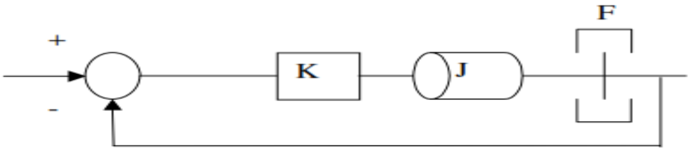
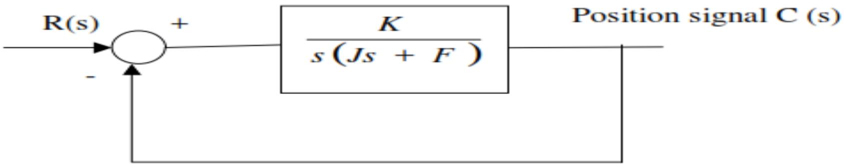
Second Order Systems

Block Diagram





Transfer function:

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Fs + K}$$

$$\frac{C(s)}{R(S)} = \frac{K/J}{s^2 + \left(\frac{F}{J}\right)s + \left(\frac{K}{J}\right)}$$

$$= \frac{\frac{K}{J}}{\left[s + \frac{F}{2J} + \sqrt{\left(\frac{F}{2J}\right)^2 - \frac{K}{J}}\right]\left[s + \frac{F}{2J} - \sqrt{\left(\frac{F}{2J}\right)^2 - \frac{K}{J}}\right]}$$

Substitute in the transfer function:

$$\frac{K}{J} = \omega_n{}^2$$

$$\frac{F}{J} = 2\,\zeta\omega_n = 2\sigma$$

$$\zeta = \frac{F}{2\sqrt{JK}}$$

ζ: damping ratio
ω_n: undamped natural frequency
σ: stability ratio

to obtain

$$\frac{C(s)}{R(s)} = \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

- ***Underdamped*** case: $0 < \zeta < 1$

  $F^2 - 4JK < 0$ two *complex conjugate* poles

- ***Critically damped*** case: $\zeta = 1$

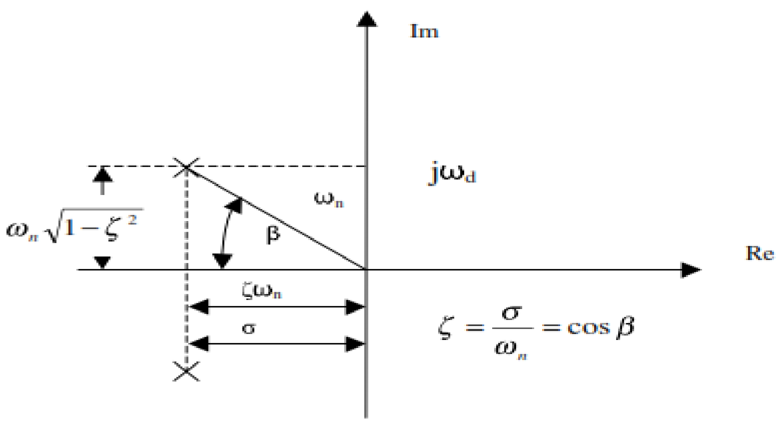
  $F^2 - 4JK = 0$ two *equal real* poles

- ***Overdamped*** case: $\zeta > 1$

  $F^2 - 4JK > 0$ two *real* poles

Under damped case $(0 < \zeta < 1)$:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^{\ 2}}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$



$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

ω_n : undamped natural frequency
ω_d : damped natural frequency
ζ : damping ratio

Unit step response:

R(s) = 1/s

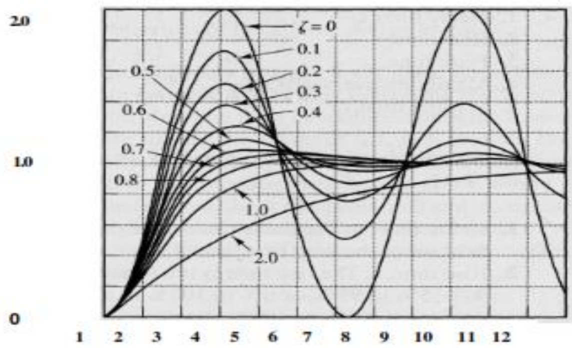
$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^{\ 2}} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^{\ 2}}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) \qquad t \geq 0$$

$$c(t) = 1 - \frac{1}{\beta}e^{-\zeta\omega_n t}\sin(\omega_n \beta t + \theta) \qquad t \geq 0$$

$$\beta = \sqrt{1-\zeta^2} \qquad \theta = \tan^{-1}\frac{\beta}{\zeta}$$

$$\mathrm{e(t) = r(t) - c(t)} = e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) \qquad t \geq 0$$



Unit step response curves of a second order system

Undamped case (ζ = 0):

Unit step response:

$$c(t) = 1 - \cos\omega_n t \qquad t \geq 0$$

Critically damped case ($\zeta = 1$):

Unit step Response: R(s) = 1/s

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2}$$

$$C(s) = \frac{1}{s(s+\omega_n)^2}$$

$$c(t) = 1 - e^{-\omega_n t}(1+\omega_n t) \qquad t \geq 0$$

Overdamped case ($\zeta > 1$):

Unit step Response: R(s) = 1/s

$$C(s) = \frac{\omega_n^2}{\left(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)\left(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)} \cdot \frac{1}{s}$$

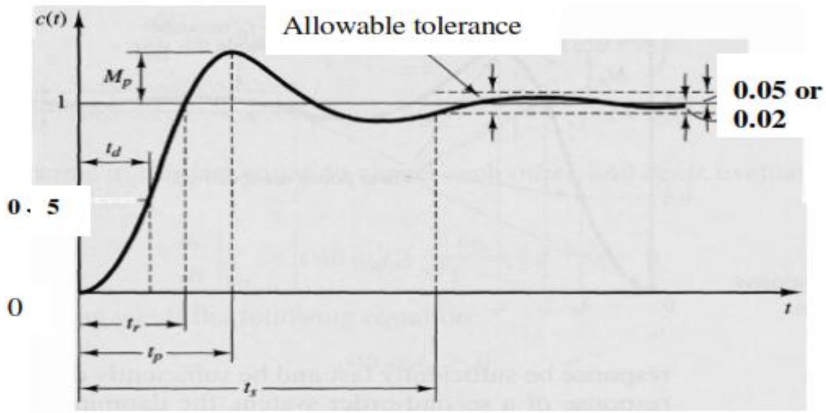
$$c(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right) \qquad t \geq 0$$

with $s_1 = \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$

$s_2 = \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$

Transient Response Specifications

Unit step response of a 2^{nd} order underdamped system:



t_d *delay time*: time to reach 50% of $c(\infty)$ for the first time.

t_r *rise time* : time to rise from 0 to 100% of $c(\infty)$.

t_p *peak time* : time required to reach the first peak.

M_p *maximum overshoot* : $\frac{c(t_p) - c(\infty)}{c(\infty)} \cdot 100\%$

t_s *settling time* : time to reach and stay within a 2% (or 5%) tolerance of the final value $c(\infty)$.

$$0.4 < \zeta < 0.8$$

Gives a good step response for an underdamped system

<u>Peak time t_p:</u> time to reach the first peak of c(t)

$$\frac{dc(t)}{dt}\bigg|_{t=t_p} = 0 \quad \Rightarrow (\sin \omega_d t_p)\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_p} = 0$$

$$\sin \omega_d t_p = 0$$

$$t_p = \frac{\pi}{\omega_d}$$

Maximum overshoot M_p:

$$t = t_p = \frac{\pi}{\omega_d}$$

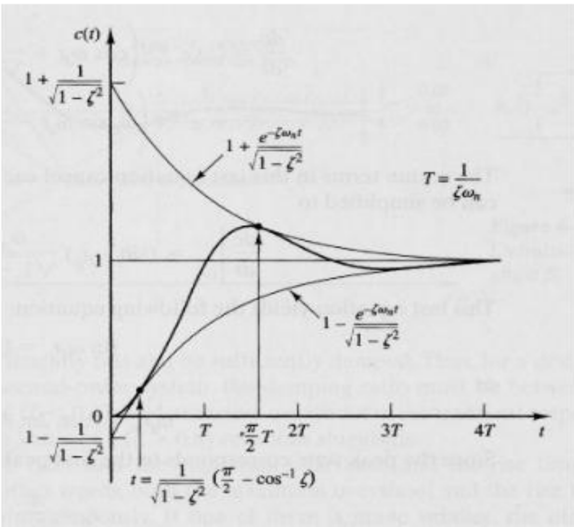
$$M_p = c(t_p) = 1 - e^{-\zeta\omega_n(\pi/\omega_d)}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right)$$

$$= e^{-\frac{\zeta\omega_n\pi}{\omega_d}} = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{\frac{-\sigma\pi}{\omega_d}}$$

Settling time t_s:

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

approximate t_s using envelope curves: $env(t) = 1 \pm \dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$



Pair of envelope curves for the unit-step response curve

2% band: $t_s = \dfrac{4}{\sigma} = \dfrac{4}{\zeta\omega_n}$ 5% band $t_s = \dfrac{3}{\sigma} = \dfrac{3}{\zeta\omega_n}$