Conditional Probability

Often it is required to find the probability of an event B under the condition that an event A occurs. This probability is called the conditional probability of B $given$ A, denoted by $P(B\backslash A)$.It is defined by

$$P(B\backslash A)=\frac{P(A\cap B)}{P(A)}\ ,P(A)\neq 0$$

Similarly, the conditional probability $of\ A\ given\ B,\ denoted\ by\ P(A\backslash B)$, is defined by

$$P(A\backslash B)=\frac{P(A\cap B)}{P(B)}\ ,P(B)\neq 0$$

Note (5)

The conditional probability $P(A\backslash B)$ satisfies all three axioms of a probability measure:

$(a) P(A\backslash B)\geq 0\ for\ all\ event\ A.$

$(b)\ P(B\backslash B)=1.$

(c) If $A_1, A_2, \ldots$ $are\ mutually\ exclusive\ events$,

$$P(\cup_{k=1}^{\infty} A_k\backslash B)=\sum_{k=1}^{\infty} P(A_k\backslash B).$$

Note (6)

If A & B are two events such that $P(A)\neq 0$ & $P(B)\neq 0$, then

$$P(A\cap B)=P(A)P(B\backslash A)=P(B)P(A\backslash B).$$

Note (7)

If B is an event in S such that $P(B)\neq 0$, then $P(S\backslash B)=1$.

Note (8)

Let A_1, A_2 & B are events in S such that A_1 & A_2 are disjoint events & $P(B)\neq 0$. Then $P(A_1\cup A_2\backslash B)=P(A_1\backslash B)+P(A_2\backslash B)$.

Note (9)

If A &B are two events in S such that $A\cap B=\emptyset$ & $P(B)\neq 0$, then $P(A\backslash B)=0$.

Note (10)

If A & B are two events in S such that $A\subset B$ & $P(B)\neq 0$, then

$$P(A\backslash B)=\frac{P(A)}{P(B)}.$$

Note (11)

If A & B are two events in S such that $B\subset A$ & $P(B)\neq 0$, then $P(A\backslash B)=1$.

Note (12)

For any three events A_1, A_2 & A_3. Then

$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2 \backslash A_1)P(A_3 \backslash A_1 \cap A_2)$.

Example (5)

In a class 25 boys and 10 girls, if we choose 3 names at random. What is the probability that: (a) $all\ boys$? (b) $all\ girls$?

Solution

We denote the boys by B_1, B_2, B_3 and the girls by G_1, G_2, G_3 .

(a) $P(B_1 \cap B_2 \cap B_3) = P(B_1)P(B_2 \backslash B_1)P(B_3 \backslash B_1 \cap B_2)$

$$= \left(\frac{25}{35}\right)\left(\frac{24}{34}\right)\left(\frac{23}{33}\right) = 0.3514$$

(b) $P(G_1 \cap G_2 \cap G_3) = P(G_1)P(G_2 \backslash G_1)P(G_3 \backslash G_1 \cap G_2)$

$$= \left(\frac{10}{35}\right)\left(\frac{9}{34}\right)\left(\frac{8}{33}\right) = 0.01833. \blacksquare$$

Example (6)

Consider all families "with two children (not twins)". Let E be the event "two boys" and F be the event "at least one boy". Calculate $P(E \backslash F)$.

Solution

$We\ denote\ the\ boy\ by\ B\ and\ the\ girl\ by\ G$. **The sample space is** $S = \{BB, BG, GB, GG\}$, $the\ events\ E = \{BB\}$, $F = \{BB, BG, GB\}$ & $E \cap F = \{BB\}$. **Therefore** $P(E) = \frac{1}{4}$, $P(F) = \frac{3}{4}$ & $P(E \cap F) = \frac{1}{4}$.

Then $P(E \backslash F) = \frac{P(E \cap F)}{P(F)} = \frac{1}{3}. \blacksquare$

Example (7)

Assume that a certain school contains equal number of female and male students. 5% of the male population is a football players. Find the probability that a randomly selected student is a football player male.

Solution

We denote the male by M & $the\ football\ player\ by\ F$. We want to calculate $P(M \cap F)$.

Then $P(M \cap F) = P(M)P(F \backslash M) = \left(\frac{1}{2}\right)\left(\frac{5}{100}\right) = \frac{1}{40}. \blacksquare$

Exercises (1-4)

1: Let A & B are two events such $that$ $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ & $P(A \cap B) = \frac{1}{4}$.

Find: $P(A \backslash B)$, $P(B \backslash A)$, $P(A \cup B)$ & $P(A' \backslash B')$.

2: Let A & B are two events $such\ that\ P(A \cup B) = \frac{3}{4}$, $P(B) = \frac{5}{8}$ & $P(A) = \frac{3}{8}$.

Find $P(A \backslash B)$ & $P(B \backslash A)$.

3: If we randomly choose two television sets in succession from a shipment of 240 television sets of which 15 are defective. What is the probability that they will be both defective?

4: A box of fuses contains 20 fuses, of which 5 are defective. If 3 of fuses are selected at random and removed from a box in succession without replacement. What is the probability that all three fuses are defective?

Definition (13): (Partition)

Let S be a set and let $\mathbb{P} = \{A_i\}_{i=1}^{n}$ be a collection of $subsets\ of\ S$. Then $\mathbb{P}$ is called a partition of S if: $S = \cup_{i=1}^{n} A_i$ and $A_i \cap A_j = \emptyset\ for\ i \neq j$.

Theorem (8): (Total Probability Theorem)

If the events $\{A_i\}_{i=1}^{n}$ constitute a partition of the sample space S & $P(A_i) \neq 0$, $i = 1, 2, \dots, n$. Then for any event $B\ in\ S$: $P(B) = \sum_{i=1}^{n} P(A_i) P(B \backslash A_i)$.

Proof:

Since $B = B \cap S = B \cap (A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n)$.

Then $B = (B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3) \cup \dots \cup (B \cap A_n)$.

But $B \cap A_i$, $i = 1, 2, 3, \dots, n$ are mutually exclusive.

$Then\ P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(A_i) P(B \backslash A_i)$. ■

Theorem (9): (Bayes' Theorem)

If the events $\{A_i\}_{i=1}^{n}$ constitute a partition of the sample space S & $P(A_i) \neq 0$, $i = 1, 2, 3, \dots, n$ .Then for any event B in S such that $P(B) \neq 0$,

$$P(A_i \backslash B) = \frac{P(A_i) P(B \backslash A_i)}{\sum_{i=1}^{n} P(A_i) P(B \backslash A_i)}$$

Proof:

By total probability theorem, we have

$P(B) = \sum_{i=1}^{n} P(A_i) P(B \backslash A_i)\ but\ P(B) = \frac{P(A_i \cap B)}{P(A_i \backslash B)}$. Therefore

$\frac{P(A_i \cap B)}{P(A_i \backslash B)} = \sum_{i=1}^{n} P(A_i) P(B \backslash A_i) \rightarrow P(A_i \backslash B) = \frac{P(A_i \cap B)}{\sum_{i=1}^{n} P(A_i) P(B \backslash A_i)}$.

Then

$P(A_i \backslash B) = \frac{P(A_i) P(B \backslash A_i)}{\sum_{i=1}^{n} P(A_i) P(B \backslash A_i)}$. ■

Example (8)

Two boxes contains balls. Box1 has 3 red & 5 white balls. Box2 has 2 red & 4 white balls. Choose a box, then choose a ball from the chosen box.

(a)Find the probability that a white ball is chosen.
(b)If a red ball is chosen, find the probability that it is from box2.

Solution

Let A_1 denotes to the box1, A_2 denotes to the box2 and B denotes to the chosen ball. Let w denotes to the white ball and R denotes to the red ball.

$$(a)\ P(B)_w = \sum_{i=1}^{2} P(A_i)P(B\backslash A_i) = P(A_1)P(B\backslash A_1) + P(A_2)P(B\backslash A_2)$$

$$= \left(\frac{1}{2}\right)\left(\frac{5}{8}\right) + \left(\frac{1}{2}\right)\left(\frac{4}{6}\right) = \frac{5}{16} + \frac{1}{3} = \frac{31}{48}.$$

$$(b)\ P(A_2\backslash B)_R = \frac{P(A_2)P(B\backslash A_2)}{\sum_{i=1}^{2} P(A_i)P(B\backslash A_i)} = \frac{P(A_2)P(B\backslash A_2)}{P(A_1)P(B\backslash A_1) + P(A_2)P(B\backslash A_2)}$$

$$= \frac{(\frac{1}{2})(\frac{2}{6})}{\left(\frac{1}{2}\right)\left(\frac{3}{8}\right)+(\frac{1}{2})(\frac{2}{6})} = \frac{\frac{1}{6}}{\frac{3}{16}+\frac{1}{6}} = \frac{8}{17}.. \blacksquare$$

Example (9)

Three machines M_1, M_2 & M_3 produce glasses where M_1 produces 20% of glasses, M_2 produces 30% of glasses and M_3 produces 50% of glasses. Also, 1% of glasses produced by M_1 is defective, 2% of glasses produced by M_2 is defective, and 3% of glasses produced by M_3 is defective. Choose a glass, then find: (a) the probability that the glass is produced by M_3 if it is defective, (b) the probability that the glass is defective

Solution

Let g denotes to the chosen glass, and d denotes the defective glass.

$$(a)\ P(M_3\ \backslash \mathrm{g})_d = \frac{P(M_3)P(g\backslash M_3)}{\sum_{i=1}^{3} P(M_i)P(g\backslash M_i)}$$

$$= \frac{\mathrm{P}(M_3)P(g\backslash M_3)}{P(M_1)P(g\backslash M_1) + P(M_2)P(g\backslash M_2) + P(M_3)P(g\backslash M_3)}$$

$$= \frac{(\frac{50}{100})(\frac{3}{100})}{\left(\frac{20}{100}\right)\left(\frac{1}{100}\right)+\left(\frac{30}{100}\right)\left(\frac{2}{100}\right)+(\frac{50}{100})(\frac{3}{100})} = \frac{15}{23}.$$

$$(b)\ P(g)_d = \sum_{i=1}^{3} P(M_i)P(g\backslash M_i)$$

$$= P(M_1)P(g\backslash M_1) + P(M_2)P(g\backslash M_2) + P(M_3)P(g\backslash M_3)$$

$$= \left(\frac{20}{100}\right)\left(\frac{1}{100}\right) + \left(\frac{30}{100}\right)\left(\frac{2}{100}\right) + \left(\frac{50}{100}\right)\left(\frac{3}{100}\right) = \frac{23}{1000}. \blacksquare$$

Exercises (1-5)

1: Two boxes contains balls. Box1 has 3 red, 1 white & 2 black balls. Box2 has 1 red & 2 black balls. Choose a box, then choose a ball from the chosen box. Find the probability that: (a) the ball is chosen from the box2 if the ball is red, and (b) a white ball is chosen.

2: A company produces electric relays has three manufacturing plants producing 50%, 30% & 20% respectively of its product. Suppose that the probabilities that a relay manufactured by these plants is defective are 0.02, 0.05 & 0.01 respectively.

(a)If a relay is selected at random from the output of the company, what is the probability that it is defective?

(b) If a relay selected at random is found to be defective, what is the probability that it was manufactured by plant 2?

3: Two boxes contains balls. Box1 contains 7 green & 4 white balls. Box2 contains 3 green and 10 yellow balls. The boxes are arranged so that the probability of selecting box1 $is\ \frac{1}{3}$ **and the probability of selecting box2** $is\ \frac{2}{3}$**. Choose a box, then choose a ball from the chosen box. Find the probability that: (a) the ball is chosen is green, and (b) the ball is chosen from box1 if it is green.**