

The Operator ${}_r\Phi_s$ and the Polynomials K_n

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Abstract

Based on basic hypergeometric series, a new generalized q -operator ${}_r\Phi_s$ has been constructed and obtained some operator identities. Also, a new polynomial $K_n(a_1, \dots, a_r, b_1, \dots, b_s, c; a; q)$ is introduced. The generating function and its extension, Mehler's formula and its extension and the Rogers formula for the polynomials $K_n(a_1, \dots, a_r, b_1, \dots, b_s, c; a; q)$ have been achieved by using the operator ${}_r\Phi_s$. In fact, this work can be considered as a generalization of Liu work's by imposing some special values of the parameters in our results. Therefore the q^{-1} -Rogers-Szegö polynomials $h_n(a, b|q^{-1})$ can be deduced directly.

Keywords: q -operator, generating function, Mehler's formula, Rogers formula, the q^{-1} -Rogers-Szegö polynomials.

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1. Introduction

Through this paper, the notations in [2] will be used here and assuming that $|q| < 1$.

Definition 1.1. [2]. Let a be a complex variable. The q -shifted factorial is defined by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

The compact notation for the multiple q -shifted factorial will be adopted here

$$(a_1, \dots, a_r; q)_n = (a_1; q)_n \dots (a_r; q)_n,$$

where n is an integer or ∞ .

Definition 1.2. [2]. The basic hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, x) = {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, x \right)$$

$$= \sum_{k=0}^{\infty} \frac{(a_1; q)_k (a_2; q)_k \cdots (a_r; q)_k}{(q; q)_k (b_1; q)_k \cdots (b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} x^k,$$

where $r, s \in \mathbb{N}$; $a_1, \dots, a_r, b_1, \dots, b_s \in \mathbb{C}$; and none of the denominator factors evaluate to zero. The above series is absolutely convergent for all $x \in \mathbb{C}$ if $r < s + 1$, for $|x| < 1$ if $r = s + 1$ and for $x = 0$ if $r > s + 1$.

Definition 1.3. [2] . The q -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, & \text{if } 0 < k < n; \\ 0, & \text{otherwise,} \end{cases} \quad (1.1)$$

where $n, k \in \mathbb{N}$.

The following equations will be used in this paper [2]:

$$(a; q)_n = \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}}. \quad (1.2)$$

$$(q/a; q)_k = (-a)^{-k} q^{\binom{k+1}{2}} (aq^{-k}; q)_{\infty} / (a; q)_{\infty}. \quad (1.3)$$

$$\binom{n-k}{2} = \binom{n}{2} + \binom{k}{2} + k - kn, \quad (1.4)$$

$$\binom{n+k}{2} = \binom{n}{2} + \binom{k}{2} + kn, \quad (1.5)$$

where n and k are integers. Cauchy identity is given by [2]

$$\sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} x^k = \frac{(ax; q)_{\infty}}{(x; q)_{\infty}}, \quad |x| < 1. \quad (1.6)$$

The special case of Cauchy identity was founded by Euler [2] which is

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q; q)_k} x^k = (x; q)_{\infty}. \quad (1.7)$$

Definition 1.4. [3] . The operator θ is defined by

$$\theta\{f(a)\} = \frac{f(aq^{-1}) - f(a)}{aq^{-1}}. \quad (1.8)$$

Theorem 1.5. [3]. (Leibniz rule for θ). Let θ be defined as in (1.8), then

$$\theta^n\{f(a)g(a)\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \theta^k\{f(a)\}\theta^{n-k}\{g(aq^{-k})\}. \quad (1.9)$$

The following identities are easy to prove:

Theorem 1.6. [4, 5, 6] . Let θ be defined as in (1.8), then

$$\theta^k\{a^n\} = \frac{(q; q)_n}{(q; q)_{n-k}} a^{n-k} q^{\binom{k}{2} + k(1+n)}. \quad (1.10)$$

$$\theta^k\{(at; q)_\infty\} = (-t)^k (at; q)_\infty. \quad (1.11)$$

$$\theta^k \left\{ \frac{(at; q)_\infty}{(av; q)_\infty} \right\} = v^k q^{-\binom{k}{2}} (t/v; q)_k \frac{(at; q)_\infty}{(av/q^k; q)_\infty}, \quad |av| < 1. \quad (1.12)$$

In 1998, Chen and Liu [4] defined the q -exponential operator $E(b\theta)$ as follows:

Definition 1.7. [4] . The q -exponential operator $E(b\theta)$ is defined as follows:

$$E(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n q^{\binom{n}{2}}}{(q; q)_n}. \quad (1.13)$$

Chen and Liu proved the following result:

Theorem 1.8. [4] . Let $E(b\theta)$ be defined as in (1.13), then

$$E(b\theta)\{(at; q)_\infty\} = (at, btq)_\infty. \quad (1.14)$$

$$E(b\theta)\{(as, at; q)_\infty\} = \frac{(as, at, bs, btq)_\infty}{(abst/q; q)_\infty}, \quad |abst| < 1. \quad (1.15)$$

They used the q -exponential operator $E(b\theta)$ to present an extension for the Askey beta integral.

In 2006, Zhang and Liu [6] used $E(d\theta)$ to prove the following result:

Theorem 1.9. [6] . Let $E(d\theta)$ be defined as in (1.13), then

$$E(d\theta)\{a^n(as; q)_\infty\} = a^n(as; q)_\infty {}_2\phi_1 \left(\begin{matrix} q^{-n}, q/as \\ 0 \end{matrix}; q, ds \right), \quad |ds| < 1. \quad (1.16)$$

In 2007, Fang [7] defined the Cauchy operator ${}_1\Phi_0\left(\begin{smallmatrix} b \\ - \end{smallmatrix}; q, -c\theta\right)$ as follows:

Definition 1.10. [7] . The Cauchy operator ${}_1\Phi_0\left(\begin{smallmatrix} b \\ - \end{smallmatrix}; q, -c\theta\right)$ is defined by

$${}_1\Phi_0\left(\begin{smallmatrix} b \\ - \end{smallmatrix}; q, -c\theta\right) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} (-c\theta)^n. \quad (1.17)$$

Fang proved the following result:

Theorem 1.11. [7] . Let ${}_1\Phi_0\left(\begin{smallmatrix} b \\ - \end{smallmatrix}; q, -c\theta\right)$ be defined as in (1.17), then

$${}_1\Phi_0\left(\begin{smallmatrix} b \\ - \end{smallmatrix}; q, -c\theta\right) \{(as; q)_{\infty}\} = \frac{(bcs, as; q)_{\infty}}{(cs; q)_{\infty}}, \quad |cs| < 1. \quad (1.18)$$

Fang used Cauchy operator ${}_1\Phi_0\left(\begin{smallmatrix} b \\ - \end{smallmatrix}; q, -c\theta\right)$ to obtain an extension for the q -Chu-Vandermonde identity.

In 2010, Zhang and Yang [8] introduced the finite q -exponential operator with two parameters ${}_2\mathcal{E}_1\left[\begin{smallmatrix} q^{-N}, v \\ w \end{smallmatrix}; q, c\theta\right]$ as follows:

Definition 1.10. [8] . The finite q -exponential operator ${}_2\mathcal{E}_1\left[\begin{smallmatrix} q^{-N}, v \\ w \end{smallmatrix}; q, c\theta\right]$ is defined by

$${}_2\mathcal{E}_1\left[\begin{smallmatrix} q^{-N}, v \\ w \end{smallmatrix}; q, c\theta\right] = \sum_{n=0}^{\infty} \frac{(q^{-N}, v; q)_n}{(q, w; q)_n} (c\theta)^n.$$

By using this operator, , Zhang and Yang found an extension for q -Chu-Vandermonde summation formula.

In 2010, Liu [1] defined the q^{-1} -Rogers-Szegö polynomial as follows:

Definition 1.12. [1] . The q^{-1} -Rogers-Szegö polynomial $h_n(a, b|q^{-1})$ is defined by

$$h_n(a, b|q^{-1}) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k^2-nk} a^k b^{n-k}. \quad (1.19)$$

Liu used the q -difference equation to prove the following:

Theorem 1.13. [1] . Let $h_n(a, b|q^{-1})$ be defined as in (1.19), then

- The generating function for $h_n(a, b|q^{-1})$

$$\sum_{n=0}^{\infty} h_n(a, b|q^{-1}) \frac{(-t)^n q^{\binom{n}{2}}}{(q; q)_n} = (at, bt; q)_{\infty}. \quad (1.20)$$

- Mehler's formula for $h_n(a, b|q^{-1})$

$$\sum_{n=0}^{\infty} h_n(a, b|q^{-1}) h_n(c, d|q^{-1}) \frac{(-t)^n q^{\binom{n}{2}}}{(q; q)_n} = \frac{(act, adt, bct, bdt; q)_{\infty}}{(abcdt^2/q; q)_{\infty}}, \quad (1.21)$$

provided that $|abcdt^2/q| < 1$.

This paper is organized as follows: In section 2, a generalized q -operator ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, -c\theta \right)$ and some of its identities will be defined and studied. In section 3, we define a polynomial $K_n(a_1, \dots, a_r, b_1, \dots, b_s, c; a; q)$ and represent it by the operator ${}_r\Phi_s$. The generating function and its extension for $K_n(a_1, \dots, a_r, b_1, \dots, b_s, c; a; q)$ is obtained. In section 4, the Mehler's formula and its extension for $K_n(a_1, \dots, a_r, b_1, \dots, b_s, c; a; q)$ is derived. while, in section 5, the Rogers formula for $K_n(a_1, \dots, a_r, b_1, \dots, b_s, c; a; q)$ is constructed. Finally, section 6 is focused on the summary of the results and the conclusions.

2. The Operator ${}_r\Phi_s$ and it's Identities

In this section, we define the generalized q -operator ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, -c\theta \right)$ as follows:

Definition 2.1. The generalized q -operator ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, -c\theta \right)$ is defined by

$${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, -c\theta \right) = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(b_1, \dots, b_s; q)_k} \frac{(-c\theta)^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r}. \quad (2.1)$$

When $r = s = 0$, we get the q -exponential operator $E(c\theta)$ defined by Chen and Liu [4] in 1998. Also when $r = 1, s = 0$, $a_1 = b$, we obtain the q -exponential operator ${}_1\Phi_0 \left(\begin{matrix} b \\ - \end{matrix}; q, -c\theta \right)$ defined by Fang [7] in 2007. And when $r = 2, s = 1$, $a_1 = q^{-N}$, $a_2 = v$, $b_1 = w$ we obtain the

finite q -exponential operator with two parameters ${}_2\mathcal{E}_1 \left[\begin{matrix} q^{-N}, v \\ w \end{matrix} ; q, c\theta \right]$ defined by Zhang and Yang [8] in 2010. Finally, when $r = 2, s = 1$, $a_1 = u$, $a_2 = v$, $b_1 = w$, we get the generalized q -exponential operator with three parameters $\mathbb{E} \left[\begin{matrix} u, v \\ w \end{matrix} | q; c\theta \right]$ defined by Li and Tan [9] in 2016.

In this paper, we will denote to $\frac{(a_1, \dots, a_r; q)_k}{(b_1, \dots, b_s; q)_k}$ by W_k . Then the generalized q -operator ${}_r\Phi_s$ can be written as follows:

$${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, -c\theta \right) = \sum_{k=0}^{\infty} W_k \frac{(-c\theta)^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r}. \quad (2.2)$$

Theorem 2.2. Let ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, -c\theta \right)$ be defined as in (2.2), then

$$\begin{aligned} {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, -c\theta \right) \{(au, at; q)_{\infty}\} &= (au, at; q)_{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} W_{k+j} \frac{(ct)^k}{(q; q)_k} \\ &\times \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \frac{(q/at; q)_j}{(q; q)_j} (actu/q)^j \left[(-1)^j q^{\binom{j}{2}} \right]^{s-r} q^{kj(s-r)}. \end{aligned} \quad (2.3)$$

Proof. From the definition of the operator ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, -c\theta \right)$ and by using Leibniz rule (1.9), we have

$$\begin{aligned} &{}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, -c\theta \right) \{(au, at; q)_{\infty}\} \\ &= \sum_{k=0}^{\infty} W_k \frac{(-c)^k}{(q, q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \theta^k \{(au, at; q)_{\infty}\} \\ &= \sum_{k=0}^{\infty} W_k \frac{(-c)^k}{(q, q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \sum_{j=0}^k kj\theta^j \{(au; q)_{\infty}\} \theta^{k-j} \{(atq^{-j}; q)_{\infty}\} \\ &= \sum_{k=0}^{\infty} W_k \frac{(-c)^k}{(q, q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \sum_{j=0}^k \frac{(q, q)_k}{(q, q)_j (q, q)_{k-j}} (-u)^j (au; q)_{\infty} \\ &\quad \times (-tq^{-j})^{k-j} (atq^{-j}; q)_{\infty} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=0}^{\infty} W_k \frac{(-c)^k}{(q, q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \sum_{j=0}^k \frac{(q, q)_k}{(q, q)_j (q, q)_{k-j}} (-u)^j (au; q)_{\infty} \\
 &\quad \times (-t)^{k-j} q^{-kj+j^2} (-at)^j q^{-j^2-j} (q/at; q)_j (at; q)_{\infty} \quad (\text{by using (1.3)}) \\
 &= (at, au; q)_{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} W_{k+j} \frac{(-c)^{k+j}}{(q, q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \left[(-1)^j q^{\binom{j}{2}} \right]^{1+s-r} q^{kj(1+s-r)} \\
 &\quad \times (-u)^j (-t)^k q^{-kj-j^2+j^2} (-at)^j q^{-j^2-j} \frac{(q/at; q)_j}{(q, q)_j} \quad (\text{by using (1.5)}) \\
 &= (au, at; q)_{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} W_{k+j} \frac{(ct)^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \frac{(q/at; q)_j}{(q; q)_j} (actu/q)^j \left[(-1)^j q^{\binom{j}{2}} \right]^{s-r} \\
 &\quad \times q^{kj(s-r)}.
 \end{aligned}$$

By setting $r = s = 0$ in (2.3), we get Theorem 2.11. obtained in Chen and Liu [4] (equation (1.15)).

Putting $u = 0$ in (2.3), we get the following corollary:

Corollary 1. Let ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, -c\theta \right)$ be defined as in (2.2), then

$${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, -c\theta \right) \{(at; q)_{\infty}\} = (at; q)_{\infty} \sum_{k=0}^{\infty} W_k \frac{(ct)^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r}. \quad (2.4)$$

Setting $r = s = 0$ in (2.4), we get Theorem 2.9. obtained by Chen and Liu [4] (equation (1.14)). Setting $r = 1$ and $s = 0$ in (2.4), we get Theorem 1.3. obtained by Fang [7] (equation (1.18)).

Theorem 2.3. Let ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, -c\theta \right)$ be defined as in (2.2) and $n \in \mathbb{Z}^+$, then

$${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, -c\theta \right) \{a^n(at, q)_{\infty}\} = a^n(at, q)_{\infty} \sum_{j=0}^n \sum_{k=0}^{\infty} W_{k+j} \frac{(ct)^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r}$$

$$\times \frac{(q^{-n}, q/at; q)_j}{(q; q)_j} (ct)^j \left[(-1)^j q^{\binom{j}{2}} \right]^{s-r} q^{kj(s-r)}. \quad (2.5)$$

Proof. From (2.2), we have

$${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s; q, -c\theta \end{matrix} \right) \{a^n(at, q)_\infty\} = \sum_{k=0}^{\infty} W_k \frac{(-c)^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \theta^k \{a^n(at, q)_\infty\}.$$

By using Leibniz rule (1.9), we have

$$\begin{aligned} & {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s; q, -c\theta \end{matrix} \right) \{a^n(at, q)_\infty\} \\ &= \sum_{k=0}^{\infty} W_k \frac{(-c)^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \sum_{j=0}^k kj \theta^j \{a^n\} \theta^{k-j} \{(atq^{-j}; q)_\infty\} \\ &= \sum_{k=0}^{\infty} W_k \frac{(-c)^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \sum_{j=0}^k \frac{(q; q)_k}{(q; q)_j (q; q)_{k-j}} (-1)^j a^{n-j} q^j (q^{-n}; q)_j \\ &\quad \times \theta^{k-j} \{(atq^{-j}; q)_\infty\} \quad (\text{by using (1.1) and (1.10)}) \\ &= \sum_{k=0}^{\infty} W_k (-c)^k \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \sum_{j=0}^k \frac{1}{(q; q)_j (q; q)_{k-j}} (-1)^j a^{n-j} q^j (q^{-n}; q)_j (-tq^{-j})^{k-j} \\ &\quad \times (atq^{-j}; q)_\infty \quad (\text{by using (1.11)}) \\ &= \sum_{j=0}^n \sum_{k=0}^{\infty} W_{k+j} (-c)^{k+j} \left[(-1)^{k+j} q^{\binom{k+j}{2}} \right]^{1+s-r} \frac{1}{(q; q)_j (q; q)_k} (-1)^j a^{n-j} q^j (q^{-n}; q)_j (-tq^{-j})^k \\ &\quad \times (-at)^j q^{-j} q^{-j} (q/at; q)_j (at; q)_\infty \quad (\text{by using (1.3)}) \\ &= a^n(at, q)_\infty \sum_{j=0}^n \sum_{k=0}^{\infty} W_{k+j} \frac{(ct)^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \frac{(q^{-n}, q/at; q)_j}{(q; q)_j} (ct)^j \left[(-1)^j q^{\binom{j}{2}} \right]^{s-r} \\ &\quad \times q^{kj(s-r)}. \quad (\text{by using (1.5)}) \end{aligned}$$

■

Setting $r = s = 0$ in (2.5), we get Corollary 2.4. obtained in Zhang and Liu [6] (equation(1.16)).

3. The Generating Function for $K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q)$

In this section we define a polynomial $K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q)$. By using the operator

${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, -c\theta \right)$, we get the generating function and its extension for the polynomials K_n .

We give some special values to the parameters in the generating function and its extension for $K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q)$ to obtain the generating function and its extension for the q^{-1} -Rogers-Szegö polynomials $h_n(a, b|q^{-1})$.

Definition 3.1. The polynomial $K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q)$ is defined by

$$K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} W_k c^k \left[(-1)^k q^{\binom{k}{2}} \right]^{2+s-r} q^{k(1-n)} a^{n-k}, \quad (3.1)$$

where $W_k = \frac{(a_1, \dots, a_r; q)_k}{(b_1, \dots, b_s; q)_k}$.

Setting $r = s = 0$, $a = b$, $c = a$ in (3.1), we get the q^{-1} -Rogers-Szegö polynomials $h_n(a, b|q^{-1})$ (2.12) defined by Liu [1] (equation (1.19)).

Theorem 3.2. Let the polynomials $K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q)$ be defined as in (3.1), then

$${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, -c\theta \right) \{a^n\} = K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q). \quad (3.2)$$

Proof.

$$\begin{aligned} & {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, -c\theta \right) \{a^n\} \\ &= \sum_{k=0}^{\infty} W_k \frac{(-c\theta)^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \{a^n\} \\ &= \sum_{k=0}^{\infty} W_k \frac{(-c)^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \theta^k \{a^n\} \\ &= \sum_{k=0}^{\infty} W_k \frac{(-c)^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \frac{(q; q)_n}{(q; q)_{n-k}} a^{n-k} q^{\binom{k}{2} - nk + k} \quad (\text{by using (1.10)}) \\ &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} W_k c^k \left[(-1)^k q^{\binom{k}{2}} \right]^{2+s-r} q^{k(1-n)} a^{n-k} \\ &= K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q). \quad \blacksquare \end{aligned}$$

Theorem 3.3. (The generating function for K_n). Let $K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q)$ be defined as in (3.2), then

$$\sum_{n=0}^{\infty} K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q) \frac{(-u)^n q^{\binom{n}{2}}}{(q; q)_n} = (au; q) {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cu \right), \quad (3.3)$$

provided that the series is absolutely convergent $\forall cu \in \mathbb{C}$ if $s > r - 1$, $cu = 0$ if $s < r - 1$ and $|cu| < 1$ if $s = r - 1$.

Proof.

$$\begin{aligned} & \sum_{n=0}^{\infty} K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q) \frac{(-u)^n q^{\binom{n}{2}}}{(q; q)_n} \\ &= \sum_{n=0}^{\infty} {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, -c\theta \right) \{a^n\} \frac{(-u)^n q^{\binom{n}{2}}}{(q; q)_n} \quad (\text{by using (3.2)}) \\ &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, -c\theta \right) \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q; q)_n} (au)^n \right\} \\ &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, -c\theta \right) \{(au; q)_{\infty}\} \quad (\text{by using (1.7)}) \\ &= (au; q)_{\infty} \sum_{k=0}^{\infty} W_k \frac{(cu)^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \quad (\text{by using (2.4)}) \\ &= (au; q) {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cu \right). \end{aligned}$$

■

Setting $r = s = 0$, $a = b$, $c = a$ in (3.3) we obtain the generating function for the polynomials $h_n(a, b|q^{-1})$ (2.13) obtained by Liu [1] (equation (1.20)).

Theorem 3.4. (Extension of the generating function for K_n).

Let $K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q)$ be defined as in (3.2), then

$$\begin{aligned} & \sum_{n=0}^{\infty} K_{n+l}(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q) \frac{(-u)^n q^{\binom{n}{2}}}{(q; q)_n} = a^l (au; q)_{\infty} \\ & \times \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(q^{-l}, q/au; q)_j}{(q; q)_j} \left[(-1)^j q^{\binom{j}{2}} \right]^{s-r} (cu)^j W_{i+j} \frac{(cu)^i}{(q; q)_i} \left[(-1)^i q^{\binom{i}{2}} \right]^{1+s-r} q^{ij(s-r)}. \quad (3.4) \end{aligned}$$

Proof.

$$\sum_{n=0}^{\infty} K_{n+l}(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q) \frac{(-u)^n q^{\binom{n}{2}}}{(q; q)_n}$$

$$\begin{aligned}
 &= \sum_{n=0}^{\infty} {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, -c\theta \right) \{a^{l+n}\} \frac{(-u)^n q^{\binom{n}{2}}}{(q; q)_n} && \text{(by using (3.2))} \\
 &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, -c\theta \right) \left\{ a^l \sum_{n=0}^{\infty} \frac{(-1)^n (au)^n q^{\binom{n}{2}}}{(q; q)_n} \right\} \\
 &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, -c\theta \right) \{a^l (au; q)_{\infty}\} && \text{(by using (1.7))} \\
 &= a^l (au; q)_{\infty} \sum_{j=0}^l \sum_{i=0}^{\infty} \frac{(q^{-l}, q/au; q)_j}{(q; q)_j} \left[(-1)^j q^{\binom{j}{2}} \right]^{s-r} (cu)^j W_{i+j} \frac{(cu)^i}{(q; q)_i} \\
 &\quad \times \left[(-1)^i q^{\binom{i}{2}} \right]^{1+s-r} q^{ij(s-r)}. && \text{(by using (2.5))}
 \end{aligned}$$

■

Setting $r = s = 0$, $a = b$, $c = a$ in (3.4) we obtain an extension of the generating function for the polynomials $h_n(a, b|q^{-1})$ as follows:

$$\begin{aligned}
 \sum_{n=0}^{\infty} h_{n+l}(a, b|q^{-1}) \frac{(-t)^n q^{\binom{n}{2}}}{(q; q)_n} &= b^l (bu; q)_{\infty} \sum_{j=0}^l \sum_{i=0}^{\infty} \frac{(q^{-l}, q/bu; q)_j}{(q; q)_j} (au)^j \frac{(au)^i}{(q; q)_i} (-1)^i q^{\binom{i}{2}} \\
 &= b^l (bu; q)_{\infty} \sum_{j=0}^l \frac{(q^{-l}, q/bu; q)_j}{(q; q)_j} (au)^j \sum_{i=0}^{\infty} \frac{(au)^i}{(q; q)_i} (-1)^i q^{\binom{i}{2}} \\
 &= b^l (au, bu; q)_{\infty} \sum_{j=0}^l \frac{(q^{-l}, q/bu; q)_j}{(q; q)_j} (au)^j.
 \end{aligned}$$

4. Mehler's Formula for $K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q)$

In the section, we will derive Mehler's formula and its extension for the polynomials K_n by using the operator ${}_r\Phi_s$. We give some special values to the parameters in the Mehler's formula and its extension for K_n to obtain Mehler's formula and its extension for the q^{-1} -Rogers-Szegő polynomials $h_n(a, b|q^{-1})$.

Theorem 4.1. (Mehler's formula for K_n). Let $K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q)$ be defined as in (3.2), then

$$\sum_{n=0}^{\infty} K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q) K_n(a_1', \dots, a_r'; b_1', \dots, b_s', c'; a'; q) \frac{(-u)^n q^{\binom{n}{2}}}{(q; q)_n}$$

$$\begin{aligned}
&= (aua'; q)_\infty \sum_{k=0}^{\infty} W_k \frac{(cua')^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(q^{-k}, q/a'au; q)_j}{(q; q)_j} \\
&\quad \times \left[(-1)^j q^{\binom{j}{2}} \right]^{s-r} (cau)^j W_{i+j} \frac{(cau)^i}{(q; q)_i} \left[(-1)^i q^{\binom{i}{2}} \right]^{1+s-r} q^{ij(s-r)}, \tag{4.1}
\end{aligned}$$

provided that $|cua'| < 1$.

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty} K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q) K_n(a_1', \dots, a_{r'}; b_1', \dots, b_{s'}, c'; a'; q) \frac{(-u)^n q^{\binom{n}{2}}}{(q; q)_n} \\
&= \sum_{n=0}^{\infty} K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q) {}_r\Phi_s \left(\begin{matrix} a_1', \dots, a_{r'} \\ b_1', \dots, b_{s'} \end{matrix}; q, -c'\theta \right) \{(a')^n\} \frac{(-u)^n q^{\binom{n}{2}}}{(q; q)_n} \\
&= {}_r\Phi_s \left(\begin{matrix} a_1', \dots, a_{r'} \\ b_1', \dots, b_{s'} \end{matrix}; q, -c'\theta \right) \left\{ \sum_{n=0}^{\infty} K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q) \times \frac{(-a'u)^n q^{\binom{n}{2}}}{(q; q)_n} \right\} \\
&= {}_r\Phi_s \left(\begin{matrix} a_1', \dots, a_{r'} \\ b_1', \dots, b_{s'} \end{matrix}; q, -c'\theta \right) \left\{ (aua'; q)_\infty {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cua' \right) \right\} \quad (\text{by using (3.3)}) \\
&= {}_r\Phi_s \left(\begin{matrix} a_1', \dots, a_{r'} \\ b_1', \dots, b_{s'} \end{matrix}; q, -c'\theta \right) \left\{ (aua'; q)_\infty \sum_{k=0}^{\infty} W_k \frac{(cua')^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \right\} \\
&= \sum_{k=0}^{\infty} W_k \frac{(cu)^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} {}_r\Phi_s \left(\begin{matrix} a_1', \dots, a_{r'} \\ b_1', \dots, b_{s'} \end{matrix}; q, -c'\theta \right) \{(a')^k (aua'; q)_\infty\} \\
&= (aua'; q)_\infty \sum_{k=0}^{\infty} W_k \frac{(cua')^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(q^{-k}, q/a'au; q)_j}{(q; q)_j} \\
&\quad \times \left[(-1)^j q^{\binom{j}{2}} \right]^{s-r} (c'au)^j W_{i+j} \frac{(c'au)^i}{(q; q)_i} \left[(-1)^i q^{\binom{i}{2}} \right]^{1+s-r} q^{ij(s-r)}. \quad (\text{by using (2.5)})
\end{aligned}$$

Setting $r = s = 0$, $c' = c$, $a = b$, $a' = d$, $c = a$ and $u = t$ in equation (4.1) we get Mehler's formula for the polynomials $h_n(a, b|q^{-1})$ (2.14) obtained by Liu [1] (equation (1.21)) as we see in the following corollary:

Corollary 2. (Mehler's formula for $h_n(a, b|q^{-1})$). Let $h_n(a, b|q^{-1})$ be defined as in (1.19), then

$$\sum_{n=0}^{\infty} h_n(a, b|q^{-1}) h_n(c, d|q^{-1}) \frac{(-t)^n q^{\binom{n}{2}}}{(q; q)_n} = \frac{(act, adt, bct, bdt; q)_\infty}{(abcdt^2/q; q)_\infty},$$

provided that $|abcdt^2/q| < 1$.

Proof. Setting $r = s = 0$, $c' = c$, $a = b$, $a' = d$, $c = a$ and $u = t$ in equation (4.1) we get

$$\begin{aligned}
 & \sum_{n=0}^{\infty} h_n(a, b|q^{-1}) h_n(c, d|q^{-1}) \frac{(-t)^n q^{\binom{n}{2}}}{(q; q)_n} \\
 &= (btd; q)_{\infty} \sum_{k=0}^{\infty} \frac{(atd)^k}{(q; q)_k} (-1)^k q^{\binom{k}{2}} \sum_{j=0}^k \sum_{i=0}^{\infty} \frac{(q^{-k}, q/dbu; q)_j}{(q; q)_j} (cbt)^j \frac{(cbt)^i}{(q; q)_i} (-1)^i q^{\binom{i}{2}} \\
 &= (btd; q)_{\infty} \sum_{k=0}^{\infty} \frac{(atd)^k}{(q; q)_k} (-1)^k q^{\binom{k}{2}} \sum_{j=0}^k \frac{(q/dtb; q)_j}{(q; q)_j} (-1)^j q^{\binom{j}{2}-kj} \frac{(q; q)_k}{(q; q)_{k-j}} (cbt)^j (cbt; q)_{\infty} \\
 &= (btd, cbt; q)_{\infty} \sum_{j=0}^{\infty} \frac{(q/dbt; q)_j}{(q; q)_j} (cbt)^j (-1)^j q^{\binom{j}{2}-j^2} \sum_{k=0}^{\infty} \frac{(dta)^{k+j}}{(q; q)_k} (-1)^{k+j} q^{\binom{k+j}{2}-kj} \\
 &= (btd, cbt; q)_{\infty} \sum_{j=0}^{\infty} \frac{(q/dbt; q)_j}{(q; q)_j} (cbt)^j (atd)^j q^{-j} \sum_{k=0}^{\infty} \frac{(dcu)^k}{(q; q)_k} (-1)^k q^{\binom{k}{2}} \quad (\text{by using (1.5)}) \\
 &= (btd, cbt; q)_{\infty} \frac{(cat; q)_{\infty}}{(acbd t^2/q; q)_{\infty}} (adt; q)_{\infty} \quad (\text{by using (1.6) and (1.7)}) \\
 &= \frac{(btd, btc, atc, atd; q)_{\infty}}{(acbd t^2/q; q)_{\infty}}.
 \end{aligned}$$

■

Theorem 4.2. (Extension of Mehler's formula for K_n). Let $K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q)$ be defined as in (3.1), then

$$\begin{aligned}
 & \sum_{n=0}^{\infty} K_{n+m}(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q) K_n(a_1', \dots, a_r'; b_1', \dots, b_s', c'; a'; q) \frac{(-u)^n q^{\binom{n}{2}}}{(q; q)_n} \\
 &= a^m \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(q^{-m}, q/aa'u; q)_j}{(q; q)_j} \left[(-1)^j q^{\binom{j}{2}} \right]^{s-r} (cu)^j W_{i+j} \frac{(cu)^i}{(q; q)_i} \left[(-1)^i q^{\binom{i}{2}} \right]^{1+s-r} \\
 & \quad \times q^{ij(s-r)} (a')^{i+j} (a'au, q)_{\infty} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q^{-(i+j)}, q/a'au; q)_l}{(q; q)_l} \left[(-1)^l q^{\binom{l}{2}} \right]^{s-r} (c'au)^l W_{k+l}
 \end{aligned}$$

$$\times \frac{(c'au)^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} q^{kl(s-r)}. \quad (4.2)$$

Proof.

$$\begin{aligned} & \sum_{n=0}^{\infty} K_{n+m}(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q) K_n(a_{1'}, \dots, a_{r'}; b_{1'}, \dots, b_{s'}, c'; a'; q) \frac{(-u)^n q^{\binom{n}{2}}}{(q; q)_n} \\ &= \sum_{n=0}^{\infty} K_{n+m}(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q) {}_r\Phi_s \left(\begin{matrix} a_{1'}, \dots, a_{r'} \\ b_{1'}, \dots, b_{s'} \end{matrix}; q, -c'\theta \right) \{(a')^n\} \\ & \quad \times \frac{(-u)^n q^{\binom{n}{2}}}{(q; q)_n} \quad (\text{by using (3.2)}) \\ &= {}_r\Phi_s \left(\begin{matrix} a_{1'}, \dots, a_{r'} \\ b_{1'}, \dots, b_{s'} \end{matrix}; q, -c'\theta \right) \left\{ \sum_{n=0}^{\infty} K_{n+m}(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q) \frac{(-a'u)^n q^{\binom{n}{2}}}{(q; q)_n} \right\} \\ &= {}_r\Phi_s \left(\begin{matrix} a_{1'}, \dots, a_{r'} \\ b_{1'}, \dots, b_{s'} \end{matrix}; q, -c'\theta \right) \left\{ a^m (aa'u; q)_{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(q^{-m}, q/aa'u; q)_j}{(q; q)_j} \right. \\ & \quad \times \left[(-1)^j q^{\binom{j}{2}} \right]^{s-r} (cua')^j W_{i+j} \frac{(cua')^i}{(q; q)_i} \left[(-1)^i q^{\binom{i}{2}} \right]^{1+s-r} q^{ij(s-r)} \left. \right\} \quad (\text{by using (3.4)}) \\ &= a^m \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(q^{-m}, q/aa'u; q)_j}{(q; q)_j} \left[(-1)^j q^{\binom{j}{2}} \right]^{s-r} (cu)^j W_{i+j} \frac{(cu)^i}{(q; q)_i} \left[(-1)^i q^{\binom{i}{2}} \right]^{1+s-r} q^{ij(s-r)} \\ & \quad \times {}_r\Phi_s \left(\begin{matrix} a_{1'}, \dots, a_{r'} \\ b_{1'}, \dots, b_{s'} \end{matrix}; q, -c'\theta \right) \{(a')^{i+j} (aa'u; q)_{\infty}\} \\ &= a^m \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(q^{-m}, q/aa'u; q)_j}{(q; q)_j} \left[(-1)^j q^{\binom{j}{2}} \right]^{s-r} (cu)^j W_{i+j} \frac{(cu)^i}{(q; q)_i} \left[(-1)^i q^{\binom{i}{2}} \right]^{1+s-r} q^{ij(s-r)} \\ & \quad \times (a')^{i+j} (a'au, q)_{\infty} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q^{-(i+j)}, q/a'au; q)_l}{(q; q)_l} \left[(-1)^l q^{\binom{l}{2}} \right]^{s-r} (c'au)^l W_{k+l} \\ & \quad \times \frac{(c'au)^k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} q^{kl(s-r)}. \quad (\text{by using (2.5)}) \end{aligned}$$

■

Setting $r = s = 0$, $a = b$, $c = a$, $c' = c$ and $a' = d$ in equation (4.2) we get an extension of Mehler's formula for the polynomials $h_n(a, b|q^{-1})$ as we see in the following corollary:

Corollary 3. (Extension of Mehler's formula for $h_n(a, b|q^{-1})$). Let $h_n(a, b|q^{-1})$ be defined as in (1.19), then

$$\sum_{n=0}^{\infty} h_{n+m}(a, b|q^{-1}) h_n(c, d|q^{-1}) \frac{(-u)^n q^{\binom{n}{2}}}{(q; q)_n} \\ = b^m(aud, cbu, dbu, q)_{\infty} \sum_{j=0}^{\infty} \frac{(q^{-m}, q/bdu; q)_j}{(q; q)_j} (aud/q)^j \sum_{l=0}^{\infty} \frac{(q^{-(i+j)}, q/dbu; q)_l}{(q; q)_l} (cbu)^l.$$

Proof.

Setting $r = s = 0$, $a = b$, $c = a$, $c' = c$ and $a' = d$ in equation (4.2) we get

$$\sum_{n=0}^{\infty} h_{n+m}(a, b|q^{-1}) h_n(c, d|q^{-1}) \frac{(-u)^n q^{\binom{n}{2}}}{(q; q)_n} \\ = b^m(dbu, q)_{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(q^{-m}, q/bdu; q)_j}{(q; q)_j} (aud/q)^j \frac{(aud)^i}{(q; q)_i} (-1)^i q^{\binom{i}{2}} \\ \times \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q^{-(i+j)}, q/dbu; q)_l}{(q; q)_l} (cbu)^l \frac{(cbu)^k}{(q; q)_k} (-1)^k q^{\binom{k}{2}} \\ = b^m(dbu, q)_{\infty} \sum_{j=0}^{\infty} \frac{(q^{-m}, q/bdu; q)_j}{(q; q)_j} (aud/q)^j \sum_{i=0}^{\infty} \frac{(aud)^i}{(q; q)_i} (-1)^i q^{\binom{i}{2}} \\ \times \sum_{l=0}^{\infty} \frac{(q^{-(i+j)}, q/dbu; q)_l}{(q; q)_l} (cbu)^l \sum_{k=0}^{\infty} \frac{(cbu)^k}{(q; q)_k} (-1)^k q^{\binom{k}{2}} \\ = b^m(aud, cbu, dbu, q)_{\infty} \sum_{j=0}^{\infty} \frac{(q^{-m}, q/bdu; q)_j}{(q; q)_j} (aud/q)^j \sum_{l=0}^{\infty} \frac{(q^{-(i+j)}, q/dbu; q)_l}{(q; q)_l} (cbu)^l. \\ \text{(by using (1.7))}$$

■

5. Rogers Formula for $K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q)$

We will derive, in this section, Roger's formula for the polynomials K_n by using the operator ${}_r\Phi_s$. We give some special values to the parameters in Rogers formula for $K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q)$ to obtain Rogers formula for the q^{-1} -Rogers-Szegő polynomials

$h_n(a, b|q^{-1})$.

Theorem 5.1. (Rogers formula for K_n). Let $K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q)$ be defined as in (3.2), then

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} K_{n+m}(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q) \frac{(-t)^n q^{\binom{n}{2}}}{(q; q)_n} \frac{(-u)^m q^{\binom{m}{2}}}{(q; q)_m} \\ &= (at, au; q)_{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q/au; q)_j}{(q; q)_j} (actu/q)^j \left[(-1)^j q^{\binom{j}{2}} \right]^{s-r} W_{k+j} \frac{(cu)^k}{(q; q)_k} \\ & \quad \times \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} q^{kj(s-r)}, \end{aligned} \quad (5.1)$$

provided that $|actu/q| < 1$.

Proof.

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} K_{n+m}(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q) \frac{(-t)^n q^{\binom{n}{2}}}{(q; q)_n} \frac{(-u)^m q^{\binom{m}{2}}}{(q; q)_m} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, -c\theta \right) \{a^{n+m}\} \frac{(-t)^n q^{\binom{n}{2}}}{(q; q)_n} \frac{(-u)^m q^{\binom{m}{2}}}{(q; q)_m} \\ &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, -c\theta \right) \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n (at)^n q^{\binom{n}{2}}}{(q; q)_n} \sum_{m=0}^{\infty} \frac{(-1)^m (au)^m q^{\binom{m}{2}}}{(q; q)_m} \right\} \\ &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, -c\theta \right) \{(at; q)_{\infty} (au; q)_{\infty}\}. \quad (\text{by using (1.7)}) \\ &= (at, au; q)_{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q/ua; q)_j}{(q; q)_j} (actu/q)^j \left[(-1)^j q^{\binom{j}{2}} \right]^{s-r} W_{k+j} \frac{(cu)^k}{(q; q)_k} \\ & \quad \times \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} q^{kj(s-r)}. \quad (\text{by using (2.3)}) \end{aligned}$$

Setting $r = s = 0$, $a = b$ and $c = a$ in equation (5.1) we obtain Rogers formula for the polynomials $h_n(a, b|q^{-1})$ as we see in the following corollary:

Corollary 4. (Rogers formula for $h_n(a, b|q^{-1}; q)$). Let $h_n(a, b|q^{-1})$ be defined as in (1.19), then

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(a, b|q^{-1}; q) \frac{(-t)^n q^{\binom{n}{2}}}{(q; q)_n} \frac{(-u)^m q^{\binom{m}{2}}}{(q; q)_m} = \frac{(at, au, bt, bu; q)_{\infty}}{(abtu/q; q)_{\infty}},$$

provided that $|abtu/q| < 1$.

Proof.

Setting $r = s = 0$, $a = b$ and $c = a$ in equation (5.1) we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(a, b|q^{-1}; q) \frac{(-t)^n q^{\binom{n}{2}}}{(q; q)_n} \frac{(-u)^m q^{\binom{m}{2}}}{(q; q)_m} \\ &= (bt, bu; q)_{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q/bu; q)_j}{(q; q)_j} (batu/q)^j \frac{(au)^k}{(q; q)_k} (-1)^k q^{\binom{k}{2}} \\ &= (bt, bu; q)_{\infty} \sum_{j=0}^{\infty} \frac{(q/au; q)_j}{(q; q)_j} (batu/q)^j \sum_{k=0}^{\infty} \frac{(au)^k}{(q; q)_k} (-1)^k q^{\binom{k}{2}} \\ &= \frac{(bt, at, bu, au; q)_{\infty}}{(batu/q; q)_{\infty}}. \quad (\text{by using (1.6) and (1.7)}) \end{aligned}$$

■

6. Conclusions

This paper devoted to study a new generalized q -operator ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, -c\theta \right)$. Also, a new polynomial $K_n(a_1, \dots, a_r, b_1, \dots, b_s, c; a; q)$ is constructed. The generating function and its extension for $K_n(a_1, \dots, a_r, b_1, \dots, b_s, c; a; q)$ is studied. Also, the Mehler's formula and its extension for $K_n(a_1, \dots, a_r, b_1, \dots, b_s, c; a; q)$ is investigated. While, the Rogers formula for $K_n(a_1, \dots, a_r, b_1, \dots, b_s, c; a; q)$ is constructed. In order to explore the results, one can imposing some special values of the parameters. So, by setting $r = s = 0$, $a = b$, $c = a$ in the generating function and its extension for $K_n(a_1, \dots, a_r, b_1, \dots, b_s, c; a; q)$, the generating function and its extension for the q^{-1} -Rogers-Szegő polynomials $h_n(a, b|q^{-1})$ is obtained directly. Also, by setting $r = s = 0$, $a = b$, $c = a$ in Mehler's formula and its extension for $K_n(a_1, \dots, a_r, b_1, \dots, b_s, c; a; q)$, the Mehler's formula and its extension for the polynomials $h_n(a, b|q^{-1})$ is achieved directly. Finally, by setting $r = s = 0$, $a = b$, $c = a$ in Rogers formula for $K_n(a_1, \dots, a_r, b_1, \dots, b_s, c; a; q)$, the Rogers formula for the polynomials $h_n(a, b|q^{-1})$ is created.

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المؤثر ${}_r\Phi_s$ ومتعددة الحدود K_n

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المستخلص

باستخدام تعريف الدالة الهندسية الفوقية الاساسية، قمنا بتعريف المؤثر q -العام ${}_r\Phi_s$ وحصلنا على بعض المتطابقات للمؤثر ${}_r\Phi_s$. أيضاً، عرفنا متعددة حدود جديدة $K_n(a_1, \dots, a_r, b_1, \dots, b_s, c; a; q)$. وجدنا الدالة المولدة وتوسيعها، صيغة Mehler وتوسيعها وصيغة Rogers لمتعددة الحدود $K_n(a_1, \dots, a_r, b_1, \dots, b_s, c; a; q)$ باستخدام المؤثر ${}_r\Phi_s$. في الحقيقة، يمكن اعتبار هذا العمل بمثابة تعميم لعمل Liu [1] عن طريق فرض بعض القيم الخاصة للمعاملات في نتائجنا. لذلك يمكن الحصول على متعدّدات حدود روجرز- زيغو- q^{-1} $h_n(a, b|q^{-1})$ مباشرة.