Gosper's Algorithm and Rational Solutions of First Order Recurrences

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Abstract

In this paper we give two approaches for Gosper's algorithm. Also we give analysis to the degree of rational solutions of certain first order difference equation. Furthermore, we present an approach to find rational solutions of first order recurrences.

Keywords: Gosper's algorithm, hypergeometric solution, rational solution.

1. Notations

Let $\mathbb{N}$ be the set of natural numbers, K be the field of characteristic zero, $K(n)$ be the field of rational functions over K, $K[n]$ be the ring of polynomials over K, deg(p) denotes the polynomial degree (in n) of any $p \in K[n]$, $p \neq 0$. We define $\deg(0) = -1$. We assume the result of any gcd (greatest common divisor) computation in $K[n]$ as being normalized to a monic polynomial p, i.e., the leading coefficient of p being 1. Recall that a non-zero term t_n is called a hypergeometric term over K if there exists a rational function $r(n) \in K(n)$ such that

$$\frac{t_{n+1}}{t_n} = r(n).$$

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Gosper's algorithm has been extensively studied and widely used to prove hypergeometric identities see, for example, [6, 7, 8, 10, 11, 13, 14]. Given a hypergeometric term t_n, Gosper's algorithm is a procedure to find a hypergeometric term z_n satisfying

$$z_{n+1} - z_n = t_n, \qquad (1.1)$$

if it exists, or confirm the nonexistence of any solution of (1.1). The key idea of the Gosper's algorithm lies in the representation of the rational functions. Gosper showed that any rational function $r(n)$ can be written in the following form, called the Gosper representation:

$$r(n) = \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)}, \qquad (1.2)$$

where a, b and c are polynomials over K and

$\gcd(a(n), b(n+h)) = 1$ for all $h \in \mathrm{N}$.

Petkovšek [8] realized that the Gosper representation becomes unique, which is called the Gosper-Petkovšek representation, or GP representation, for short, if we further require that b, c are monic polynomials such that

$\gcd(a(n), c(n)) = 1,$

$\gcd(b(n), c(n+1)) = 1.$

In [11], Paule and Strehl gave a derivation of Gosper's algorithm by using GP representation, also see [5]. In [10], Equipped with the Greatest Factorial Factorization (GFF), Paule present a new approach for Gosper's algorithm. Paule's approach leads to the same algorithm Gosper, but in a new setting. In [9], Lisoněk and et al., gave a detailed study of the degree setting for Gosper's algorithm.

The problem of computing rational solutions is quite important in computer algebra because some interesting problems can be reduced to it. One of the applications of computing rational solutions is a generalization of Gosper's algorithm. In [1] an algorithm to find rational solutions has been proposed. That algorithm is quite complicated. Conceivably such an algorithm could give a denominator of smaller degree than the algorithm proposed in [3]. Undoubtedly the algorithm described in [3] is more elementary in its structure. Additionally, it can be adapted for the case of

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q-difference equations. For more details about rational solutions see [2, 4, 15, 16].

The contents of this paper are as follows:

In sections 2 and 3, we give two approaches for Gosper's algorithm. In section 4, we give analysis to the degree of rational solutions to the equation that appeared in the derivation of Gosper's algorithm. Finally, in section 5, we give an approach to find rational solutions of first order recurrences.

2. An Approach for Gosper's Algorithm

In this section we present an approach for Gosper's algorithm. To do this we need the following lemma:

Lemma 2.1. Let $a,b,c,A,B,C \in K[n]$ such that

gcd($a(n),c(n)$)=gcd($b(n),c(n+1)$)=gcd($A(n),B(n+h)$=1 $\forall\ h \in N$.
If

$$\frac{a(n)}{b(n)}\frac{c(n+1)}{c(n)} = \frac{A(n)}{B(n)}\frac{C(n+1)}{C(n)},$$

then $c(n)$ divides $C(n)$.

Proof. See [12].
□

Given a hypergeometric term t_n and suppose that there exists a hypergeometric term z_n satisfying (1.1). By using (1.1) we get

$$\frac{z_n}{t_n} = \frac{z_n}{z_{n+1} - z_n} = \frac{1}{\frac{z_{n+1}}{z_n} - 1}.$$

Let $y(n) = {z_n}/{t_n}$. It follows that $y(n)$ is a rational function of n. Substituting $y(n)t_n$ for z_n in (1.1) to obtain

$$r(n)y(n+1) - y(n) = 1, \qquad (2.1)$$

where $r(n) = {t_{n+1}}/{t_n}$ is a rational function of n. Let

$$y(n) = \frac{f(n)}{g(n)}, \qquad (2.2)$$

where $f(n), g(n)$ are two unknown relatively prime polynomials. Substituting (2.2) into (2.1) to obtain

$$r(n) = \frac{f(n)+g(n)}{f(n+1)}\frac{g(n+1)}{g(n)}. \qquad (2.3)$$

Let n_0 be the largest $n \in N$ such that n is a pole of r, or $-\infty$ if no such n exists. Then, clearly

$$t_n = C\prod_{k=n_0+1}^{n-1} r(k) \quad \text{for} \quad n > n_0,$$

satisfies

$$t_{n+1} = r(n)t_n,$$

for almost all n, where $C \in K$ is an arbitrary constant. Let $(a)_n$ denote the rising factorial for a,
namely

$$(a)_n = a(a+1)\ldots(a+n-1)$$

If $r(n)$ factors into linear factors over K:

$$r(n) = z \cdot n^u \cdot \frac{(n-a_1)(n-a_2)\ldots(n-a_r)}{(n-b_1)(n-b_2)\ldots(n-b_s)}, \qquad (2.5)$$

where $z, a_i, b_j \in K, u \in Z,$ and $a_i, b_j \neq 0, n$, for all $n \in N$, then by iterating (2.4) and using equation (2.5) we get that

$$t_n = Cz^{n-1}((n-1)!)^u \frac{b_1 b_2 \ldots b_s}{a_1 a_2 \ldots a_r}\frac{(-a_1)_n(-a_2)_n \ldots (-a_r)_n}{(-b_1)_n(-b_2)_n \ldots (-b_s)_n},$$

where $C \in K$ is an arbitrary constant. On the Other hand

$$r(n) = \frac{t_{n+1}}{t_n}$$

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$$= \frac{z \cdot n^u}{(-b_1+n)(-b_2+n)\cdots(-b_s+n)} \frac{(-a_1)_{n+1}(-a_2)_{n+1}\cdots(-a_r)_{n+1}}{(-a_1)_n(-a_2)_n\cdots(-a_r)_n}$$

$$= \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)} \qquad (2.6)$$

where $c(n) = (-a_1)_n(-a_2)_n\cdots(-a_r)_n$ and $a(n) = z \cdot n^u, b(n) = (-b_1+n)(-b_2+n)\cdots(-b_s+n)$ if $u \geq 0$ or $a(n) = z, b(n) = n^{-u}(-b_1+n)(-b_2+n)\cdots(-b_s+n)$ otherwise. From equations (2.3) and
(2.6) we obtain

$$\frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)} = \frac{(f(n)+g(n))}{f(n+1)} \frac{g(n+1)}{g(n)}, \qquad (2.7)$$

By Lemma 2.1, $g(n) \mid c(n)$, so $c(n)$ is a suitable denominator for $y(n)$. Write $y(n) = \frac{v(n)}{c(n)}$, where $v(n)$ is an unknown polynomial, and substitute this together with (2.6) into (2.1) to obtain $a(n)v(n+1) = (v(n)+c(n))b(n)$. This shows that $b(n)$ divides $v(n+1)$, hence we have

$$y(n) = \frac{b(n-1)x(n)}{c(n)}, \qquad (2.8)$$

where $x(n)$ is a polynomial of n. Substitution of (2.6) and (2.8) into (2.1) shows $x(n)$ satisfies

$$a(n)x(n+1) - b(n-1)x(n) = c(n). \qquad (2.9)$$

Now if $x(n)$ is a polynomial solution of (2.9) then

$$z_n = \frac{b(n-1)x(n)}{c(n)} t_n$$

is a hypergeometric solution of (1.1) . □

3. Another Approach for Gosper's Algorithm

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In this section we present another approach for Gosper's algorithm. Let us define the hypergeometric term t_n as

$$t_n = \frac{(a_1)_n (a_2)_n \ldots (a_r)_n}{(b_1)_n (b_2)_n \ldots (b_s)_n} \frac{z^n}{n!},$$

where $r, s \geq 0$, z, $a_i, b_j \in K$, $(1 \leq i \leq r,\ 1 \leq j \leq s)$, $z \neq 0$, and $a_i, b_j \neq -n$, for all $n \in N$. We start with equation (2.3) (there is no difference at the beginning): Find relatively prime polynomials $f(n)$, $g(n)$ satisfying

$$r(n) = \frac{(f(n) + g(n))}{f(n+1)} \frac{g(n+1)}{g(n)},$$

where $r(n) = t_{n+1} / t_n$ is a rational function of n. By using the above definition of $t_n, r(n)$ can be viewed as follows:

$$r(n) = \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)}, \qquad (3.1)$$

where $a(n) = z, b(n) = (n+1)(b_1 + n)(b_2 + n) \ldots (b_s + n),\ c(n) = (a_1)_n (a_2)_n \ldots (a_r)_n$. From equations (3.1) and (2.3) we obtain

$$\frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)} = \frac{(f(n) + g(n))}{f(n+1)} \frac{g(n+1)}{g(n)}, \qquad (3.2)$$

By Lemma 2.1, $g(n) \mid c(n)$, so $c(n)$ is a suitable denominator for $y(n)$. Write $y(n) = v(n) / c(n)$, where $v(n)$ is an unknown polynomial, and substitute this together with (3.1) into (2.1) to obtain $a(n)v(n+1) = (v(n) + c(n))b(n)$. This shows that $b(n)$ divides $v(n+1)$, hence we have

$$y(n) = \frac{b(n-1)x(n)}{c(n)} t_n, \qquad (3.3)$$

where $x(n)$ is a polynomial of n. Substitution of (3.1) and (3.3) into (2.1) shows $x(n)$ satisfies

$$a(n)x(n+1) - b(n-1)x(n) = c(n). \qquad (3.4)$$

Now if $x(n)$ is a polynomial solution of (3.4) then

$$z_n = \frac{b(n-1)x(n)}{c(n)} t_n$$

is a hypergeometric solution of (1.1). □

4. The Degree of the Rational Solution

In this section we consider the degree of the rational solution $y(n)$ of equation (2.1). Let $r(n)$ and $y(n)$ be as in (1.2) and (2.2) respectively. By substituting $r(n)$ and $y(n)$ into (2.1) we obtain

$$a(n)c(n+1)f(n+1)g(n) - b(n)c(n)f(n)g(n+1) = b(n)c(n)g(n)g(n+1). \qquad (4.1)$$

Let $\bar{a}(n) = a(n)c(n+1)$, $\bar{b}(n) = b(n)c(n)$ then we obtain

$$\bar{a}(n)f(n+1)g(n) - \bar{b}(n)f(n)g(n+1) = \bar{b}(n)g(n)g(n+1). \qquad (4.2)$$

Assume that $\deg f(n) = p$, $\deg g(n) = q$. We distinguish two cases:

Case 1: $\deg \bar{a}(n) \neq \deg \bar{b}(n)$ or lc $\bar{a}(n) \neq$ lc $\bar{b}(n)$.

The leading terms on the left hand side of (4.2) do not cancel. Hence the degree of the left hand side of (4.2) is $p + q + \max\{\deg \bar{a}(n), \deg \bar{b}(n)\}$. Since the degree of the right hand side is $\deg \bar{b}(n) + 2q$, it follows that

$$p - q = \deg \bar{b}(n) - \max\{\deg \bar{a}(n), \deg \bar{b}(n)\}$$

is the only candidate for the degree of a nonzero rational solution of (4.2).

Case 2: $\deg \bar{a}(n) = \deg \bar{b}(n)$ or lc $\bar{a}(n) =$ lc $\bar{b}(n) = l$.

The leading terms on the left hand side of (4.2) cancel. Again there are two cases to consider.

(2a) The terms of the second-highest degree on the left hand side of (4.2) do not cancel. Then the degree of the left hand side of (4.2) is $p+q+\deg\bar{a}(n)-1=\deg\bar{a}(n)+2q$, thus

$p-q=1$.

(2b) The terms of the second-highest degree on the left hand side of (4.2) cancel. Let

$$\bar{a}(n)=l\,n^{k}+An^{k-1}+o(n^{k-2}), \qquad (4.3)$$

$$\bar{b}(n)=l\,n^{k}+Bn^{k-1}+o(n^{k-2}), \qquad (4.4)$$

$$f(n)=c_0n^{p}+c_1n^{p-1}+o(n^{p-2}),$$

$$g(n)=d_0n^{q}+d_1n^{q-1}+o(n^{q-2}),$$

where $c_0,d_0\neq 0$. Then, expanding the terms on left of (4.2) successively, we find that

$$f(n+1)=c_0n^{p}+(c_0p+c_1)n^{p-1}+o(n^{p-2}),$$

$$g(n+1)=d_0n^{q}+(d_0q+d_1)n^{q-1}+o(n^{q-2}),$$

$$\bar{a}(n)f(n+1)g(n)=c_0d_0\,l\,n^{k+p+q}+(d_0(l(c_0p+c_1)+Ac_0)+d_1c_0\,l)n^{k+p+q-1}+o(n^{k+p+q-2}),$$

$$\bar{b}(n)f(n)g(n+1)=c_0d_0\,l\,n^{k+p+q}+(c_0(l(d_0p+d_1)+Bd_0)+c_1d_0\,l)n^{k+p+q-1}+o(n^{k+p+q-2}),$$

$$\bar{a}(n)f(n+1)g(n)-\bar{b}(n)f(n)g(n+1)=c_0d_0(l(p-q)+A-B)n^{k+p+q-1}+o(n^{k+p+q-2}). \qquad (4.5)$$

By assumption, the coefficient of $n^{k+p+q-1}$ on the right side of (4.5) vanishes, therefore $c_0d_0(l(p-q)+A-B)=0.$ It follows that

$$p-q=\frac{B-A}{l}.$$

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Thus in Case 2 the only possible degrees of nonzero rational solutions of (4.2) are 1 and $(B-A)/l$, where A and B are defined in (4.3) and (4.4), respectively. Of course, only nonnegative integer candidates need be considered. When there are two candidates we can use the larger of the two as an upper bound for the degree. Note that, in general, both Cases (2a) and (2b) can occur since equation (4.2) may in fact have nonzero rational solutions of two distinct degrees.

5. Rational Solutions of First Order Recurrences

In this section we consider the problem of finding rational solutions $R(n)$ of the equation

$$p_1(n)R(n+1) - p_0(n)R(n) = p(n), \qquad (5.1)$$

where $p_0, p_1, p \in K[n]$ are given non-zero polynomials. Equation (5.1) can be written as

$$r(n)y(n+1) - y(n) = 1, \qquad (5.2)$$

where

$$r(n) = \frac{p_1(n)}{p_0(n+1)}\frac{p(n+1)}{p(n)}, \qquad (5.3)$$

and

$$y(n) = \frac{p_0(n)R(n)}{p(n)}, \qquad (5.4)$$

are rational functions of n. Hence finding rational solutions of (5.1) is therefore equivalent to finding rational solutions of (5.2). Note that equation (5.2) is the same as equation (2.1). Equation
(5.2) or (2.1) is a well known equation appears in many approaches for Gosper's algorithm, see for example the above two approaches. Write $r(n)$ as in (2.6) or (3.1) as follows:

$$r(n) = \frac{A(n)}{B(n)}\frac{C(n+1)}{C(n)}.$$

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By using one of the above two approaches for Gosper's algorithm we get

$$y(n) = \frac{B(n-1)x(n)}{C(n)}, \qquad (5.5)$$

where $x(n)$ is a polynomial of n satisfying

$$A(n)x(n+1) - B(n-1)x(n) = C(n). \qquad (5.6)$$

Now if equation (5.6) can be solved for $x \in K[n]$ then

$$R(n) = \frac{p(n)B(n-1)x(n)}{p_0(n)C(n)}$$

is a rational solution of (5.1), otherwise no rational solution of (5.1) exists.
□

Example 5.1. Given
$p_1(n)R(n+1) - p_0(n)R(n) = p(n),$

with $p_1(n) = (n-1)(2n+1), \quad p_0(n) = (n-3)(2n-1), \quad p(n) = 4n(n+2)$, then

$$r(n) = \frac{p_1(n)}{p_0(n+1)}\frac{p(n+1)}{p(n)} = \frac{1}{1}\frac{(n-1)(n+1)(n+3)}{(n-2)n(n+2)},$$

Hence $A(n) = 1, \quad B(n) = 1, \quad C(n) = (n-2)n(n+2).$ By (5.6), $x(n)$ is a polynomial which satisfies

$$x(n+1) - x(n) = (n-2)n(n+2)$$

The polynomial $x(n) = k + \frac{1}{4}n(n-1)(n^2 - n - 8)$ is a solution of this equation, where k is constant. Therefore, $R(n) = \frac{4k + n(n-1)(n^2 - n - 8)}{(n-2)(n-3)(2n-1)}$ is a rational solution to the equation (5.7).

□

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خوارزمية جوسبر والحلول النسبية للمعادلة التكرارية
من الرتبة الاولى

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الخلاصة

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في هذا البحث نعطي اسلوبين لخوارزمية جوسبر. كذلك نعطي تحليل لدرجة نوع محدد من الدوال النسبية لمعادلة تكرارية من الرتبة الاولى. علاوة على ذلك نقدم اسلوب لايجاد الحلول النسبية لمعادلة تكرارية **من الرتبة الاولى .**

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