Infinite Sequences:

A sequence is a list of numbers $a_1, a_2, a_3, \cdots, a_n, \cdots$ in a *given order*. For example the sequence $3, 6, 12, 24, 48, \ldots, 3(2^{n-1}), \ldots$ has 1st term $a_1 = 3$, 2nd term $a_2 = 6$, 3rd term $a_3 = 12$,…., the nth term $a_n = 3(2^{n-1})$.

Definition (1): (Sequence)

An infinite sequence of numbers is a function whose domain is the set of positive integers.

How can described Sequences?

1: By writing rules that specify their terms, such as $a_n = \sqrt{n}$, $b_n = \frac{1}{n}$.

2: By listing terms:

$$\{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \cdots, \sqrt{n}, \cdots\}, \{b_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots . \frac{1}{n}, \cdots\right\}.$$

3: By writing its rules as: $\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}$, $\{b_n\} = \{\frac{1}{n}\}_{n=1}^{\infty}$.

Representation Sequences Graphically

There are two ways:

1: Marks the first few points $a_1, a_2, a_3, \cdots, a_n, \cdots$ on the real axis.

2: Shows the graph of the function defining the sequence.

The function is defined only on integer inputs, and located at $(1, a_1)$, $(2, a_2)$,
$(3, a_3), \cdots, (n, a_n), \cdots$.

Example (1): Graph the following sequences:

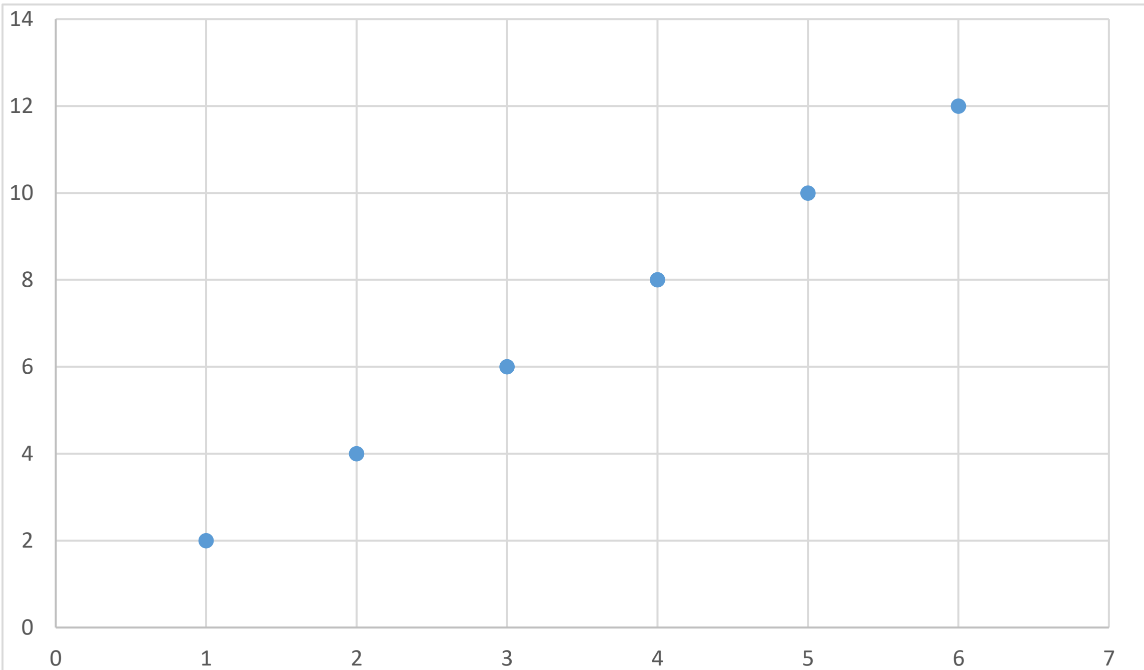
A: $\{a_n\} = \{2n\}$

Solution:

(1):

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
0	2	4	6	8	10	12	14	16

(2):



B: $\{b_n\} = \{n^2\}$

Solution: (H.W.)

Convergence and Divergence

Definition (2):

The sequence $\{a_n\}$ converges to the number L if for every $\epsilon > 0$ there is an integer N such that $|a_n - L| < \epsilon$ whenever $n > N$.

If no such number L exists, we say that $\{a_n\}$ diverges.

Note: If $\{a_n\}$ converges to L, we write $\lim_{n \to \infty} a_n = L$ or simply $a_n \to L$ and call L the limit of the sequence.

Definition (3):

The sequence $\{a_n\}$ diverges to $(+\infty)$ if for every number M there is an integer N such that for all $n > N, a_n > M$. If this condition holds, we write $\lim_{n\to\infty} a_n = \infty$ or $a_n \to \infty$.

Definition (4):

The sequence $\{a_n\}$ diverges to $(-\infty)$ if for every number m there is an integer N such that for all $n > N, a_n < m$. If this condition holds, we write $\lim_{n\to\infty} a_n = -\infty$ or $a_n \to -\infty$.

Calculating Limits of Sequences

Theorem (1):

Let $\{a_n\}$ *and* $\{b_n\}$ be sequences of real numbers and A & B are real numbers such that $\lim_{n\to\infty} a_n = A$ & $\lim_{n\to\infty} b_n = B$. Then the following rules hold.

1: $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$.

2: $\lim_{n\to\infty} k\,a_n = k(\lim_{n\to\infty} a_n) = kA$. (($k$ is a constant))

3: $\lim_{n\to\infty}(a_n . b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = AB$.

4: $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$ *if* $B \neq 0$.

Example (2): Find the following limits:

1: $\lim_{n\to\infty}\left\{\frac{2}{n}\right\} = 2\left(\lim_{n\to\infty}\{\frac{1}{n}\}\right) = 2(0) = 0$.

2: $\lim_{n\to\infty}\left(\frac{n-1}{2n}\right) = \lim_{n\to\infty}\left(\frac{n}{2n}\right) - \lim_{n\to\infty}\left(\frac{1}{2n}\right) = \lim_{n\to\infty}\left(\frac{1}{2}\right) - \frac{1}{2}\left(\lim_{n\to\infty}\frac{1}{n}\right) = \frac{1}{2}$.

3: $\lim_{n\to\infty}\left(\frac{4n+6n^2}{2n^2+5}\right) = \lim_{n\to\infty}\frac{\frac{4}{n}+6}{2+\frac{5}{n^2}} = \frac{0+6}{2+0} = 3$.

Notes:

1: There are two divergent sequences but their sum converge.
For example, $\{a_n\} = \{1, 2, 3, \dots, n, \dots\}$ and $\{b_n\} = \{-1, -2, -3, \dots, -n, \dots\}$ are divergent sequences but the sequence $\{c_n\} = \{a_n + b_n\} = \{0, 0, 0, \dots, 0, \dots\}$ converges to the number zero.

2: If the sequence $\{a_n\}$ diverges and the number $k \neq 0$ then the sequence $\{ka_n\}$ diverges also.