Triple Scalar or Box Product

Definition (15):

The product $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$ is called the triple scalar product of $\boldsymbol{u}, \boldsymbol{v}$ & $\boldsymbol{w}$ and is calculating as:

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Note:

We can find the volume of the parallelepiped determined by the vectors $\boldsymbol{u}, \boldsymbol{v}$, & $\boldsymbol{w}$ as:

$$\boldsymbol{Volume} = |(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}|.$$

Example (20):

Find the volume of the box determined by $\boldsymbol{u} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{v} = -2\boldsymbol{i} + 3\boldsymbol{k}$ and $\boldsymbol{w} = 7\boldsymbol{j} - 4\boldsymbol{k}$.

Solution:

$$\boldsymbol{Volume} = |(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}| = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = 23\ \boldsymbol{cube\ units}.$$

Note:

We can show that the three vectors $\boldsymbol{u}, \boldsymbol{v}$, & $\boldsymbol{w}$ are coplanar if $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = 0$.

Example (21):

Determine whether the points $A(1,1,1), B(-1,0,4), C(0,2,1)$ & $D(2,-2,3)$ *are coplanar*.

Solution:

First, we determine three vectors as

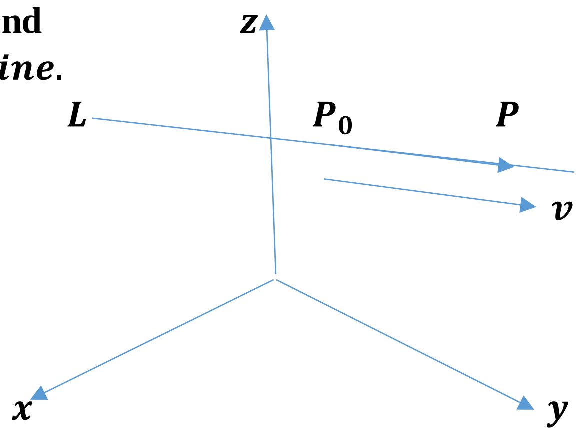
$$\overrightarrow{AB} = -2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}\ , \overrightarrow{AD} = \boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}\ \&\ \overrightarrow{BC} = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}\ .$$

$$\therefore \left(\overrightarrow{AB} \times \overrightarrow{AD}\right) \cdot \overrightarrow{BC} = \begin{vmatrix} -2 & -1 & 3 \\ 1 & -3 & 2 \\ 1 & 2 & -3 \end{vmatrix} = 0\ .$$

$\therefore$The given points are coplanar.

5: Lines and Planes in Space:

In space a line is determined by a point and *a vector given the direction of the line.*
Suppose that L is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to the vector $v = v_1 i + v_2 j + v_3 k$.
Then L is a set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to v .
$\therefore \overrightarrow{P_0P} = tv$ *for some scalar parameter* t .



$\therefore (x - x_0)i + (y - y_0)j + (z - z_0)k = t(v_1 i + v_2 j + v_3 k)$
which can be written as:

$xi + yj + zk = x_0 i + y_0 j + z_0 k + t(v_1 i + v_2 j + v_3 k)$ .... (1)

If $r(t)$ is the position vector of a point $P(x, y, z)$ on the line and r_0 is the position vector of the *point* $P_0(x_0, y_0, z_0)$, then Equation (1) gives the following vector form for the equation of a line in space.
Hence, a vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to v is:

$r(t) = r_0 + tv \,, -\infty < t < \infty$ ... ... (2)

Parametric Equations for a Line:

Equating the corresponding components of the two sides of Equation (1) gives three scalar equations involving the parameter t

$x = x_0 + tv_1 \,, y = y_0 + tv_2 \,, z = z_0 + tv_3$ .... (3)

Example (22):

Find parametric equations of the line through $(-2, 0, 4)$ parallel to $v = 2i + 4j - 2k$.

Solution:

The parametric equations of the line is

$$x = x_0 + tv_1 \,, y = y_0 + tv_2 \,, z = z_0 + tv_3$$

$$\therefore \quad x = -2 + 2t \,, y = 4t, z = 4 - 2t \,.$$

Example (23):

Find parametric equations for the line through the points $P(-3, 2, -3)$ & $Q(1, -1, 4)$.

Solution:

The vector $\overrightarrow{PQ} = 4i - 3j + 7k$ is parallel to the line.

The parametric equations of the line is

$x = x_0 + tv_1 \,, y = y_0 + tv_2 \,, z = z_0 + tv_3$

$\therefore x = -3 + 4t \,, y = 2 - 3t, z = -3 + 7t$.

Note:

The parameterizations are not unique.

Note:

To parameterize a line segment joining two points:

1: Parameterize the line through the points.

2: Find $the\ t - values\ for$ the end points and restrict t to line in the closed interval bounded by these values.

This line equations together with this added restriction parametrize the segment.

Example (24):

Parametrize the line segment joining the points $P(-3, 2, -3)$ & $Q(1, -1, 4)$.

Solution:

The parametric equations for a line is

$x = -3 + 4t \,, y = 2 - 3t, z = -3 + 7t$.

$\therefore (x, y, z) = (-3 + 4t, 2 - 3t, -3 + 7t)$

$\therefore (-3 + 4t, 2 - 3t, -3 + 7t) = (-3, 2, -3) \rightarrow t = 0$.

$(-3 + 4t, 2 - 3t, -3 + 7t) = (1, -1, 4) \rightarrow t = 1$.

$\therefore 0 \le t \le 1$.

The parametrization for a line segment is

$x = -3 + 4t \,, y = 2 - 3t, z = -3 + 7t \,;\ 0 \le t \le 1$.

Note:

The vector form for a line in space is given by

$$r(t) = r_0 + tv = r_0 + t|v|\frac{v}{|v|} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (4)$$

$Initial \qquad Time \quad Speed \qquad Direction$

$\boldsymbol{Position}$

Equation (4) means:

The position of the particle at time t is its initial position plus its distance moved $(speed \times time)$ in the direction $\frac{v}{|v|}$ of its straight line motion.

Example (25):

A helicopter is to fly directly from helipad at the origin in the direction of the point $(1, 1, 1)$ at a speed of 60.What is the position of the helicopter $after$ $10\ sec$?

Solution:

We place the origin at the starting position (helipad) of the helicopter. Then the unit vector is

$u = \frac{1}{\sqrt{3}} i + \frac{1}{\sqrt{3}} j + \frac{1}{\sqrt{3}} k$ *gives the flight direction of the helicopter.*

The position of the helicopter at any time t is

$$r(t) = r_0 + t(speed)u = O + t(60)\left(\frac{1}{\sqrt{3}} i + \frac{1}{\sqrt{3}} j + \frac{1}{\sqrt{3}} k\right) = 20\sqrt{3}\, t(i + j + k)$$

When $t = 10\ sec$,

$$r(10) = 200\sqrt{3}\,(i + j + k) = \langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3} \rangle \ .$$

After 10 *sec of flight from the origin toward* $(1, 1, 1)$**, the helicopter is located at the point $(200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3})$ in space. It has traveled a distance of 600, which is the length of the vector $r(10)$.**