Percentiles for Continuous Random Variables

Definition (7):

Let p be a real number between 0 and 1. A $100\,p^{th}$ percentile of the distribution of a random variable X is any real number q satisfyies $P(X \leq q) = p$ & $P(X > q) = 1 - p$.

Example (14)

Let the random variable X has probability density function $f(x) = \begin{cases} e^{x-2} & , for\ x < 2 \\ 0 & , otherwise \end{cases}$. Find the 75^{th} $percentile\ of\ X$.

Solution

Here, $p = \frac{75}{100} = 0.75$. By definition (7), we have

$$0.75 = P(X \leq q) = \int_{-\infty}^{q} f(x)dx = \int_{-\infty}^{q} e^{x-2}dx = [e^{x-2}]_{-\infty}^{q} = e^{q-2}.$$

$$\ln(0.75) = q - 2 \rightarrow q = 2 + ln(0.75) \simeq 1.712 . ■$$

Definition (8):

The 25^{th} & 75^{th} percentiles are called the first & third quartiles.

Example (15)

Let the probability density function of the random variable X is $f(x) = \begin{cases} \frac{x}{32} & , 0 < x \leq 8 \\ 0 & , otherwise \end{cases}$. Find the first quartile for this distribution.

Solution

$Here, p = 0.25$. By definition (7), we have

$$0.25 = P(X \leq q) = \int_{-\infty}^{q} f(x)dx = \int_{-\infty}^{q} \frac{x}{32}\,dx \rightarrow 0.25 = [\frac{x^2}{64}]_0^q = \frac{q^2}{64} \rightarrow$$

$$q^2 = 16 \rightarrow q = 4 . ■$$

Definition (9)

The 50^{th} percentile is called the median of the distribution.

Example (16)

Let the probability density function of the random variable X is $f(x) = \begin{cases} -\frac{1}{2}\,e^{-\frac{x}{2}} & if\ x \geq 0 \\ 0 & , otherwise \end{cases}$. Find the median of this distribution.

Solution

$Here, p = 0.5$. By definition (7) we have $0.5 = P(X \leq q) = \int_{-\infty}^{q} f(x)dx \rightarrow$

$0.5=\int_{-\infty}^{q}-\frac{1}{2}\,e^{-\frac{x}{2}}\,dx \rightarrow 0.5=[e^{-\frac{x}{2}}]_{-\infty}^{q} \rightarrow 0.5=e^{-\frac{q}{2}} \rightarrow ln(0.5)=-\frac{q}{2} \rightarrow$
$q=-2ln(0.5)=1.386$. ■

Note (9)

If a probability density function of the random variable X is symmetric about y-axis, then the median is always zero.

Example (17)

Let $f(x)=\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2}$ $,-\infty<x<\infty$ is the probability density function of the random variable X.What is the median of this distribution?

Solution

Since $f(x)$ is symmetric about the y-axis, then the median is equal to zero. ■

Definition (10)

A mode of the distribution of a continuous random variable X is the value of x where the probability density function $f(x)$ attains a relative maximum.

Note (10)

The distribution can have more than one mode.

Example (18)

Let $f(x)=\begin{cases} x^2\,e^{-x}\,,x\geq 0 \\ 0\,,otherwise \end{cases}$ be the probability density function of the random variable X.Find the mode of this distribution.

Solution

Put $f'(x)=0 \rightarrow\ -x^2e^{-x}+2xe^{-x}=0 \rightarrow xe^{-x}(-x+2)=0 \rightarrow$

$x(-x+2)=0 \rightarrow either\ x=0\ or\ x=2$.

$f''(x)=x^2e^{-x}-4xe^{-x}+2e^{-x}=e^{-x}(x^2-4x+2)$.

Since $f''(0)>0\ \&\ f''(2)<0$.

$\therefore\ f(x)$ has minimum value at $x=0$ and maximum value at $x=2$.

$\therefore\ The\ mode\ of\ this\ distribution\ is\ 2.$ ■

Example (19)

Let $f(x)=\begin{cases} 1 & ,0\leq x\leq 1 \\ 0\,, & otherwise \end{cases}$ be the probability density function of the random variable X.What is the mode of this distribution?

Solution

Since $f'(x) = 0$ for any real number. Therefore there are infinitely many modes. ■

Exercises (2-2)

1: Let $f(x) = \begin{cases} 3x^2, & for\ 0 \le x \le 1 \\ 0 & ,otherwise \end{cases}$ be the probability density function of the $random$ variable X. Find the median of this distribution.

2: What is the median of the distribution if its $probability\ density\ function\ of\ the\ random\ variable\ X\ is$

$$f(x) = \frac{1}{\pi(1+x^2)}, -\infty < x < \infty ?$$

3: Let $f(x) = \begin{cases} \frac{1}{b} e^{-\frac{x}{b}}, & if\ x \ge 0,\ b > 0 \\ 0 & ,otherwise \end{cases}$ be the probability density function of the random variable X.What is the 10^{th} percentile of this distribution?

4: Let $f(x) = \begin{cases} \frac{3x^2}{8}, & 0 \le x \le 2 \\ 0 & ,otherwise \end{cases}$ be the probability density function of the random variable X.Find the probability that X is greater than its 75^{th} percentile?

Percentiles for Discrete Random Variables

Definition (11)

Let $X_1, X_2, X_3, \dots, X_n$ be a random sample. The sample median is defined as:

$$M = \begin{cases} X_{\left(\frac{n+1}{2}\right)} & ,if\ n\ is\ odd\ number \\ \frac{1}{2}\left[X_{\left(\frac{n}{2}\right)} + X_{\left(\frac{n+2}{2}\right)}\right] & ,if\ n\ is\ even\ number \end{cases}$$

Definition (12)

The $100p^{th}$ sample percentile is defined as

$$\pi_p = \begin{cases} X_{([np])}, & if\ p < 0.5 \\ M & ,if\ p = 0.5\ , \\ X_{(n+1-[n(1-p)])} & ,if\ p > 0.5 \end{cases}$$

$where\ [b]\ denotes\ the\ number\ b\ rounded\ to\ the\ nearest\ integer.$

Example (20)

Let $X_1, X_2, X_3, \dots, X_{12}$ be a random sample of size 12. What is 65^{th} percentile of this sample?

Solution

$Here, p = 0.65 > 0.5$.

$\therefore \pi_p = X_{(n+1-[n(1-p)])}$. $Here\ n = 12$.

$\therefore n(1-p) = 12(1-0.65) = 12 \times 0.35 = 4.2$ & $[n(1-p)] = [4.2] = 4$.

$\therefore (n+1-[n(1-p)]) = 12+1-4 = 9$.

$\therefore \pi_{0.65} = X_9$. **Thus the** 65^{th} **percentile of the random sample** $X_1, X_2, X_3, \dots, X_{12}$ **is the** 9^{th} **order statistic.** ■