouraging. The method is simple and easy to apply.

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