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# On the Cauchy Polynomials

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**Abstract**

We give special values to the parameters in Goldman–Rota $q$-binomial identity and its inverse to get some wellknown identities such as Cauchy identities, Euler identity and Goulden and Jackson identity. We show the equivalence between Goldman–Rota $q$-binomial identity and its inverse. Using the Cauchy operator, we give an operator proof for the Goldman–Rota $q$-binomial identity and the exchange property of the Cauchy polynomials.

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**Keywords:** Cauchy polynomials, Goldman–Rota $q$-binomial identity, the Cauchy operator

## 1. Introduction

In this paper we will follow the standard notations on $q$-series in [3, 6] and we always assume that $|q| < 1$.

The $q$-shifted factorial is defined by:

$$(a;q)_k = \begin{cases} 1, & \text{if } k = 0, \\ (1-a)(1-aq)\cdots(1-aq^{k-1}), & \text{if } k = 1, 2, 3, \cdots. \end{cases}$$

We also define

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

We shall adopt the following notation of multiple $q$-shifted factorials:

$$(a_1, a_2, \cdots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n.$$

The $q$-binomial coefficient is defined by:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & \text{if } 0 \leqslant k \leqslant n, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

When $n \to \infty$ in (1.1), we get

$$\lim_{n\to\infty} \begin{bmatrix} n \\ k \end{bmatrix} = \frac{1}{(q;q)_k}.$$

The inverse pair is given by [8]:

$$\begin{aligned} a_n &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} b_k, \quad \text{for } n \geqslant 0, \\ b_n &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} a_{n-k}, \quad \text{for } n \geqslant 0. \end{aligned}$$

One of the most wellknown identities in $q$-series is Cauchy identity

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}. \tag{1.2}$$

The Cauchy polynomials is defined by

$$\begin{aligned} P_n(x,y) &= (x-y)(x-qy)(x-q^2y)\cdots(x-q^{n-1}y) \\ &= (y/x;q)_n x^n. \end{aligned} \tag{1.3}$$

The homogeneous version of the Cauchy identity (the generating function of $P_n(x,y)$) is given by

$$\sum_{n=0}^{\infty} P_n(x,y) \frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}. \tag{1.4}$$

Setting $y = 0$ in (1.4), we get Euler's identity:

$$\sum_{n=0}^{\infty} \frac{(xt)^n}{(q;q)_n} = \frac{1}{(xt;q)_\infty}, \quad |xt| < 1. \tag{1.5}$$

The Cauchy polynomials $P_n(x,y)$ was studied by Andrews [1, 2], Goldman and Rota [7], Goulden and Jackson [8] and Roman [9].

In 1970, Goldman and Rota [7] have shown the $q$-binomial identity

$$P_n(x,y) \;=\; \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x,z)P_{n-k}(z,y). \tag{1.6}$$

Setting $z=0$ in (1.6), one obtains the following identity:

$$P_n(x,y) \;=\; \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} y^k x^{n-k}. \tag{1.7}$$

Goldman and Rota, by Möbius inversion, obtain the following identity:

$$P_n(x,y) \;=\; \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} P_k(y,1)P_{n-k}(x,q^k). \tag{1.8}$$

In 1983, Goulden and Jackson [8] gave the following exchange property of $P_n(x,y)$:

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x,y)P_{n-k}(w,z) \;=\; \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x,z)P_{n-k}(w,y). \tag{1.9}$$

Setting $w=z$, the exchange property of $P_n(x,y)$ becomes the $q$-binomial identity (1.6). Also, they have found the following basic relations:

$$\begin{aligned} P_n(q^{n-1}y,x) &= (-1)^n q^{\binom{n}{2}} P_n(x,y). \\ P_k(q^{n-1}y,q^{n-1}) &= q^{(n-1)k} P_k(y,1). \\ P_{n-k}(q^{n-1},x) &= (-1)^{n-k} q^{\binom{n}{2}+\binom{k}{2}+k-nk} P_{n-k}(x,q^k). \end{aligned} \tag{1.10}$$

They used these relations to give a derivation of the inverse Goldman-Rota $q$-binomial identity (1.8) from Goldman-Rota $q$-binomial identity (1.6).

In 2003, Chen et al. [5] have found the following relations:

$$P_n(x,y) \;=\; (-1)^n q^{\binom{n}{2}} P_n(y,q^{1-n}x), \tag{1.11}$$

$$P_{n-k}(x,q^{1-n}y) \;=\; (-1)^{n-k} q^{\binom{k}{2}-\binom{n}{2}} P_{n-k}(y,q^k x). \tag{1.12}$$

They used these relations to give a similar derivation of (1.8) from (1.6). Also, they introduced the homogeneous Rogers-Szegö polynomials defined by:

$$h_n(x,y|q) \;=\; \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x,y). \tag{1.13}$$

In 2010, Saad and Sukhi [10] used (1.11) and (1.12) to derive (1.6) from (1.8). Also, they gave the following new formula for the homogeneous Rogers-Szegö polynomials $h_n(x, y|q)$:

$$h_n(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (y; q)_k x^{n-k}. \tag{1.14}$$

This paper is organized as follows. In Section 2, we give special values to the parameters in Goldman–Rota $q$-binomial identity (1.6) and its inverse (1.8) to get some wellknown identities. In Section 3, we show the equivalence between Goldman–Rota $q$-binomial identity (1.6) and its inverse (1.8). Finally, in Section 4, we give an operator proof for the Goldman–Rota $q$-binomial identity (1.6) and the exchange property of the Cauchy polynomials (1.9).

## 2. Special Values

We give special values to the parameters in Goldman–Rota $q$-binomial identity (1.6) and its inverse (1.8) to get some wellknown identities.

- Setting $x \to \frac{1}{x}$, $y \to \frac{1}{y}$ and $z \to \frac{1}{z}$ in (1.6) to obtain

$$(x/y; q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (x/z; q)_{n-k} (z/y; q)_k \left(\frac{x}{z}\right)^k. \tag{2.1}$$

When $n \to \infty$ in (2.1), we get

$$\frac{(x/y; q)_\infty}{(x/z; q)_\infty} = \sum_{k=0}^{\infty} \frac{(z/y; q)_k}{(q; q)_k} \left(\frac{x}{z}\right)^k \tag{2.2}$$

Setting $z/y \to a$ and $x/z \to x$ in (2.2), we get Cauchy identity (1.2).

- Setting $y = 0$ and $x \to 1/x$ in (1.8), we get

$$\begin{aligned} 1 &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k^2-k} (q^k x; q)_{n-k} x^k \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k^2-k} \frac{(x; q)_n}{(x; q)_k} x^k. \end{aligned} \tag{2.3}$$

When $n \to \infty$ in (2.3), we get

$$\sum_{k=0}^{\infty} \frac{q^{k^2-k} x^k}{(q, x; q)_k} = \frac{1}{(x; q)_\infty}. \tag{2.4}$$

Equation (2.4) is due to Cauchy.

- Setting $x = 0$ and then $y \to x$ in (1.8), we get

$$x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x, 1). \tag{2.5}$$

Identity (2.5) is due to Cauchy. Setting $x \to 1/x$ in (2.5), we get

$$1 = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (x; q)_{n-k} x^k. \tag{2.6}$$

When $n \to \infty$ in (2.6), we get Euler identity (1.5).

- Setting $y = 1$ in (1.13) and by using (2.5), we get

$$h_n(x, 1|q) = x^n. \tag{2.7}$$

By the inverse pair on (1.13), we get

$$P_n(x, y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} h_{n-k}(x, y|q). \tag{2.8}$$

Setting $y = 1$ in (2.8) and by using (2.7), we get

$$\prod_{i=0}^{n-1} (x - q^i) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} x^{n-k}. \tag{2.9}$$

Identity (2.9) is given by Goulden and Jackson [8].

# 3. The Goldman-Rota identities

In this section, we show the equivalence between Goldman–Rota $q$-binomial identity (1.6) and its inverse (1.8). We have found the following basic relations for the Cauchy polynomials $P_n(x, y)$ which are easy to verify:

$$P_k(q^{n-1}x, q^{n-1}y) = q^{(n-1)k} P_k(x, y). \tag{3.1}$$

$$P_{n-k}(q^{n-1}y, x) = (-1)^{n-k} q^{\binom{n}{2}+\binom{k}{2}+k-nk} P_{n-k}(x, q^k y). \tag{3.2}$$

From (3.1) and (3.2), we get the following relation:

$$P_{n-k}(1, x) = (-1)^{n-k} q^{-\binom{n}{2}+\binom{k}{2}} P_{n-k}(q^{n-1}x, q^k). \tag{3.3}$$

**Theorem 3.1.** *We have*

$$P_n(xy,1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} P_k(x,1) P_{n-k}(y,1). \tag{3.4}$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} y^{k} P_k(x,1) P_{n-k}(y,1). \tag{3.5}$$

*Proof.* By the homogeneous version of the Cauchy identity (1.4), we find

$$\frac{(t;q)_\infty}{(xyt;q)_\infty} = \frac{(t;q)_\infty}{(xt;q)_\infty}\frac{(xt;q)_\infty}{(xyt;q)_\infty}$$

$$\sum_{n=0}^{\infty} P_n(xy,1)\frac{t^n}{(q;q)_n} = \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} P_k(x,1)P_{n-k}(xy,x)$$

$$= \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} x^{n-k}P_k(x,1)P_{n-k}(y,1).$$

By comparing coefficients of $t^n$ in the above equation, we get (3.4).

Again, by the homogeneous version of the Cauchy identity (1.4), we find

$$\frac{(t;q)_\infty}{(xyt;q)_\infty} = \frac{(t;q)_\infty}{(yt;q)_\infty}\frac{(yt;q)_\infty}{(xyt;q)_\infty}$$

$$\sum_{n=0}^{\infty} P_n(xy,1)\frac{t^n}{(q;q)_n} = \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} P_k(y,1)P_{n-k}(xy,y)$$

$$= \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} y^{k}P_k(x,1)P_{n-k}(y,1).$$

By comparing coefficients of $t^n$ in the above equation, we get (3.5). ∎

Setting $x \to 1/x$ and $y \to 1/y$ in (3.4) and (3.5), we get

$$P_n(1,xy) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} y^{k} P_k(1,x) P_{n-k}(1,y).$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} P_k(1,x) P_{n-k}(1,y). \tag{3.6}$$

**Theorem 3.2.** *The following statements are equivalent:*

1. $P_n(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x,z) P_{n-k}(z,y).$

2. $P_n(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} P_k(y,1) P_{n-k}(x,q^k)$.

3. $P_n(xy,1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} P_k(x,1) P_{n-k}(y,1)$.

4. $P_n(1,xy) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} P_k(1,x) P_{n-k}(1,y)$.

*Proof.* $1 \Longrightarrow 2$

$$\begin{aligned}
P_n(y,x) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(y,1) P_{n-k}(1,x) \quad \text{(by using (1.6))} \\
&= (-1)^n q^{-\binom{n}{2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} P_k(y,1) P_{n-k}(q^{n-1}x, q^k) \quad \text{(by using (3.3))} \\
P_n(q^{n-1}x,y) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} P_k(y,1) P_{n-k}(q^{n-1}x, q^k) \quad \text{(by using (1.10))} \\
P_n(x,y) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} P_k(y,1) P_{n-k}(x, q^k). \quad \text{(by setting } x \to q^{1-n}x).
\end{aligned}$$

$2 \Longrightarrow 3$

Let $x \to 1/x$ in (1.8), we get

$$\begin{aligned}
\frac{P_n(1,xy)}{x^n} &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} P_k(y,1) \frac{P_{n-k}(1,q^k x)}{x^{n-k}}. \\
P_n(q^{n-1}xy,1) &= (-1)^n q^{\binom{n}{2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} x^k P_k(y,1) P_{n-k}(1,q^k x) \quad \text{(by using (1.10))} \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k P_k(y,1) q^{-k+nk} P_{n-k}(q^{n-1}x,1) \quad \text{(by using (3.2))} \\
P_n(xy,1) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k P_k(y,1) P_{n-k}(x,1) \quad \text{(by setting } x \to q^{1-n}x) \\
P_n(xy,1) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} P_k(x,1) P_{n-k}(y,1).
\end{aligned}$$

$3 \Longrightarrow 4$

Let $x \to 1/y$, $y \to 1/x$ in (3.4), we get

$$\begin{aligned}
\frac{P_n(1,xy)}{x^n y^n} &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{y^{n-k}} \frac{P_k(1,y)}{y^k} \frac{P_{n-k}(1,x)}{x^{n-k}} \\
P_n(1,xy) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k P_k(1,y) P_{n-k}(1,x) \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} P_k(1,x) P_{n-k}(1,y).
\end{aligned}$$

$4 \Longrightarrow 1$

Let $x = 1/y$, $y = x$ in (3.6), we get

$$\begin{aligned}
\frac{P_n(y,x)}{y^n} &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{y^{n-k}} \frac{P_k(y,1)}{y^k} P_{n-k}(1,x) \\
P_n(y,x) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(y,1) P_{n-k}(1,x) \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(y,1)(-1)^{n-k} q^{\binom{k}{2}-\binom{n}{2}} P_{n-k}(q^{n-1}x, q^k) \quad \text{(by using (3.3))} \\
P_n(y, q^{1-n}x) &= (-1)^n q^{-\binom{n}{2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} P_k(y,1) P_{n-k}(x, q^k) \quad \text{(by setting } x \to q^{1-n}x) \\
P_n(x,y) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} P_k(y,1) P_{n-k}(x, q^k). \quad \text{(by using (1.11))}
\end{aligned}$$

∎

Theorem 3.2 shows the equivalence between Goldman-Rota $q$-binomial identity (1.6) and its inverse (1.8).

# 4. The Cauchy operator and the Cauchy polynomials

In this section, we use Cauchy operator to give an operator proof for the Goldman–Rota $q$-binomial identity (1.6) and the exchange property of the Cauchy polynomials (1.9).

The $q$-differential operator, or $q$-derivative, $D_q$ is defined by:

$$D_q\{f(a)\} = \frac{f(a) - f(aq)}{a}.$$

In 2008, Chen and Gu [4] defined the Cauchy operator as follows:

$$\mathbb{T}(a,b;D_q) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}(bD_q)^n.$$

The following lemma is easy to verify.

**Lemma 4.1.** *We have*

$$\begin{aligned}
D_q^k\{x^n\} &= \frac{(q;q)_n}{(q;q)_{n-k}}x^{n-k}.\\
D_q^k\{P_n(x,y)\} &= \frac{(q;q)_n}{(q;q)_{n-k}}P_{n-k}(x,y).\\
\mathbb{T}(a,b,D_q)\{x^n\} &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(a;q)_k b^k x^{n-k}. && (4.1)\\
\mathbb{T}(a,b,D_q)\{P_n(x,y)\} &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(a;q)_k b^k P_{n-k}(x,y). && (4.2)\\
\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}n-k\\i\end{bmatrix} &= \begin{bmatrix}n\\i\end{bmatrix}\begin{bmatrix}n-i\\k\end{bmatrix}. && (4.3)
\end{aligned}$$

**Corollary 4.1.1.** *We have*

$$\mathbb{T}(y/z,z,D_q)\{x^n\} = z^n h_n(\frac{x}{z},\frac{y}{z}|q). \tag{4.4}$$

*Proof.* by using (4.1), we get

$$\begin{aligned}
\mathbb{T}(y/z,z,D_q)\{x^n\} &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(y/z;q)_k z^k x^{n-k}\\
&= z^n\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(y/z;q)_k\left(\frac{x}{z}\right)^{n-k}\\
&= z^n h_n(\frac{x}{z},\frac{y}{z}|q). \quad \text{(by using (1.14))}
\end{aligned}$$

∎

Now we are ready to give an operator proof for the Goldman–Rota $q$-binomial identity (1.6) and the exchange property of $P_n(x,y)$ (1.9).

*Proof of* (1.6). By using (4.2), we get

$$\begin{aligned}\mathbb{T}(y/z,z,D_q)\left\{P_n(x,z)\right\} &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(y/z;q)_k z^k P_{n-k}(x,z)\\ &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}P_k(z,y)P_{n-k}(x,z) \quad \text{(by using (1.3))}\\ &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}P_k(x,z)P_{n-k}(z,y). \qquad (4.5)\end{aligned}$$

$$\begin{aligned}\mathbb{T}(y/z,z,D_q)\left\{P_n(x,z)\right\} &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}} z^k \mathbb{T}(y/z,z,D_q)\left\{x^{n-k}\right\} \quad \text{(by using (1.7))}\\ &= z^n \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}} h_{n-k}(\frac{x}{z},\frac{y}{z}|q) \quad \text{(by using (4.4))}\\ &= z^n P_n(\frac{x}{z},\frac{y}{z}) \quad \text{(by using (2.8))}\\ &= P_n(x,y). \qquad (4.6)\end{aligned}$$

From (4.5) and (4.6), we get (1.6). ∎

*Proof of* (1.9). By using (4.2), we get

$$\begin{aligned}\mathbb{T}(z/w,w,D_q)\left\{P_n(x,y)\right\} &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(z/w;q)_k w^k P_{n-k}(x,y)\\ &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}P_k(w,z)P_{n-k}(x,y) \quad \text{(by using (1.3))}\\ &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}P_k(x,y)P_{n-k}(w,z). \qquad (4.7)\end{aligned}$$

$$
\begin{aligned}
\mathbb{T}(z/w, w, D_q)\left\{P_n(x,y)\right\} &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} y^k \mathbb{T}(z/w, w, D_q)\left\{x^{n-k}\right\} \quad \text{(by using (1.7))} \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} y^k w^{n-k} h_{n-k}(\frac{x}{w}, \frac{z}{w}|q) \quad \text{(by using (4.4))} \\
&= \sum_{k=0}^{n} \sum_{i=0}^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n-k \\ i \end{bmatrix} (-1)^k q^{\binom{k}{2}} y^k w^{n-i-k} P_i(x,z) \\
&= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} P_i(x,z) \sum_{k=0}^{n-i} \begin{bmatrix} n-i \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} y^k w^{n-i-k} \quad \text{(by using (4.3))} \\
&= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} P_i(x,z) P_{n-i}(w,y). \quad \text{(by using (1.7))} \qquad (4.8)
\end{aligned}
$$

From (4.7) and (4.8), we get (1.9). ∎

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