

Sheet 2

Q1:- Define a metric d_1 on $C([a, b])$ by

$$d_1(f, g) = \int_a^b |f(x) - g(x)| dx, f, g \in C([a, b]).$$

Show that d_1 is indeed metric on $C([a, b])$. Dose the function $\|f\|_1 = \int_a^b |f(x)| dx$ define a norm on $C([a, b])$.

Q2:- Consider linear vector space

$$C^1([a, b]) = \{f : [a, b] \rightarrow \mathbb{R} \mid f \text{ and } \dot{f} \text{ are continuous on } [a, b]\}$$

a) Verify that

$$\|f\|_{\infty, 1} := \max_{x \in [a, b]} |f(x)| + \max_{x \in [a, b]} |\dot{f}(x)|$$

defines a norm on $C^1([a, b])$.

b) Let $f(x) = \sin(\pi x)$. Find $\|f\|_{\infty, 1}$ in $C^1([0, 1])$.

Q3:- Let A, B, C, D, E be subsets of the Euclidean space \mathbb{R}^2 . Find their boundary, their interior, and their exterior. Conclude from here whether these sets are open, closed, or neither.

- $A = \{x \in \mathbb{R}^2 \mid d_2(x, x_0) \leq 2\}$, where $x_0 \in \mathbb{R}^2$.
- $B = \mathbb{R} \times [a, b)$, where $a, b \in \mathbb{R}$, $a < b$.
- $C = (a, b)^2 = (a, b) \times (a, b)$, where $a, b \in \mathbb{R}$, $a < b$.
- $D = \{a\} \times [b, c)$, where $a, b, c \in \mathbb{R}$, $b < c$.
- $E = \{a\} \times \{b, c\}$, where $a, b, c \in \mathbb{R}$, $b \neq c$.

Remark: In this problem, it is sufficient to sketch the set and to give the correct answers without justification.

Q4:- In the Euclidean space \mathbb{R}^3 , find the set

$$A = \bigcap_{n=1}^{\infty} \left[-1 - \frac{1}{n}, 2 + \frac{1}{n} \right] \times \left[-\frac{1}{n^2}, \frac{1}{n^2} \right] \times [0, e^{-n}].$$

Is this set open, closed, or neither?

Q5:- Let (X, d) be a discrete metric space and let $x_0 \in X$ be a point. Describe the open ball $B_r(x_0)$ where (a) $0 < r \leq 1$. (b) $r > 1$. (c) $r > 1$ and $r \neq 0$

Q6:- Consider the following sets

1. $A_n = [\frac{1}{n}, \infty)$ closed for each $n = 1, 2, \dots$. However $\bigcup_{n=1}^{\infty} A_n =$
2. $B_n = [\frac{1}{n}, 1 - \frac{1}{n}]$ closed for each $n = 1, 2, \dots$. However $\bigcup_{n=1}^{\infty} B_n =$
3. $C_n = [\frac{1}{n}, 1]$ closed for each $n = 1, 2, \dots$. However $\bigcup_{n=1}^{\infty} C_n =$
4. $D_n = [-n, n]$ closed for each $n = 1, 2, \dots$. However $\bigcup_{n=1}^{\infty} D_n =$

What you can conclude from the union of the above sets?