Independent Events

Definition (14): (Independent Events)

Two events A & B of a sample space S are called independent if and only if $P(A \cap B) = P(A)P(B)$.

$For\ example, if\ P(A) = \frac{2}{5}, P(B) = \frac{1}{2}\ \&\ P(A \cap B) = \frac{1}{5}, then\ A\ \&\ B\ are\ independent\ events.$

Theorem (10)

If $A\ \&\ B$ *are two independent events such that* $P(B) \neq 0$. Then $P(A \backslash B) = P(A)$.

Proof:

$$P(A \backslash B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A). \blacksquare$$

Theorem (11)

If $A\ \&\ B$ *are two independent events. Then*:

$(a)\ A\ \&\ B'\ are\ independent.\ (b)\ A'\ \&\ B\ are\ independent.$

$(c)\ A'\ \&\ B'\ are\ independent.$

Proof:

(a) $Since\ P(A) = P(A \cap B) + P(A \cap B') \rightarrow P(A \cap B') = P(A) - P(A \cap B) \rightarrow$

$P(A \cap B') = P(A) - P(A)P(B) = P(A)[1 - P(B)] = P(A)P(B')$.

Therefore A & B' are independent.

(b) (Home Work)

$(c)\ Since\ A' \cap B' = (A \cup B)'$. **Therefore**

$$P(A' \cap B') = P[(A \cup B)'] = 1 - P(A \cup B)$$
$$= 1 - [P(A) + P(B) - P(A \cap B)] = 1 - P(A) - P(B) + P(A)P(B)$$
$$= [1 - P(A)] - P(B)[1 - P(A)] = [1 - P(A)][1 - P(B)]$$
$$= P(A')P(B').$$

Therefore A' & B' are independent. $\blacksquare$

Example (10)

Tossing a coin three times. Let A be the event "there are more heads than tails", B be the event "the results of the first two tosses are the same" and C be the event "heads on the first toss". (a) Does A & B are independent? Why? (b) Does A & C are independent? Why? (c) Does B & C are independent? Why?

Solution

$The\ sample\ space\ S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$

$A = \{HHH, HHT, HTH, THH\}, B = \{HHH, HHT, TTH, TTT\},$

$C = \{HHH, HHT, HTH, HTT\}$.

$(a)\ A \cap B = \{HHH, HHT\}$.

$P(A) = \frac{4}{8} = \frac{1}{2}\ , P(B) = \frac{4}{8} = \frac{1}{2}\ , P(A \cap B) = \frac{2}{8} = \frac{1}{4}$.

$Since\ P(A)P(B) = P(A \cap B), therefore\ A\ \&\ B$ **are independent.**

$(b)\ A \cap C = \{HHH, HHT, HTH\}$.

$P(A) = \frac{1}{2}\ , P(C) = \frac{4}{8} = \frac{1}{2}\ , P(A \cap C) = \frac{3}{8}$.

$Since\ P(A)P(C) \neq P(A \cap C), therefore\ A\ \&\ C$ **are not independent.**

$(c)\ B \cap C = \{HHH, HHT\}$.

$P(B) = \frac{1}{2}\ , P(C) = \frac{1}{2}\ , P(B \cap C) = \frac{2}{8} = \frac{1}{4}$.

$Since\ P(B)P(C) = P(B \cap C), therefore\ B\ \&\ C\ are\ independent.$ ■

Theorem (12)

$If\ A\ \&\ B\ are\ mutually\ exclusive\ events\ such\ that\ P(A) \neq 0\ \&\ P(B) \neq 0$. **Then** $A\ \&\ B\ are\ not\ independent(\ dependent)$.

Proof:

Suppose $A\ \&\ B$ **are independent.** $Therefore\ P(A \cap B) = P(A)P(B)$.

Since $A\ \&\ B$ **are mutually exclusive, therefore** $A \cap B = \emptyset$.

Hence $P(A \cap B) = P(\emptyset) = 0 = P(A)P(B) \rightarrow either\ P(A) = 0\ or\ P(B) = 0$.

This is a contradiction. Then $A\ \&\ B$ **are not independent.** ■

Theorem (13)

$If\ A\ \&\ B\ independent\ events\ such\ that\ P(A) \neq 0\ \&\ P(B) \neq 0. Then\ A\ \&\ B\ are\ not\ mutually\ exclusive\ events.$

Proof

$Since\ A\ \&\ B\ are\ independent\ events, therefore\ P(A \cap B) = P(A)P(B)$.

$Suppose\ A\ \&\ B\ are\ mutually\ exclusive\ events, therefore\ P(A \cap B) = 0$

Hence $P(A)P(B) = 0 \rightarrow either\ P(A) = 0\ or\ P(B) = 0$.

$This\ is\ contradiction.$

$Then\ A\ \&\ B\ are\ not\ mutually\ exclusive\ events.$ ■

Definition (15)

Three events $A, B\ \&\ C$ **are said to be independent if and only if:**

$(a)\ P(A \cap B) = P(A)P(B)$. $(b)\ P(A \cap C) = P(A)P(C)$.

$(c)\ P(B \cap C) = P(B)P(C)$. $(d)\ P(A \cap B \cap C) = P(A)P(B)(P(C)$.

Example (11)

Rolling a die and a coin. Let A be the event that "tails and odd numbers", B be the event that "tails and prime numbers" and C be the event that "tails and even numbers". Does A, B & C are independent? Why?

Solution

The sample space $S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$.

$A = \{T1, T3, T5\}$, $\quad B = \{T2, T3, T5\}$, $\quad C = \{T2, T4, T6\}$

$A \cap B = \{T3, T5\}$, $\quad A \cap C = \emptyset$, $\quad B \cap C = \{T2\}$, $\quad A \cap B \cap C = \emptyset$.

$P(A) = P(B) = P(C) = \frac{3}{12} = \frac{1}{4}$. $P(A \cap B) = \frac{2}{12} = \frac{1}{6}$, $P(A \cap C) = 0$,

$P(B \cap C) = \frac{1}{12}$, $P(A \cap B \cap C) = 0$.

We notice that

$P(A \cap B) \neq P(A)P(B), P(A \cap C) \neq P(A)P(C), P(B \cap C) \neq P(B)P(C)$ &

$P(A \cap B \cap C) \neq P(A)P(B)P(C)$. **Then**

A, B & C ***are not independent.*** ■

Theorem (14)

If A, B & C are independent events. Then

(a) A & $(B \cup C)$ are independent events.

(b) A & $(B \cap C)$ are independent events.

Proof

$$(a)\ P[A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$$
$$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$$
$$= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C)$$
$$= P(A)[P(B) + P(C) - P(B)P(C)]$$
$$= P(A)[P(B) + P(C) - P(B \cap C)] = P(A)P(B \cup C).$$

Then A & $(B \cup C)$ are independent events.

(b) $P[A \cap (B \cap C)] = P(A \cap B \cap C) = P(A)P(B)P(C) = P(A)P(B \cap C)$.

Then A & $(B \cap C)$ are independent events. ■

Exercises (1-6)

1: Let A & B be independent events with $P(A) = P(B)$ & $P(A \cup B) = 0.5$. Find $P(A)$.

2: Let A & B *be independent events with* $P(A \cap B) = 0.16$ & $P(A \cup B) = 0.64$. Find $P(A)$ & $P(B)$.

3: Let A & B be independent events such that the probability that at least one

$of\ them\ occurs\ is\ \frac{1}{3}\ and\ the$ **probability that** A **occurs but** B **does not** $occur\ is\ \frac{1}{9}$**. Calculate** $P(B)$.

4: $Let\ the\ event\ A = \{a\ family\ has\ childern\}\ and\ the\ event$ $B = \{a\ family\ has\ at\ most\ one\ boy\}$. **Show that** $A\ \&\ B$ **are independent** $if\ the$ **family has three children. If the family has two children determine** $whether\ the\ events\ A\ \&\ B\ independent\ or\ not$.

5: Rolling 2 dice. Let A **be the event that the sum of two faces greater than 6,** $B\ be\ the$ **event that the sum of two faces greater than 8 and** C **be the event** $that\ the$ **sum of two faces greater than 10. Does** $A, B\ \&\ C$ **independent? Explain your answer.**