Ministry of High Education and Scientific Research
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Finite Difference Method (FDM)

A paper Submitted to
The College of Science Basrah University
In Partial Fulfillment of the Requirements for the
Degree of Bachelor of Science in
Mathematics

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2019 1440

بسم الله الرحمن الرحيم

﴿ وَعَلَّمَكَ مَا لَمْ تَكُنْ تَعْلَمُ وَكَانَ فَضْلُ اللهِ عَلَيْكَ عَظِيماً ﴾

صدق الله العليّ العظيم

(النساء ١١٣)

الإهداء

بدانا بأكثر من يد وقاسينا أكثر من هم وعانينا الكثير من الصعوبات وها نحن اليوم والحمد لله نطوي سهر الليالي وتعب الإيام وخلاصة مشوارنا بين دفتي هذا العمل المتواضع.

إلى الينبوع الذي لا يمل العطاء إلى من حاكت سعادتي بخيوط منسوجة من قلبها إلى

(والدتي العزيزة)

إلى من سعى وشقى لأنعم بالراحة والهناء الذي علمني أن أرتقي سلم الحياة بحكمة وصبر

(والدي العزيز)

إلى من حبهم يجري في عروقي ويلهج بذكراهم فؤادي

(أخي واخواتي)

الى من علمتني وعانت الصعاب لأصل الى ما أنا فيه الى من تتسابق الكلمات معبرة عن مكنون ذاتها (جدتي الغالية)

الى اجمل نعمة اعطاها الله لي والى من وقف بجانبي لأصل الى هذه المرحلة الى نور عيني ومصباح طريقي (زوجي العزيز)

إلى من صاغوا لنا علمهم حروفا ومن فكرهم منارة تنير لنا سيرة العلم والنجاح إلى أساتذتنا الكرام واخص منهم من علمتني حروفا من ذهب وكلمات من درر وعبارات من اسمى عبارات العلم مشرفتي العزيزة م.م بشائر كاظم

Abstract

In this study, we discussed the finite difference method, it is techniques used to solve differential equations. The first chapter explains the mechanism of using the finite difference method for partial differential equation (heat equation) by applying each of finite difference methods as an explanatory example and showed a table with the results we obtained.

In the second chapter, we discussed the problem of different equation (1-D) with boundary condition. The results were compared between the analytic solution and numerical result and the results for the two methods were almost identical.

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Introduction

Partial Differential Equation **(PDE)** is an equation involving an Unknown functions of two or more variables and some partial derivatives special cases of two dimensional second-order equation.

$$a\frac{d^2\varphi}{dx^2}+ b\frac{d^2\varphi}{dxdy}+ c\frac{d^2\varphi}{dy^2}+ d\frac{d^2\varphi}{dx}+ e\frac{d^2\varphi}{dy}+ f\varphi + g = 0,$$

where a , b , c , d , e , f and g may be function of the independent variables x and y and of the dependent variable φ.

In applications the function usually represent physical quantities, the derivatives represent their rates of change and the equation defines a relations are extremely common , the differential equations play a prominent role in many disciplines including engineering, physics, economics and biology.

Generally, these equation are classified into three types of equations, elliptic , when $b^2 - 4ac < 0$, Parabolic when $b^2 - 4ac = 0$ and hyperbolic when $b^2 - 4ac > 0$.

Also, the equation can be classified on the order and degree and it way be Linear or Non-Linear [For more details [1]].

The mathematical model is complete and for practical applications, the solution of these differential equation is very important.

Therefore, many researchers concentrated on the solution of the differential equations.

The solutions are divided into two types: the first is analytical, which is used to find the exact solution for a specific problem and this includes separation of variables and the transformation such as a Laplace transform, Fourier transform. The other is numerically.

There will be times when solving the exact solution for the equation it will be unavailable, since the complex nature of problem. At the present time, there are many approaches that can be employed to solve the differential equations and found the approximation solution of these equation.

Basically, the main methods are like finite difference method **(FDM)**, finite volume method **(FVM)** and finite element method **(FEM)**.

However, as presented in numerous paper of numerical method, the finite difference method has emerged as available tool for the solution of partial differential equation .

Chapter 1

Finite difference method

1.1 Introduction

The finite difference approximation derivatives are one of the simplest and of the oldest methods to solve differential equation. It was already known by L .Euler **(1707-1783)** is one dimension of space and was probably extended to dimension two by C. Runge **(1856-1927)**. The advent of finite difference techniques in numerical application began in the early 1950s and their development was stimulated by the emergence of computes that offered a convenient frame work for dealing with complex problem of science and technology .

1.2 Finite difference approximation to derivatives

As a first step in developing a method of calculating the values of u at each interior grid point, the space and time derivatives of u at the (i , j) grid point must be expressed in terms of values of u at nearby grid points (see Figure 1). Taylor series expansions of u about the (i , j) grid point will be used in this process.

$$u(x + h , t) = u(x , t) + h\, u'(x , t) + \frac{h^2}{2!}\, u''(x , t) + \frac{h^3}{3!}\, u'''(x , t) \qquad (1.1)$$

$$u(x + h , t) = u(x , t) + h\, u'(x , t) + O(h^2) \qquad (1.2)$$

with this equation (1.2) the forward difference approximation for $\frac{\partial u}{\partial x}$

$$\frac{\partial u}{\partial x} = \frac{u(x+h ,t) - u(x ,t)}{h} + O(h) \qquad (1.3)$$

with a leading error of O(h).

With this notation the forward difference approximation for $\frac{\partial u}{\partial t}$.

$$\frac{\partial u}{\partial t} = \frac{u(x,t+k) - u(x,t)}{K} + O(k) \tag{1.4}$$

with a leading error of O(k).

The Taylor series for u(i – 1 , j) about (i , j) is

$$u(x-h,t) = u(x,t) - h\,u'(x,t) + \frac{h^2}{2!}u''(x,t) - \frac{h^3}{3!}u'''(x,t) \tag{1.5}$$

$$u(x-h,t) = u(x,t) - h\,u'(x,t) + O(h^2) \tag{1.6}$$

with this equation (1.6) back ward difference approximation for $\frac{\partial u}{\partial t}$

$$\frac{\partial u}{\partial x} = \frac{u(x,t) - u(x-h,t)}{h} + O(h) \tag{1.7}$$

with a leading error of O(h)

with this notation the backward difference approximation for $\frac{\partial u}{\partial t}$

$$\frac{\partial u}{\partial t} = \frac{u(x,h) - u(x,t-k)}{k} + O(k) \tag{1 . 8}$$

Addition of these (1.2) and (1.6) (giving central differences approximation to $\frac{\partial^2 u}{\partial x^2}$

$$u(x+h,t) + u(x-h,t) = u(x,t) + h\,u'(x,t) + \frac{h^2}{2!}u''(x,t) + u(x,t) - h\,u'(x,t) + \frac{h^2}{2!}u''(x,t) + O(h^4) \tag{1 . 9}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} O(h^2). \tag{1.10}$$

Similarly

$$\frac{\partial^2 u}{\partial t^2} = \frac{u(x,t+k) - 2u(x,t) + u(x,t-k)}{k^2} + O(k^2) \tag{1.11}$$

approximation to $\frac{\partial u}{\partial x}$ is to subtract equations (1.6) from equation (1.2), where all derivatives are evaluated at (i , j) , is

$$u(x+h,t) - u(x-h,t) = u(x,t) + h\,u'(x,t) - u(x,t) + h\,u'(x,t) + o(h^2).$$

$$\frac{\partial u}{\partial x} = \frac{u(x+h,t) - u(x-h,t)}{2h} + O(h) \tag{1.12}$$

(centeral $\frac{\partial u}{\partial x}$) , with a leading error of O(h).

$$\frac{\partial u}{\partial t} = \frac{u(x,t+k) - u(x,t-k)}{2k} + O(k) \tag{1.13}$$

(centeral $\frac{\partial u}{\partial t}$), with a leading error of O(k).

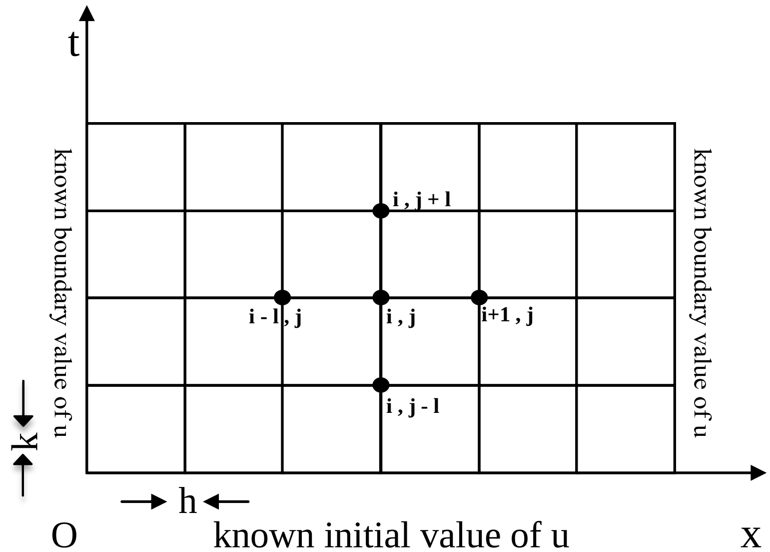


Figure (1)

There are methods of finite difference for solving the differential equations. [see [2] , [3] , [4]].

1.3 The explicit method

Is one of the methods used in numerical analysis for obtaining numerical approximation to solution of time –dependent ordinary and partial differential equations and from advantage the solution algorithm is simple to set up.

The solution $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ of heat equation.

In this approaches using a forward difference at time t and a second order central difference for the space derivatives.

$$\frac{ui\,,j+1-ui\,,j}{k} = \frac{ui+l\,,j-2ui\,,j+ui-1\,,j}{h^2}$$

$u_{i\,,\,j\,+\,1} - u_{i\,,\,j} = \frac{k}{h^2}\,(u_{i\,+\,1\,,\,j} - 2u_{i\,,\,j} + u_{i\,-\,1\,,\,j})$

suppose $\qquad r = \frac{\Delta\, t}{(\,\Delta\, x\,)^2} = \frac{k}{h^2}$

$u_{i\,,\,j\,+\,1} - u_{i\,,\,j} = \frac{k}{h^2}\,(u_{i\,+\,1\,,\,j} - 2u_{i\,,\,j} + u_{i\,-\,1\,,\,j})$

$$u_{i\,,\,j\,+\,1} = r\,u_{i\,+\,1\,,\,j} + (\,1 - 2r)\,u_{i\,,\,j} + r\,u_{i\,-\,1\,,\,j} \qquad (1.14)$$

This is an explicit method for solving the heat equation.

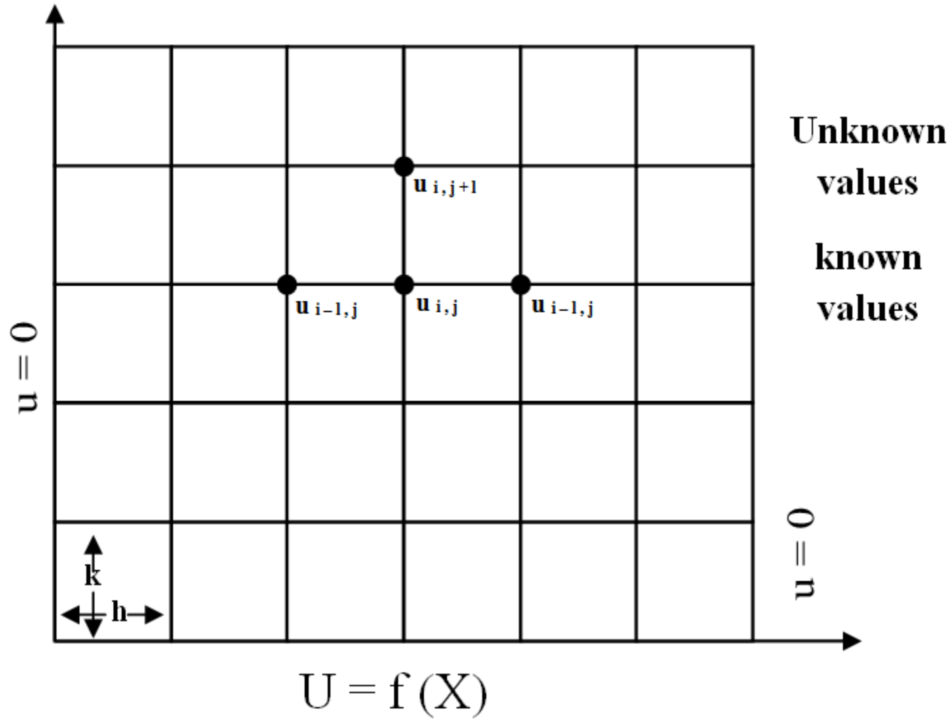


Figure (2)

This explicit method is known to be numerically stable and convergent whenever $\mathbf{r} \leq \frac{\mathbf{1}}{\mathbf{2}}$ and this conditions is one of the disadvantages of this method.

<u>Example :-</u> consider the following heat equation :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 \boldsymbol{u}}{\partial x^2} \qquad\qquad 0 < x < 1$$

where

i) u = 0 at x = 0 and x = 1 , t > 0 **(the boundary condition)**

ii) $u = 2x$, for $0 \leq x \leq \frac{1}{2}$
$u = 2(1 - x)$, for $\frac{1}{2} \leq x \leq 1$ } t = 0 **(the initial condition)**

$\Delta\ x = h = 0.1$ $\Delta t = k = 0.001$

$$r = \frac{k}{h^2} = \frac{1}{10}$$

$u_{0,0} = 0$, $u_{1,0} = 0.2$, $u_{3,0} = 0.6$, $u_{2,0} = 0.4$
$u_{4,0} = 0.8$, $u_{5,0} = 1$, $u_{6,0} = 0.8$, $u_{7,0} = 0.6$
$u_{8,0} = 0.4$, $u_{9,0} = 0.2$, $u_{10,0} = 0$

$$u_{i,j+1} = \frac{1}{10} u_{i-1,j} + 0.8 \; u_{i,j} + \frac{1}{10} u_{i+1,j}$$

when $i = 1$, $j = 0$

$u_{1,1} = \frac{1}{10} u_{0,0} + 0.8 \; u_{1,0} + \frac{1}{10} u_{2,0} = 0,2$

$u_{2,1} = \frac{1}{10} u_{1,0} + 0.8 \, u_{2,0} + \frac{1}{10} u_{3,0} = 0,4$

$u_{3,1} = \frac{1}{10} u_{2,0} + 0.8 \, u_{3,0} + \frac{1}{10} u_{4,0} = 0,6$

$u_{4,1} = \frac{1}{10} u_{3,0} + 0.8 \, u_{4,0} + \frac{1}{10} u_{5,0} = 0,8$

$u_{5,1} = \frac{1}{10} u_{4,0} + 0.8 \, u_{5,0} + \frac{1}{10} u_{6,0} = 1$

$u_{6,1} = \frac{1}{10} u_{5,0} + 0.8 \, u_{6,0} + \frac{1}{10} u_{7,0} = 0,8$

Application of equation (1.14) to the data of problem is shown in table (1)

Table (1)

	i=0 x=0	i=1 0,1	i=2 0,2	i=3 0,3	i=4 0,4	i=5 0,5	i=6 0,6
(i=0)t=0,000	0	0,2000	0,4000	0,6000	0,8000	1,000	0,8000
(i=1)= 0,001	0	0,2000	0,4000	0,6000	0,8000	0,9600	0,8000
(i=2)= 0,002	0	0,2000	0,4000	0,6000	0,7960	0,9280	0,7960
(i=3)= 0,003	0	0,2000	0,4000	0,5996	0,7896	0,9016	0,7896
(i=4)= 0,004	0	0,2000	0,4000	0,5986	0,7818	0,8792	0,7818
(i=5)= 0,005	0	0,2000	0,3999	0,5971	0,7732	0,8597	0,7732

(i=10)= 0,01	0	0,1996	0,3968	0,5822	0,7281	0,7867	0,7281
. . .							
(i=20)= 0,02	0	0,1938	0,3781	0,5373	0,6486	0,6891	0,6486

1.4 Crank – Nicolson implicit method

Al though the explicit method is computationally simple it has one serious drawback. The time step $\Delta t = k$ is necessarily very small because the process is valid only for $0 < \frac{k}{h^2} \leq \frac{1}{2}$, $k \leq \frac{1}{2} h^2$, and $h = \Delta x$ must be kept small in order to attain reasonable a ccuracy.

Crank and Nicolson (1947) proposed, and used a method that reduces the total volume of calculation and is valid (i.e convergent and stable) for all finite values of r.

They considered the partial differential equation as being satisfied at the midpoint (ih, $(j + \frac{1}{2}) k$) and replaced $\frac{\partial^2 u}{\partial x^2}$ by the mean of its finite difference approximation at the ith and (j + 1)th time – levels.

In other words they approximated the equation.

$$\left[\frac{\partial u}{\partial t}\right]_{i,\ j+\frac{1}{2}} = \left[\frac{\partial^2 u}{\partial x^2}\right]_{i,\ j+\frac{1}{2}}$$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{2}\left[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}\right]$$

$$2u_{i,j+1} - 2u_{i,j} = \frac{k}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\right)$$

$$r = \frac{k}{h^2} \quad , \quad r > 0$$

$$\boxed{-r\,u_{i+1,j+1} + (2+2r)\,u_{i,j+1} - r\,u_{i-1,j+1} = r\,u_{i+1,j} - (2-2r)\,u_{i,j} + r\,u_{i-1,j} \qquad (1.15)}$$

In general , the left side of equation (1.15) contains three unknown and the right side three known, pivotal values of u (Fig (3))

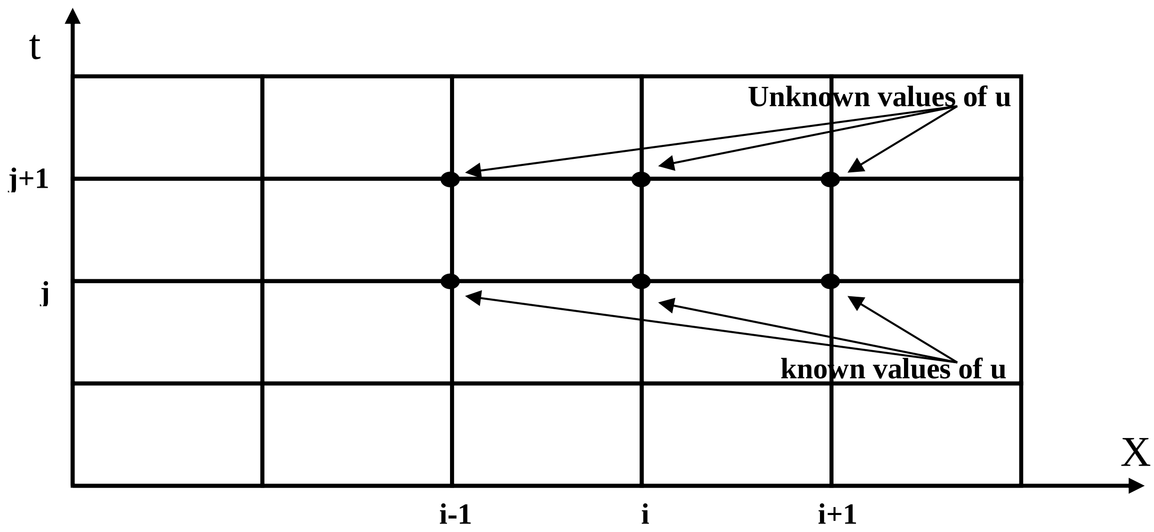


Figure (3)

Example:-

Use the Crank -Nicolson method to calculate a numerical solution of the previous worked

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{dx^2} \qquad 0 < x < 1 \text{ , } t > 0$$

let $h = 0.1$, $k = 0.01$

$$r = \frac{k}{h^2} = 1$$

$$- u_{i+1,j+1}\ 4u_{i,j+1} - u_{i-1,j+1} = u_{i+1,j} + u_{i-1,j}$$

let $j = 0$, $i = 1$

$- u_{2,1} + 4u_{1,1} - u_{0,1} = u_{2,0} + 4_{0,0}$

$- u_{2,1} + 4u_{1,1} = 0.4$ (1)

$j=0$, $i=2$

$- u_{3,1} + 4u_{2,1} - u_{1,1} = u_{3,0} + u_{1,0}$

$- u_{3,1} + 4u_{2,1} - u_{1,1} = 0.8$ (2)

$j=0$, $i=3$

$- u_{4,1} + 4u_{3,1} – u_{2,1} = u_{3,0} + u_{1,0}$

$- u_{3,1} + 4u_{2,1} – u_{2,1} = 1.2$ (3)

j= 0 , i = 4

$-u_{5,1}+4u_{4,1}–u_{3,1}=u_{5,0}+u_{3,}-u_{5,1}+4u_{4,1}–u_{3,1}=1.6$ (4)

j = 0 , i = 5

$- u_{4,1} + 4u_{5,1} – u_{4,1} = u_{6,0} + u_{4,0}$

$- u_{6,1} + 4u_{5,1} – u_{4,1} = 1.6$ (5)

$- u_{2,1} + 4u_{1,1} = 0.4$…………(1)

$- u_{3,1} + 4u_{2,1} – u_{1,1} = 0.8$ ………..(2)

$-u_{4,1} + 4u_{3,1} – u_{2,1} = 1.2$………..(3)

$-u_{5,1} + 4u_{4,1} – u_{3,1} = 1.6$ ………(4)

$- u_{6,1} + 4u_{5,1} – u_{4,1} = 1.6$ ……..(5)

Au = b

AS indicated :

$$\begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0,4 \\ 0,8 \\ 1,2 \\ 1,6 \\ 1,6 \end{bmatrix}$$

For the first time level and for each time level we have to solve the system Au=b, are easily solved by systematic Jacobi to give:

$u_{1=}$ 0,1989 , $u_{2=}$0,3956 , $u_{3=}$0,5834 , $u_{4=}$ 0,7381, $u_{5=}$ 0,7691.

Application of equation (1.15) to the data of problem is shown in table (2)

Table (2)

	i=0 x=0	i=1 0,1	i=2 0,2	i=3 0,3	i=4 0,4	i=5 0,5
t=0,00	0	0,2	0,4000	0,6000	0,8000	1,000
t= 0,01	0	00,1989	0,3956	0,5834	0,7381	0,7691

t= 0,02	0	00,1936	0,3989	0,5400	0,6461	0,6921
. . .						
t= 0,10	0	00,0948	0,1803	0,2482	0,2918	0,3021

1.5 The fully implicit method

The simplest implicit method which solve the differential equation was suggested by Brienetal. (1951) which approximate u_{xx} in the (j +1) level

$$\frac{\partial u}{\partial t} = \left.\frac{\partial^2 u}{\partial x^2}\right]_{i\,,\,j\,+\,1}$$

$$\frac{ui\,,j{+}1{-}ui\,,j}{k} = \frac{ui{-}1\,,j{+}2ui\,,j{+}1{+}ui{+}1\,,j{+}1}{h^2}$$

$$u_{i,\,j\,+1} - u_{i\,,\,j} = \frac{k}{h^2}\,(u_{i\,-\,1\,,j\,+1} - 2u_{i\,,\,j\,+1} + u_{i\,+\,1,\,j\,+1})$$

$$\boxed{r\,u_{i\,-\,1\,,\,j\,+1} - (1+2r)\,u_{i\,,j\,+1} + r\,u_{i\,+\,1\,,j\,+1} = -\,u_{i\,,j}}$$

The general, the left side of equation contains three unknown and the right side only one known , pivotal values of (fig (4))

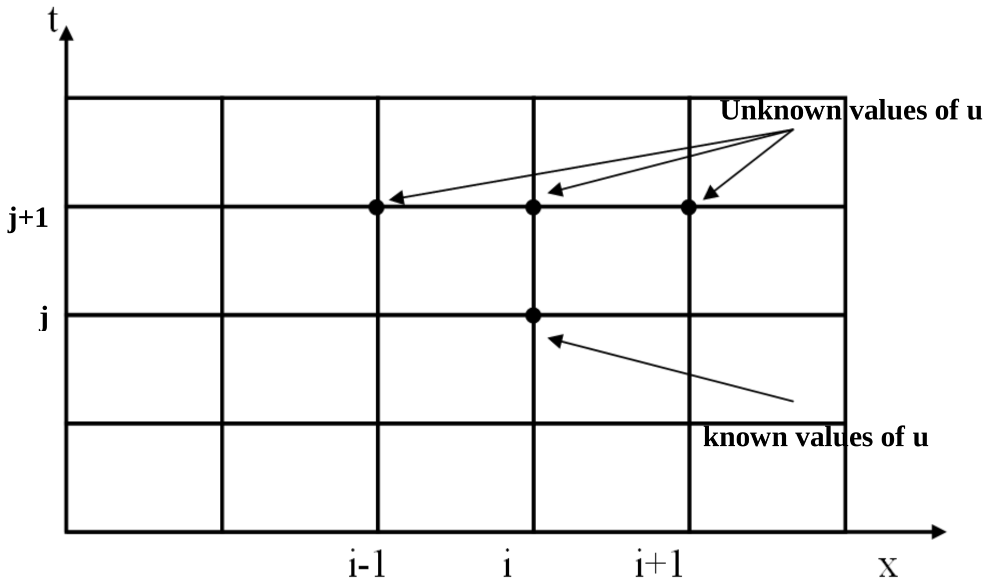


Figure (4)

Example :- consider the folling heat equation :

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} \qquad 0 < x < 1$$

h = 0.1 , k = 0.001

where

(i) u = 0 at x = 0 and x = 1 , t > 0 (The boundary condition)

(ii) $\left.\begin{array}{lll} u = 2x & \text{for} & 0 \le x \le \frac{1}{2} \\ u = 2(1 - x) & \text{for} & \frac{1}{2} \le x \le 1 \end{array}\right\} \quad t = 0$

let i = 1 , j = 0

$-u_{0,1} + 3u_{1,1} - u_{2,1} = u_{1,0}$

$3u_{1,1} - u_{2,1} = 0.2$ (1)

i = 2 , j = 0

$-u_{1,1} + 3u_{2,1} - u_{3,1} = u_{2,0}$

$-u_{1,1} + 3u_{2,1} - u_{3,1} = 0.4$ (2)

i = 3 , j = 0

$-u_{2,1} + 3u_{3,1} - u_{4,1} = u_{3,0}$

$-u_{2,1} + 3u_{3,1} - u_{4,1} = 0.6$ (3)

$i = 4 \quad , \quad j = 0$
$- u_{3,1} + 3u_{4,1} – u_{5,1} = u_{4,0}$
$- u_{3,1} + 3u_{4,1} – u_{5,1} = 0.8$ (4)

$i = 5 \quad , \quad j = 0$
$- u_{4,1} + 3u_{5,1} – u_{6,1} = u_{5,0}$
$- u_{4,1} + 3u_{5,1} – u_{6,1} = 1$ (5)
$3u_{1,1} – u_{2,1} = 0.2$

$-u_{1,1} + 3u_{2,1} – u_{3,1} = 0.4$
$- u_{2,1} + 3u_{3,1} – u_{4,1} = 0.6$
$- u_{3,1} + 3u_{4,1} – u_{5,1} = 0.8$
$- u_{4,1} + 3u_{5,1} – u_{6,1} = 1$

$$\begin{bmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.6 \\ 0.8 \\ 1 \end{bmatrix}$$

For the first time level and for each time level we have to solve the system Au = b, are easily solved by systematic ($u=A^{-1} * b$) to give:

$u_1 = 0.1917$, $u_2 = 0.3750$, $u_3 = 0.5333$, $u_4 = 0.6250$
$u_5 = 0.5417$

The last system of equations can be solved by any iterative method (Jacobi , Gauss – seidel or foR) or any direct method .

Chapter 2

Application

2.1 Introduction

The finite difference method are useful to obtain approximate solution to differential governing equation. In order to explain the finite difference method and comparison with exact solution, We consider the following sample problem.

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{x} \frac{\partial u}{\partial x} - \frac{u}{x^2} = 0$$

With Boundary condition

u (2) = 0.008 , u (6,5) = 0.003 at h = 1.5

2.2 Exact Solution of problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{x} \frac{\partial u}{\partial x} - \frac{u}{x^2} = 0$$

with boundary condition

u(2) = 0.008 , u(6,5) = 0.003 at h = 1.5

$u = x^r$

$$\frac{\partial u}{\partial x} = r\, x^{r-1}$$

$$\frac{\partial^2 u}{\partial x^2} = r\,(r-1)\, x^{r-2}$$

$$r\,(r-1)\, x^{r-2} + r\, x^{r-1}\, x^{-1} - x^{-2}\, x^{r} = 0$$

$$r^2 x^{r-2} - r\, x^{r-2} + r\, x^{r-2} - x^{r-2} = 0$$

$$(r^2 - 1)\, x^{r-2} = 0$$

$$r^2 - 1 = 0 \longrightarrow r = \mp 1$$

$$u = c_1 x + \frac{c_2}{x}$$

$u = 2c_1 + \frac{c_2}{2}$, $6.5\, c_1 + \frac{c_2}{6{,}5}$

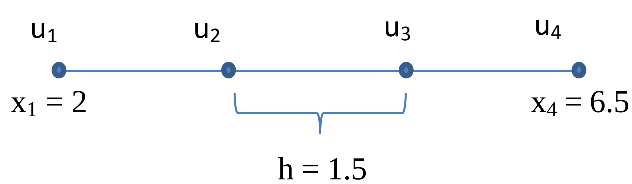
$0{,}008 = 2c_1 + \frac{1}{2}\, c_2 \ldots\ldots(1)$ $(*6.5$
$0{,}003 = 6{,}5c_1 + \frac{1}{6{,}5}\, c_2 \ldots\ldots(2)$ $(*2$

$0{,}052 = 13c_1 + 3{,}25c_2 \ldots\ldots.(3)$
$0{,}006 = 13c_1 + \frac{2}{6{,}5}\, c_2 \ldots\ldots\ldots(4)$

$0{,}046 = 2{,}9424c_2$
$c_2 = 0{,}015633$
$c_1 = 9{,}1503 * 10^{-5}$
$u = 9.1503 * 10^{-5}\, x + \frac{0{,}015634}{x}$
$u_2 = 4\,(3{,}5) = 0{,}0047871$
$u_3 = u(5) = 0{,}0035843$

2.3 Numerical Solution of problem

$\frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} - \frac{u}{x^2} = 0$



With Boundary condition

$u\,(2) = 0.008$, $u\,(6{,}5) = 0.003$, $h = 1.5$

$\frac{\partial^2 u}{\partial x^2} = \frac{ui{+}1{-}2ui{+}ui{-}1}{h^2}$

$\frac{\partial u}{\partial x} = \frac{ui{+}1{-}ui}{h}$

$\frac{ui{+}1{-}2ui{+}ui{-}1}{h^2} + \frac{1}{xi}\ \frac{ui{+}1{-}ui}{h} - \frac{ui}{(xi)^2} = 0$

$i = 1$ $u\,(2) = 0.008$

$i = 2$

$\frac{u3{-}2u2{+}u1}{(1{,}5)^2} + \frac{1}{3.5}\ \frac{u3{-}42}{1{,}5} - \frac{u2}{(3{,}5)^2} = 0$

$0{,}44444u_1 - 1{,}1610\, u_2 + 0{,}6349213 - 0$

i = 3

$$\frac{u4-2u3+u2}{(1.5)^2}+\frac{1}{5}\,\frac{u4-u3}{1.5}-\frac{u3}{(\,5\,)^2}$$

$0{,}44444u_2 - 1{,}0622u_3 + 0{,}57778u_4 = 0$

i = 4

$u_4 = 0{,}003$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0{,}44444 & -1{,}1610 & 0{,}63442 & 0 \\ 0 & 0{,}44444 & -1{,}0622 & 0{,}57778 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0{,}008 \\ 0 \\ 0 \\ 0{,}003 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0{,}44444 & -1{,}1610 & 0{,}63442 & 0 \\ 0 & 0{,}44444 & -1{,}0622 & 0{,}57778 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0{,}008 \\ 0 \\ 0 \\ 0{,}003 \end{bmatrix}$$

$u_1 = 0{,}008$, $u_2 = 0{,}005128$, $u_3 = 0{,}003778$, $u_4 = 0{,}003$

The comparison in results between the analytic solution and numerical results is illustrated in Figure (5) .

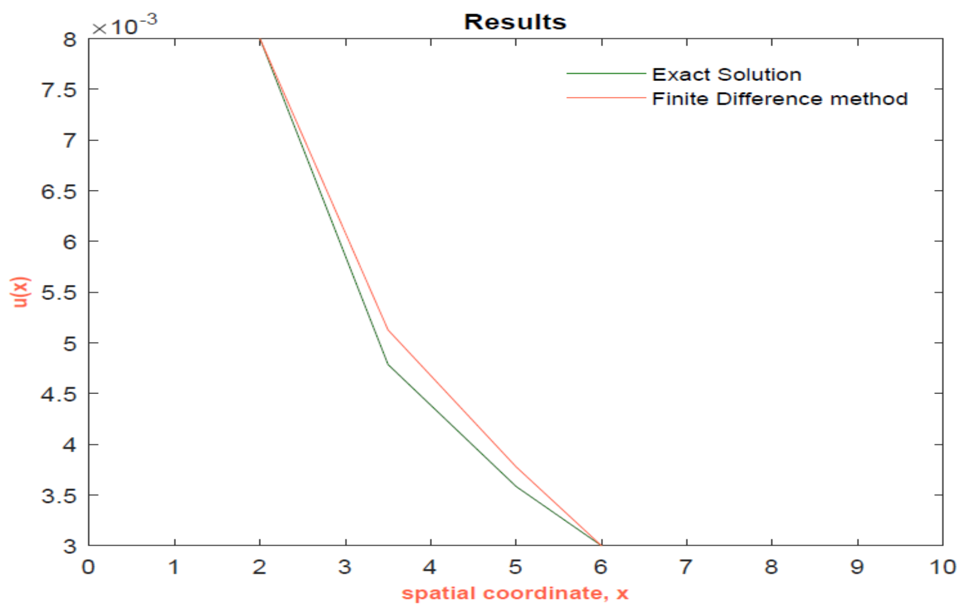


Figure (5)

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[4] J. M. Ortega , W.G. Poole , An introduction to numerical methods for differential equations.